# The radiation condition at infinity for the high-frequency Helmholtz equation with source term: a wave packet approach

François Castella

IRMAR - Université de Rennes 1
Campus Beaulieu - 35042 Rennes Cedex - France
francois.castella@univ-rennes1.fr

**J. Funct. Anal., Vol. 223, pp. 204-257 (2005).**

## Contents

**Abstract:** We consider the high-frequency Helmholtz equation with a given source term, and a small absorption parameter $\alpha > 0$. The high-frequency (or: semi-classical) parameter is $\varepsilon > 0$. We let $\varepsilon$ and $\alpha$ go to zero simultaneously. We assume that the zero energy is non-trapping for the underlying classical flow. We also assume that the classical trajectories starting from the origin satisfy a transversality condition, a generic assumption.

Under these assumptions, we prove that the solution $u^\varepsilon$ radiates in the outgoing direction, **uniformly** in $\varepsilon$. In particular, the function $u^\varepsilon$, when conveniently rescaled at the scale $\varepsilon$ close to the origin, is shown to converge towards the **outgoing** solution of the Helmholtz equation, with coefficients frozen at the origin. This provides a uniform version (in $\varepsilon$) of the limiting absorption principle.

Writing the resolvent of the Helmholtz equation as the integral in time of the associated semi-classical Schrödinger propagator, our analysis relies on the following tools: (i) For very large times, we prove and use a uniform version of the Egorov Theorem to estimate the time integral; (ii) for moderate times, we prove a uniform dispersive estimate that relies on a wave-packet approach, together with the above mentioned transversality condition; (iii) for small times, we prove that the semi-classical Schrödinger operator with variable coefficients has the same dispersive properties as in the constant coefficients case, uniformly in $\varepsilon$.

**2000 Mathematics Subject Classification number:** Primary 35Q40; Secondary 35J10, 81Q20.

# 1 Introduction

In this article, we study the asymptotics $\varepsilon \to 0^+$ in the following scaled Helmholtz equation, with unknown $w^\varepsilon$,

$$i\varepsilon\, \alpha_\varepsilon\, w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S\,(x)\,. \tag{1.1}$$

In this scaling, the absorption parameter $\alpha_\varepsilon > 0$ is small, i.e.

$$\alpha_\varepsilon \to 0^+ \ \text{ as } \ \varepsilon \to 0.$$

The limiting case $\alpha_\varepsilon = 0^+$ is actually allowed in our analysis. Also, the index of refraction $n^2(\varepsilon x)$ is almost constant,

$$n^2(\varepsilon x) \approx n^2(0).$$

The competition between these two effects is the key difficulty of the present work.

In all our analysis, the variable $x$ belongs to $\mathbb{R}^d$, for some $d \geq 3$. The index of refraction $n^2(x)$ is assumed to be given, smooth and non-negative[1]

$$\forall x \in \mathbb{R}^d, \quad n^2(x) \geq 0, \quad \text{ and } \ n^2(x) \in C^\infty(\mathbb{R}^d). \tag{1.2}$$

[1] Our analysis is easily extended to the case where the refraction index is a function that changes sign. The only really important assumption on the sign of $n$ is $n^2_\infty > 0$, see Proposition 4. Otherwise, all the arguments given in this paper are easily adapted when $n^2(x)$ changes sign, the analysis being actually simpler when $n^2(x)$ has the wrong sign because contribution of terms involving $\chi_\delta(H_\varepsilon)$ vanishes in that case (see below for the notations).

It is also supposed that $n^2(x)$ goes to a constant at infinity,

$$n^2(x) = n_\infty^2 + O\left(\langle x\rangle^{-\rho}\right) \quad \text{as} \quad x \to \infty, \tag{1.3}$$

for some, possibly small, exponant $\rho > 0$[2]. In the language of Schrödinger operators, this means that the potential $n_\infty^2 - n^2(x)$ is assumed to be either short-range or long range. Finally, the source term in (1.1) uses a function $S(x)$ that is taken sufficiently smooth and decays fast enough at infinity. We refer to the sequel for the very assumptions we need on the refraction index $n^2(x)$, together with the source $S$ (see the statement of the main Theorem below).

