DYNAMICAL BOREL-CANTELLI LEMMA FOR HYPERBOLIC SPACES

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Abstract. We prove that almost every (resp. almost no) geodesic rays in a finite volume hyperbolic manifold of real dimension n intersects for arbitrary large times t a decreasing family of balls of radius r_t, provided the integral $\int_0^\infty r_t^{n-1}dt$ diverges (resp. converges).
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1. Introduction

In this paper, we study the shrinking target problem on hyperbolic manifolds of finite volume. Let V be a hyperbolic manifold (possibly complex, quaternionic or Cayley hyperbolic) of real dimension n. Write T^1V for the unitary tangent bundle over V, $\pi : T^1V \to V$ the canonical projection, $(\phi^t)_{t\in\mathbb{R}}$ the geodesic flow on T^1V, μ the Liouville measure on T^1V, and d the Riemannian distance on V. We are interested in the Liouville measure of the unit vectors that generate a geodesic rays that keeps intersecting a shrinking family of balls for arbitrary large times. We show

Theorem 1. *Let $(B_t)_{t\geq 0}$ be a decreasing family of closed balls in V (with respect to the metric d), of radius $(r_t)_{t\geq 0}$. Then for μ-almost every (resp. μ-almost no) v in T^1V, the set*

$$\{t \geq 0 \,:\, \pi(\phi^t v) \in B_t\},$$

is unbounded (resp. bounded) provided $\int_0^\infty r_t^{n-1}dt$ diverges (resp. converges).

See also [CK01] for general results of this type. As a consequence of Theorem 1, a generic geodesic ray approach the point p at a speed which is of the order of $t^{-1/(n-1)}$. More precisely,

Corollary 1. *For all p in V and μ-almost every v in T^1V,*

$$\limsup_{t\to+\infty} \frac{-\log d(p, \pi(\phi^t v))}{\log t} = \frac{1}{n-1}.$$

This is the analog of Sullivan's *logarithm law for geodesics* (see [Su82], and also [KM99] in the context of symmetric spaces), when one considers balls in the manifold rather that horospherical neighborhoods of one cusp. When V is compact and of negative curvature, Hersonsky and Paulin [HP01] provided sharp estimate of the Hausdorff dimension of geodesic rays that accumulate on a point p in V exponentially fast, depending on the exponent.

Theorem 1 is deduced from the following general proposition. Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting on a probability space (X,μ), preserving μ, and such that the flow is ergodic with respect to μ. Let $F=(f_t)_{t\geq 0}$ be a family of functions $f_t : X \to \mathbb{R}$, such that $F : [0,+\infty[\times X \to \mathbb{R}_+$ is measurable. Such a family will be said *decreasing* if for any positive real numbers $s \geq t$ then $f_s \leq f_t$, and *positive* if $f_t \geq 0$ for any $t \geq 0$. Denote by $L^p(\mu)$ the space of measurable functions of integrable p-power, endowed with the classical L^p norm. We will write

$$S_T[F](x) = \int_0^T f_t(\phi^t x)dt,$$

$$I_T[F] = \int_0^T \left(\int_X f_t d\mu\right) dt.$$

Proposition 1. *Let p be in $]1,+\infty[$, $F=(f_t)_{t\geq 0}$ be a measurable, positive and decreasing family of functions such that f_t is in $L^p(\mu)$ for all $t \geq 0$. Assume that $\lim_{T\to+\infty} I_T[F] = +\infty$, and that $S_T[F]/I_T[F]$ remains bounded in L^p norm as T goes to infinity. Then $S_T[F]/I_T[F]$ converges weakly in $L^p(\mu)$ to the constant function 1, and there exists $c \in [1,+\infty]$, such that for μ-almost every x in X, we have*