Upon the $L^2$-unitary rescaling

$$w^\varepsilon(x) = \varepsilon^{d/2} u^\varepsilon(\varepsilon x),$$

the study of (1.1) is naturally linked to the analysis of the high-frequency Helmholtz equation,

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right), \tag{1.4}$$

where the source term $S(x/\varepsilon)$ now plays the role of a concentration profile at the scale $\varepsilon$. In this picture, the difficulty now comes from the interaction between the oscillations induced by the source $S(x/\varepsilon)$, and the ones due to the semiclassical operator $\varepsilon^2\Delta/2 + n^2(x)$. We give below more complete motivations for looking at the asymptotics in (1.1) or (1.4).

The goal of this article is to prove that the solution $w^\varepsilon$ to (1.1) converges (in the distributional sense) to the **outgoing solution** of the natural constant coefficient Helmholtz equation, i.e.

$$\lim_{\varepsilon\to 0} w^\varepsilon = w^{\text{out}}, \quad \text{where} \quad w^{\text{out}} \quad \text{is defined as the solution to}$$

$$i0^+ w^{\text{out}}(x) + \frac{1}{2}\Delta_x w^{\text{out}}(x) + n^2(0) w^{\text{out}}(x) = S(x). \tag{1.5}$$

In other words,

$$\begin{aligned} w^{\text{out}} &= \lim_{\delta\to 0^+} \left(i\delta + \frac{1}{2}\Delta_x + n^2(0)\right)^{-1} S \\ &= i\int_0^{+\infty} \exp\left(it\left(\frac{1}{2}\Delta_x + n^2(0)\right)\right) S\, dt. \end{aligned} \tag{1.6}$$

It is well-known that $w^{\text{out}}$ can also be defined as the unique solution to $(\Delta_x/2 + n^2(0))w^{\text{out}} = S$ that satisfies the Sommerfeld radiation condition at infinity

$$\frac{x}{\sqrt{2}|x|}\cdot\nabla_x w^{\text{out}}(x) + in(0)w^{\text{out}}(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty. \tag{1.7}$$

The main geometric assumptions we need on the refraction index to ensure the validity of (1.5) are twofolds. First, we need that the trajectories of the Hamiltonian $\xi^2/2 - n^2(x)$ at the zero energy are **not trapped**. This is a standard assumption in this context. It somehow prevents accumulation of energy in bounded regions of space. Second, it turns out that the trajectories that really

[2] Here and below we use the standard notation $\langle x\rangle := (1 + x^2)^{1/2}$.

matter in our analysis, are those that start from the origin $x = 0$, with zero energy $\xi^2/2 = n^2(0)$. In this perspective, we need that these trajectories satisfy a **transversality condition**: in essence, each such ray can self-intersect, but we require that the self-intersection is then "tranverse" (see assumption **(H)** page 35, i.e. (7.23), (7.24), in section 7 below). This second assumption prevents accumulation of energy at the origin.

We wish to emphasize that the statement (1.5) is not obvious. In particular, if the transversality assumption **(H)** page 35 is not fullfilled, our analysis shows that (1.5) becomes false in general. We also refer to the end of this paper for "counterexamples".