$$\limsup_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]} = c.$$

Proposition 1 fails for $p=1$, see [CK01], Proposition 1.6 and remark after. Note that it is possible to deduce from Fatou's Lemma (see [KM99], 2.3, and Lemma 3) that if F is a measurable, positive and decreasing family of $L^1(\mu)$-functions such that $\lim_{T\to+\infty} I_T[F] = +\infty$, then we have

$$\liminf_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]} \leq 1.$$

As in the Borel-Cantelli Lemma, the divergence case is the difficult one. We show that for some functions f_t on T^1V related to the balls B_t, $S_T[F]/I_T[F]$ remains bounded in L^2 norm, then apply the preceding proposition to conclude. The arguments to bound the L^2-norm are purely geometrical, and do not involve the rate of mixing of the geodesic flow, unlike [KM99], but however make use of the very particular symmetries of rank 1 globally symmetric spaces. In the case when one replace f_t in the proof by the characteristic function of the unitary tangent space of the ball B_t, one obtain the following

Proposition 2. *Let $(B_t)_{t\geq 0}$ be a decreasing sequence of closed balls in V, of radius $(r_t)_{t\geq 0}$. Then for μ-almost every (resp. μ-almost no) v in T^1V, the set*

$$\{t \geq 0 \,:\, \pi(\phi^t v) \in B_t\},$$

is of infinite Lebesgue measure (resp. finite Lebesgue measure) provided $\int_0^\infty r_t^n dt$ diverges (resp. converges). □

As a final remark, we wish to underline that the exponent $n-1$ in Theorem 1 is related to the dimension, and not to the critical exponent of the fundamental group, as one can see by considering the non-real hyperbolic spaces.

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2. Proof of Proposition 1

We will need the following sequence of Lemmas. We refer to [Ru95] for facts about measure theory.

Lemma 1. *Let $F = (f_t)_{t\geq 0}$ be a measurable and positive family of functions. Then if*

$$I_\infty[F] = \int_0^{+\infty} \left(\int_X f_t d\mu \right) dt < +\infty,$$

we have that $S_T[F]$ tends everywhere and in L^1-norm toward a positive and L^1 function.

Proof. Put $S_\infty[F](x) = \lim_{T\to+\infty} S_T[F](x)$, which exists because $S_T[F]$ is increasing in T. From the Lebesgue's monotone convergence Theorem, $S_\infty[F]$ is in $L^1(\mu)$ and

$$\int_X S_\infty[F] d\mu = I_\infty[F].$$

Lebesgue's dominated convergence Theorem allows us to conclude that the convergence occurs in L^1-norm. □

We recall that, if two reals $p > 1$ and $q > 1$ are such that $1/p + 1/q = 1$, the two Banach spaces $L^p(\mu)$ and $L^q(\mu)$ are dual to each other, and that a sequence $(g_n)_{n\geq 0}$ in $L^p(\mu)$ converges weakly to g means that for every h in $L^q(\mu)$, $\int_X g_n h d\mu$ tends to $\int_X g h d\mu$ as n tends to infinity.

Lemma 2. *Let $F = (f_t)_{t\geq 0}$ be a measurable, positive and decreasing family of functions, and assume that f_0 is in $L^p(\mu)$. If $I_\infty[F] = +\infty$, then every weak limit in $L^p(\mu)$ of $S_T[F]/I_T[F]$ as T tends to infinity is the constant function equal to* 1.

Proof. Let h be a weak limit in $L^p(\mu)$ of a subsequence $S_{T_n}[F]/I_{T_n}[F]$. Let $t > 0$ be fixed. We wish to prove first that $h \leq h \circ \phi^t$. We have

$$S_{T_n+t}[F](x) - S_{T_n}[F](x) = \int_{T_n}^{T_n+t} f_s(\phi^s x)ds \leq \int_0^t f_0(\phi^s(\phi^{T_n}(x)))ds.$$

So this difference is positive and bounded in L^p-norm. Dividing it by $I_{T_n}[F]$, we obtain that $S_{T_n+t}[F](x)/I_{T_n}[F]$ converges weakly to h. On the other hand, we have

$$\begin{aligned} S_{T+t}[F](x) - S_T[F](\phi^t x) &= \int_0^{T+t} f_s(\phi^s x)ds - \int_0^T f_s(\phi^{t+s} x)ds \\ &= \int_0^t f_s(\phi^s x)ds + \int_0^T (f_{s+t} - f_s)(\phi^{t+s} x)ds. \end{aligned}$$

Since f_t is decreasing in t, we have

$$S_{T+t}[F](x) - S_T[F](\phi^t x) \leq \int_0^t f_s(\phi^s x)ds,$$

And when dividing by $I_{T_n}[F]$, and replacing T by T_n, we obtain :

$$\frac{S_{T_n+t}[F](x)}{I_{T_n}[F]} - \frac{S_{T_n}[F](\phi^t x)}{I_{T_n}[F]} \leq \frac{S_t[F](x)}{I_{T_n}[F]}$$

Let E be the set of x such that $h(x) \geq h(\phi^t x)$. We can write

$$\int_E \frac{S_{T_n+t}[F](x)}{I_{T_n}[F]} - \frac{S_{T_n}[F](\phi^t x)}{I_{T_n}[F]} d\mu(x) \leq \frac{\int_E S_t[F](x)dx}{I_{T_n}[F]},$$

and since the characteristic function of the set E is in $L^q(\mu)$, we obtain as $n \to +\infty$ the inequality

$$\int_E h(x) - h(\phi^t x) d\mu(x) \leq 0.$$

From the definition of E, this means that $h \leq h \circ \phi^t$ almost everywhere. Because of the invariance of μ and that h is in $L^p(\mu)$, this implies that $h = h \circ \phi^t$ almost everywhere. Since the flow is ergodic with respect to μ, h is constant (almost everywhere). On the other hand, $\int_X S_T[F]/I_T[F]d\mu = 1$, so $h = 1$ almost everywhere. $\square$

Lemma 3. *Let $F = (f_t)_{t\geq 0}$ be a measurable, positive and decreasing family of functions, and suppose that f_0 is in $L^1(\mu)$. Assuming $I_\infty[F] = +\infty$, then the function from X to $[0, +\infty]$*

$$L(x) = \limsup_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]},$$

is constant almost everywhere. Moreover, the function from X to $[0, +\infty]$

$$l(x) = \liminf_{T\to+\infty} \frac{S_T[F](x)}{I_T[F]},$$

is constant almost everywhere, and this constant is in $[0, 1]$.

Proof. We have for any $t > 0$ and $T \geq 0$,

$$\begin{aligned} S_{T+t}[F](x) - S_T[F](\phi^t x) &= \int_0^{T+t} f_s(\phi^s x)ds - \int_0^T f_s(\phi^{t+s}x)ds \\ &= \int_0^t f_s(\phi^s x)ds + \int_0^T (f_{s+t} - f_s)(\phi^{t+s}x)ds \end{aligned}$$

Denote by $G^{(t)} = (g_s^{(t)})_{s\geq 0}$ the family of functions defined by

$$\forall s \geq 0,\ g_s^{(t)} = f_s - f_{s+t}.$$

The above equation can be rewritten

$$S_{T+t}[F](x) - S_T[F](\phi^t x) = S_t[F](x) - S_T[G^{(t)}](\phi^t x)$$

Since F is decreasing, $G^{(t)}$ is positive, and

$$I_\infty[G^{(t)}] = I_t[F] < \infty.$$

From Lemma 1, $S_T[G^{(t)}](x)$ has a finite limit $H^{(t)}(x)$ as $T \to +\infty$ for almost every x, which is moreover in $L^1(\mu)$. So

$$\frac{S_t[F](x)}{I_T[F]} \geq \frac{I_{T+t}[F]}{I_T[F]} \frac{S_{T+t}[F](x)}{I_{T+t}(x)} - \frac{S_T[F](\phi^t x)}{I_T[F]} \geq -\frac{H^{(t)}(\phi^t x)}{I_T[F]}.$$