The central difficulty is the following. On the one hand, the vanishing absorption parameter $\alpha_\varepsilon$ in (1.1) leads to thinking that $w^\varepsilon$ should satisfy the Sommerfeld radiation condition at infinity **with the variable refraction index** $n^2(\varepsilon x)$ (see (1.7)). Knowing that $\lim_{|x|\to\infty} n^2(\varepsilon x) = n_\infty^2$, this roughly means that $w^\varepsilon$ should behave like $\exp(i2^{-1/2}n_\infty|x|)/|x|$ at infinity in $x$ (in dimension $d = 3$, say). On the other hand, the almost constant refraction index $n^2(\varepsilon x)$ in (1.1) leads to observe that $w^\varepsilon$ naturally goes to a solution of the Helmholtz equation **with constant refraction index** $n^2(0)$. Hoping that we may follow the absorption coefficient $\alpha_\varepsilon$ continuously along the limit $\varepsilon \to 0$ in $n^2(\varepsilon x)$, the statement (1.5) becomes natural, and $w^\varepsilon$ should behave like $\exp(i2^{-1/2}n(0)|x|)/|x|$ asymptotically. But, since $n(0) \neq n_\infty$ in general, the last two statements are contradictory ... As we see, the strong non-local effects induced by the Helmholtz equation make the key difficulty in following the continuous dependence of $w^\varepsilon$ upon both the absorption parameter $\alpha_\varepsilon \to 0^+$ and on the index $n^2(\varepsilon x) \to n^2(0)$.

Let us now give some more detailed account on our motivations for looking at the asymptotics $\varepsilon \to 0$ in (1.1).

In [BCKP], the high-frequency analysis of the Helmholtz equation with source term is performed. More precisely, the asymptotic behaviour as $\varepsilon \to 0$ of the following equation is studied[3]

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right), \tag{1.8}$$

where the variable $x$ belongs to $\mathbb{R}^d$, for some $d \geq 3$, and the index of refraction $n^2(x)$ together with the concentration profile $S(x)$ are as before (see [BCKP]). Later, the analysis of [BCKP] was extended in [CPR] to more general oscillating/concentrating source terms. The paper [CPR] studies indeed the high-frequency analysis $\varepsilon \to 0$ in

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \\ \frac{1}{\varepsilon^q}\int_\Gamma S\left(\frac{x-y}{\varepsilon}\right) A(y)\, \exp\left(i\,\frac{\phi(x)}{\varepsilon}\right)\, d\sigma(y). \tag{1.9}$$

(See also [CRu] for extensions - see [Fou] for the case where $n^2$ has discontinuities). In (1.9), the function $S$ again plays the role of a concentration profile like in (1.8), but the concentration occurs this time around a smooth submanifold $\Gamma \subset \mathbb{R}^d$ of dimension $p$ instead of a point. On the more, the source term here

[3] note that we use here a slightly different scaling than the one used in [BCKP]. This a harmless modification that is due to mere convenience.

includes additional oscillations through the (smooth) amplitude $A$ and phase $\phi$. In these notations $d\sigma$ denotes the induced euclidean surface measure on the manifold $\Gamma$, and the rescaling exponant $q$ depends on the dimension of $\Gamma$ together with geometric considerations, see [CPR].

Both Helmholtz equations (1.8) and (1.9) modelize the propagation of a high-frequency source wave in a medium with scaled, variable, refraction index $n^2(x)/\varepsilon^2$. The scaling of the index imposes that the waves propagating in the medium naturally have wavelength $\varepsilon$. On the other hand, the source in (1.8) as well as (1.9) is concentrating at the scale $\varepsilon$, close to the origin, or close to the surface $\Gamma$. It thus carries oscillations at the typical wavelength $\varepsilon$. One may think of an antenna concentrated close to a point or to a surface, and emmitting waves in the whole space. The important phenomenon that these linear equations include precisely lies in the **resonant interaction** between the high-frequency oscillations of the source, and the propagative modes of the medium dictated by the index $n^2/\varepsilon^2$. This makes one of the key difficulties of the analysis performed in [BCKP] and [CPR].