Since F is a decreasing family and $I_\infty[F] = +\infty$, it follows that $I_{T+t}[F]/I_T[F]$ goes to 1 as T goes to infinity. From the last equation, we deduce that for any x such that $H^{(t)}(x) < \infty$, we have

$$L(x) = L(\phi^t x).$$

If $t \leq 1$, then $H^{(t)} \leq H^{(1)}$. The sets $L^{-1}([a, b])$ are invariant under ϕ^t for any t in $[0, 1]$ and $a < b$, apart from a set of 0 measure, because $H^{(1)}$ is finite almost everywhere. Because μ is ergodic, the sets $L^{-1}([a, b])$ are of measure 0 or 1. A dichotomy on the intervals $[a, b]$ then shows that L must be constant almost everywhere. The proof that l is constant almost everywhere is identical. Like in [KM99], by the Fatou Lemma

$$\int_X l(x)d\mu \leq \liminf_{T\to\infty} \left(\int_X S_T[F](x)d\mu(x) \right) / I_T[F] = 1.$$

□

Proof of Proposition 1. Let h be any function in $L^q(\mu)$. The family of integrals

$$J(T) = \int_X h(x)S_T[F](x)/I_T[F]d\mu(x)$$

is bounded, since $S_T[F]/I_T[F]$ is also bounded in L^p-norm. If one takes any sequence T_n that goes to infinity, such that $J(T)$ has a limit l, then by the Banach-Alaoglu Theorem one can find a subsequence $T_{\phi(n)}$ such that $S_T[F]/I_T[F]$ has a weak limit g in $L^p(\mu)$. From Lemma 2, $g = 1$. Then $l = \int_X hd\mu$ necessarily, and so $J(T)$ goes to $\int_X hd\mu$ as T tends to infinity. Since this is valid for any h in $L^q(\mu)$, this means exactly that $S_T[F]/I_T[F]$ converges weakly to 1 as T goes to infinity. On the other hand, Lemma 3 implies that $L(x) = \limsup_{T\to+\infty} S_T[F](x)/I_T[F]$ is constant almost everywhere. Let c be this constant. By absurd, suppose that $c < 1$. Then for any $\epsilon > 0$ sufficiently small, there exist a set $E \subset X$ of positive measure, such that for any x in E, and $T \geq 1/\epsilon$, we have $S_T[F](x) < (1 - \epsilon)I_T[F]$. Fix such an ϵ, and let 1_E be the characteristic function of E. It belongs to $L^q(\mu)$ and so we have that $\int_E S_T[F]/I_T[F]d\mu$ goes to $\mu(E)$ as T goes to infinity. But we can also show that these integrals are less that $(1 - \epsilon)\mu(E)$ provided that $T > 1/\epsilon$. This is a contradiction, and so $c \geq 1$.

□

3. Proof of Theorem 1

The universal cover $\tilde{V}$ of V is homothetic to the real hyperbolic space $\mathbb{H}^d_{\mathbb{R}}$ of real dimension d, complex hyperbolic space $\mathbb{H}^d_{\mathbb{C}}$ of real dimension $2d$, quaternionic hyperbolic space $\mathbb{H}^d_{\mathbb{H}}$ of real dimension $4d$, or the Cayley hyperbolic plane $\mathbb{H}^2_{Ca}$ of real dimension 16 (see [Mo73] for a description of these spaces). We denote by δ the volume entropy of $\tilde{V}$, which is

$$\delta = \lim_{t\to\infty} \frac{\log(\mathrm{Vol}(B(o, t)))}{t}.$$

In fact, the volume of balls of radius t and the area of spheres of radius t are both equivalent to some constant times $\exp(\delta t)$ as $t \to +\infty$. It is not necessary for what follows to normalize the sectional curvature of $\tilde{V} = \mathbb{H}^d_{\mathbb{K}}$ between -4 and -1, but if this is the case and $d \geq 2$, $\delta = d - 1 = n - 1$ in the case $\mathbb{K} = \mathbb{R}$ and the curvature is -1, $\delta = n - 2 + \dim_{\mathbb{R}}\mathbb{K}$ if $\mathbb{K} \neq \mathbb{R}$.

We recall that $T^1\tilde{V} = \mathcal{G} \times \mathbb{R}$, where $\mathcal{G}$ is the space of oriented, unpointed geodesics, and the geodesic flow ϕ^t acts on $\phi^t(g,s) = (g, s+t)$; in case one writes $\mathcal{G} = \partial\tilde{V} \times \partial\tilde{V} - diag$, this is simply the *Hopf decomposition*. Let ν be the measure on $\mathcal{G}$ such that $\mu = \nu \otimes dt$. The action of the geodesic flow on T^1V is ergodic with respect to μ, and since V is of finite volume, we can freely normalize μ to be a probability measure.

The following two estimates will be the main ingredients of the proof.

Lemma 4. *There exists a constant $c_1 > 0$ such that, for any two balls in $\tilde{V}$ of respective center and radius (o,r), (o',r') that satisfy $r < 1$, $r' < 1$, and $d(o,o') > 2$, then the ν-measure of geodesics which intersects those two balls is less than*

$$c_1 r^{n-1} r'^{n-1} \exp(-\delta d(o,o')).$$

Sketch of proof. Let $R = d(o,o')$. Consider the hyperbolic sphere $S(o,R)$ centered on o of radius R. There exists a constant $c > 0$, independent of R, such that one can find $k \geq cr'^{n-1}exp(\delta R)$ points $p_1, ..., p_k$ on S satisfying that the balls $B(p_i, 2r')$ of center p_i and radius $2r'$ are disjoints. Then the measure m_i of the geodesics which intersect both $B(o,r)$ and $B(p_i, r')$ are all equal, because there exists a isometry that sends o to o and p_i to p_j for all i, j (hyperbolic spaces are *2-points homogeneous*), and are equal to the measure m we are interested in. Let M be the measure of geodesics which intersect $B(o,r)$. Any geodesic intersecting $B(o,r)$ intersect at most two balls $B(p_i, r')$, so one have the inequality $km \leq 2M$. Since M is less that some constant times r^{n-1}, the result follows from the lower bound on k. □

Lemma 5. [Su79] *For any discrete subgroup Γ of isometries of $\tilde{V}$, for any p in V, there exists a constant $c_2 > 0$ such that for any $T \geq 0$,*

$$|\{\gamma \in \Gamma \,:\, d(p, \gamma p) \leq T\}| \leq c_2 \exp(\delta T).$$

□

The fundamental group $\Gamma = \pi_1(V)$ can be seen as a subgroup of the isometry group of $\tilde{V}$. For a point p in a Riemannian manifold W, let's write $i_W(p)$ for the injectivity radius of p in W. Given a real number $R > 0$, a ball B of center o and radius r, containing a point p, in a Riemannian manifold W, with $0 < r \leq R \leq i_W(p)/4$, we define for any

v in T^1W,

$$H_B(v) = \begin{cases} 1 & \text{if } d(\pi(\phi^\theta v), o) \leq r, \text{for some } \theta \in [-R/2, R/2], \\ 0 & \text{otherwise.} \end{cases}$$

From now on, we can (and we will) assume that the radius r_t of the ball B_t goes to 0 as t tends to infinity, because otherwise $\cap_{t\geq 0} B_t$ would contain a ball, and the result follows from Birkhoff's ergodic Theorem. Then, necessarily, $\cap_{t\geq 0} B_t$ is a point p in V. We fix R to be the minimum between $i_V(p)/4$ and 1. Moreover, one can assume that r_t is stricly less than R for any $t \geq 0$; it will be convenient to define r_t to be r_0 for all t negative. Let $\tilde{p}$ and $\tilde{B}_t$ be lifts of p, B_t, such that $\tilde{p}$ is contained in $\tilde{B}_t$ for all $t \geq 0$.

We define $f_t = H_{B_t}$ for this R and $W = V$, and $g_t = H_{\tilde{B}_t}$ for this R and $W = \tilde{V}$. Thus, the lift $\tilde{f}_t$ of f_t to $T^1\tilde{V}$ satisfies

$$\tilde{f}_t = \sum_{\gamma \in \Gamma} g_t \circ \gamma.$$

The family of functions $F = (f_t)_{t\geq 0}$ on T^1V is measurable, positive, and decreasing because the family of balls B_t is decreasing. Moreover, it can be seen easily that $\int_{T^1V} f_t d\mu$ is equivalent as $t \to +\infty$ to some constant (depending on n and R only) times r_t^{n-1}, and thus $I_T[F]$ is equivalent to the same constant times $\int_0^T r_t^{n-1} dt$, provided any these two functions tends to infinity as T goes to infinity.

The main argument is

Lemma 6. *The L^2-norm of $S_T[F]/I_T[F]$ is bounded for all $T \geq 1$.*

Proof. Using Fubini's Theorem, one can write for any $T \geq 0$, any v in T^1V,

$$S_T[F](v)^2 = 2\int_0^T \int_0^s f_t(\phi^t v) f_s(\phi^s v) dt ds.$$

Let's integrate over T^1V, and then make a change of variables $w = \phi^s v$. We obtain

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_0^s \int_{T^1V} f_s(w) f_t(\phi^{t-s} w) dw dt ds.$$

Let D be a fundamental domain of V for Γ, such that D contains the ball of center $\tilde{p}$, radius $2R$, and $T^1D = \pi^{-1}(D)$.

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_0^s \int_{T^1D} \tilde{f}_s(w) \tilde{f}_t(\phi^{t-s} w) dw ds dt.$$

The choice of D insures that for all w in T^1D and $s \geq 0$, we have $\tilde{f}_s(w) = g_s(w)$ because g_s has support contained in D, and so in the sum $\sum_{\gamma\in\Gamma} g_t(\gamma(w))$, all terms but the one corresponding to $\gamma = id$ are zero. So we have

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \sum_{\gamma\in\Gamma} \int_0^s \int_{T^1D} g_s(w)g_t(\gamma\phi^{t-s}w)dwdtds.$$

Putting $v = -w$, and using the symmetry of the functions g_t, we deduce that

$$\int_{T^1V} S_T[F](v)^2 dv = 2\int_0^T \int_{T^1D} \sum_{\gamma\in\Gamma} \int_0^s g_s(v)g_t(\gamma\phi^{s-t}v)dvdtds.$$

For any positive integer i, we define Γ_i to be the finite set of γ in Γ such that $d(\tilde{p}, \gamma\tilde{p})$ is in the interval $[i-1, i+1]$. Since the union of these sets is Γ itself, we have

$$\int_{T^1V} S_T[F](v)^2 dv \leq 2\int_0^T \sum_{i>0}\sum_{\gamma\in\Gamma_i} \int_0^s \int_{T^1D} g_s(v)g_t(\gamma\phi^{s-t}v)dvdtds.$$

Let's analyse when the integrated function can be strictly positive. For fixed γ, v, s and t, the quantity

$$g_s(v)g_t(\gamma\phi^{s-t}v)$$

is not zero if and only if there exists some $(\theta_1, \theta_2) \in [-R/2, R/2]^2$ such that $\pi(\phi^{\theta_1}v)$ is in $\tilde{B}_s$, and $\pi(\phi^{\theta_2+s-t}v)$ is in $\gamma^{-1}\tilde{B}_t$.

In this case, the quantity above is equal to 1 and one can state, in virtue of the triangle inequality, that

$$|(s-t) - d(\tilde{p}, \gamma\tilde{p})| < |\theta_1| + |\theta_2| + 2r_t + 2r_s \leq 5R, \tag{1}$$

and moreover that the geodesic ray $(\phi^u v)_{u\geq -1}$ intersects the ball of center $\gamma^{-1}\tilde{p}$ and radius $2r_{s-d(\tilde{p},\gamma\tilde{p})-5R}$ (this is because r_t is decreasing), and intersects also the ball of center $\tilde{p}$ and radius $2r_s$.

So, for fixed s, v and γ in Γ_i, Equation (1) implies that $g_s(v)g_t(\gamma\phi^{s-t}v)$ is zero when t is outside the interval $[s-d(\tilde{p},\gamma\tilde{p})-5R, s-d(\tilde{p},\gamma\tilde{p})+5R]$, of length $10R$. So we have

$$\int_0^s g_s(v)g_t(\gamma\phi^{s-t}v)dt \leq 10R, \tag{2}$$

in all cases, and that this integral vanishes if the geodesic generated by v does not intersects the two balls $B(\tilde{p}, 2r_s)$,$B(\gamma^{-1}\tilde{p}, 2r_{s-i-5R-1})$, or if $s < i - (5R+1)$. If moreover $i \geq 5R+3$, then $s - t \geq 2$ by (1), and by Lemma 4 the measure of the set of such geodesics is bounded by $c_1 4^{n-1} r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)}$. Since on a geodesic, the interval of vectors

v such that $g_s(v) > 0$ is at most of length 3, we obtain when s and $\gamma \in \Gamma_i$ are fixed and $i \geq 5R+3$,

$$\int_{T^1D} \int_0^s g_s(v) g_t(\gamma\phi^{s-t}v) dt dv \leq 30R4^{n-1} c_1 r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)},$$

and this quantity is zero if $s+5R+1 < i$. Thus, if $i \geq 5R+3$, summing over γ in Γ_i, we obtain

$$\sum_{\gamma\in\Gamma_i} \int_{T^1D} \int_0^s g_s(v) g_t(\gamma\phi^{s-t}v) dt dv \leq 30R4^{n-1} c_1 r_s^{n-1} r_{s-i-5R-1}^{n-1} e^{-\delta(i-1)} N_i,$$

where we have written N_i for the number of elements of Γ_i; moreover, the majorated sum does in fact vanish if $s+5R+1 < i$. From Lemma 5, $N_i \leq c_2 e^{\delta(i+1)}$. Let's write $[x]$ for the integer part of any real number x. It follows that there exists some positive constant c_3 (depending only on R) such that

$$\int_0^T \sum_{i>0} \sum_{\gamma\in\Gamma_i} \int_0^s \int_{T^1D} g_s(v) g_t(\gamma\phi^{s-t}v) dv dt ds$$

$$\leq c_3 \int_0^T \left(\sum_{i=[5R+4]}^{[s+5R]+1} r_s^{n-1} r_{s-i-5R-1}^{n-1} \right) ds + \int_0^T \sum_{i=1}^{[5R+3]} \sum_{\gamma\in\Gamma_i} \int_{T^1D} \int_0^s g_s(v) g_t(\phi^{s-t}v) dt dv ds.$$

Observing that $\cup_{i=1}^{[5R+3]} \Gamma_i$ is a finite set, let's write K its cardinal. Since (r_t) is decreasing, $r_{s-i-5R-1}^{n-1} \leq \int_{s-i-5R-2}^{s-i-5R-1} r_t^{n-1} dt$. Moreover, in Equation 2, we can freely multiply the upper bound by $g_s(v)$, because if $g_s(v) = 0$ then the majorated integral vanishes, else $g_s(v) = 1$. This allows us to write

$$\int_{T^1V} S_T[F](v)^2 dv \leq 2c_3 \int_0^T r_s^{n-1} \sum_{i=[5R+4]}^{[s+5R+1]} \int_{s-i-5R-2}^{s-i-5R-1} r_t^{n-1} dt ds + 2K \int_0^T \int_{T^1D} 10R g_s(v) dv,$$

Since $\int_{T^1D} g_s(v) dv$ is equivalent, as s tends to infinity, to some constant times r_s^{n-1}, there exists some positive constant c_4, depending only on R, such that

$$\int_{T^1V} S_T[F](v)^2 dv \leq 2c_3 \int_0^T r_s^{n-1} \int_{-10R-4}^{s-10R-7} r_t^{n-1} dt ds + c_4 \int_0^T r_t^{n-1} dt,$$

$$\int_{T^1V} S_T[F](v)^2 dv \leq 2c_3 \int_0^T r_s^{n-1} \int_{-10R-4}^{s} r_t^{n-1} dt ds + c_4 \int_0^T r_t^{n-1} dt,$$

and by Fubini's Theorem,

$$\int_{T^1V} S_T[F](v)^2 dv \le c_3 \left(\int_0^T r_s^{n-1} ds \right)^2 + (2(10R+4)c_3 r_0 + c_4) \left(\int_0^T r_s^{n-1} ds \right).$$

Since $I_T[F]$ is equivalent, as T tends to infinity, to some constant times $\int_0^T r_t^{n-1} dt$, we conclude that $S_T[F]/I_T[F]$ is bounded is L^2-norm. □

Proof of Theorem 1. Convergence case. We assume $\int_0^{+\infty} r_t^{n-1} dt < +\infty$. For any $w \in T^1V$ and $s \ge 0$, if $\pi(w)$ is B_s, then from the definition of f_s we have

$$\int_{-R}^{R} f_s(\phi^t w) dt \ge R.$$

Since $F = (f_s)_{s \ge 0}$ is decreasing, we can write, after a change of variable $u = t + s - R$, that

$$\int_{s-2R}^{s} f_u(\phi^{u-s+R} w) du \ge R,$$

and so if $\pi(\phi^s v)$ is in B_s, we have

$$\int_{s-2R}^{s} f_t(\phi^{t+R} v) dt \ge R.$$

However, Lemma 1 states that for almost every v,

$$\int_0^\infty f_t(\phi^t v) dt < \infty,$$

and so for a set of full measure, we can rewrite it

$$\int_0^\infty f_t(\phi^t(\phi^R v)) dt < \infty,$$

and by the preceding assertion, this implies that for t sufficiently large and v in a set of full measure, $\pi(\phi^t v)$ is not in B_t.

Divergence case. We assume $\int_0^{+\infty} r_t^{n-1} dt = +\infty$. Let v in T^1V such that there exists some $T > 0$, such that for all $t \ge T$, $\pi(\phi^t v)$ is not in B_t. Since the balls B_t are decreasing, we have that for any $s \ge t \ge T$, $\pi(\phi^t v) \notin B_s$. In particular, if $t \ge T + R$ and $t \in [s, s+R]$, $\pi(\phi^{t-R} v) \notin B_s$. From the definition of f_s, this implies that $f_s(\phi^{s-R/2} v) = 0$, for all $s \ge T + R$. This shows that $S_\infty[F](\phi^{-R/2} v) < +\infty$, provided the geodesic ray $(\phi^t v)_{t \ge 0}$ intersects B_t for bounded times t only. But from Lemma 6 and Proposition 1, we have that $S_\infty[F](v) = +\infty$ for almost every v. So the preceding situation may happen for a set of v of null measure only. □

Proof of corollary 1. This is quite classical. We consider the family of balls centered on p of radius $r_t = (t+1)^{-1/(n-1)-\epsilon}$ with arbitrary small $\epsilon > 0$, and use the convergence case of Theorem 1 to obtain the upper bound $1/(n-1)+\epsilon$. Then we use the divergence case applied to $r_t = (t+1)^{-1/(n-1)}$ in order to obtain the lower bound $1/(n-1)$. $\square$

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