A Wigner approach is used in [BCKP] and [CPR] to treat the high-frequency asymptotics $\varepsilon \to 0$. Up to a harmless rescaling, these papers establish that the Wigner transform $f^\varepsilon(x,\xi)$ of $u^\varepsilon(x)$ satisfies, in the limit $\varepsilon \to 0$, the stationnary transport equation

$$0^+ f(x,\xi) + \xi \cdot \nabla_x f(x,\xi) + \nabla_x n^2(x) \cdot \nabla_\xi f(x,\xi) = Q(x,\xi), \tag{1.10}$$

where $f(x,\xi) = \lim f^\varepsilon(x,\xi)$ measures the energy carried by rays located at the point $x$ in space, with frequency $\xi \in \mathbb{R}^d$. The limiting source term $Q$ in (1.10) describes quantitatively the resonant interactions mentioned above. In the easier case of (1.8), one has $Q(x,\xi) = \delta\left(\xi^2/2 - n^2(0)\right)\ \delta(x)\ |\widehat{S}(\xi)|^2$, meaning that the asymptotic source of energy is concentrated at the origin in $x$ (this is the factor $\delta(x)$), and it only carries resonant frequencies $\xi$ above this point (due to $\delta\left(\xi^2/2 - n^2(0)\right)$). A similar but more complicated value of $Q$ is obtained in the case of (1.9). In any circumstance, equation (1.10) tells us that the energy brought by the source $Q$ is propagated in the whole space through the transport operator $\xi \cdot \nabla_x + \nabla_x n^2(x) \cdot \nabla_\xi$ naturally associated with the semi-classical operator $-\varepsilon^2 \Delta_x/2 - n^2(x)$. The term $0^+ f$ in (1.10) specifies a radiation condition at infinity for $f$, that is the trace, as $\varepsilon \to 0$ of the absorption coefficient $\alpha_\varepsilon > 0$ in (1.8) and (1.9). It gives $f$ as the outgoing solution

$$f(x,\xi) = \int_0^{+\infty} Q\left(X(s,x,\xi), \Xi(s,x,\xi)\right)\ ds.$$

Here $(X(s,x,\xi), \Xi(s,x,\xi))$ is the value at time $s$ of the characteristic curve of $\xi \cdot \nabla_x + \nabla_x n^2(x) \cdot \nabla_\xi$ starting at point $(x,\xi)$ of phase-space (see (1.13) below). Obtaining the radiation condition for $f$ as the limiting effect of the absorption coefficient $\alpha_\varepsilon$ in (1.8) is actually the second main difficulty of the analysis performed in [BCKP] and [CPR].

It turns out that the analysis performed in [BCKP] relies at some point on the asymptotic behaviour of the scaled wave function $w^\varepsilon(x) = \varepsilon^{d/2} u^\varepsilon(\varepsilon x)$ that measures the oscillation/concentration behaviour of $u^\varepsilon$ close to the origin. Similarly, in [CPR] one needs to rescale $u^\varepsilon$ around any point $y \in \Gamma$, setting $w_y^\varepsilon(x) := \varepsilon^{d/2} u^\varepsilon(y + \varepsilon x)$ for any such $y$. We naturally have

$$i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S\left(x\right),$$

in the case of (1.8), and a similar observation holds true in the case of (1.9). Hence the natural rescaling leads to the analysis of the prototype equation (1.1). Under appropriate assumptions on $n^2(x)$ and $S(x)$, it may be proved that $w^\varepsilon$, solution to (1.1), is bounded in the weighted $L^2$ space $L^2(\langle x\rangle^{1+\delta}\,dx)$, for any $\delta > 0$, uniformly in $\varepsilon$. For a fixed value of $\varepsilon$, such weighted estimates are consequences of the work by Agmon, Hörmander, [Ag], [AH]. The fact that these bounds are uniform in $\varepsilon$ is a consequence of the recent (and optimal) estimates established by B. Perthame and L. Vega in [PV1], [PV2] (where the weighted $L^2$ space are replaced by a more precise homogeneous Besov-like space). The results in [PV1] and [PV2] actually need a virial condition of the type $2n^2(x)+x\cdot\nabla_x n^2(x) \geq c > 0$, an inequality that *implies* both our transversality assumption **(H)** page 35, and the non-trapping condition, i.e. the two hypothesis made in the present paper. We also refer to the work by N. Burq [Bu], Gérard and Martinez [GM], T. Jecko [J], as well as Wang and Zhang [WZ], for (not optimal) bounds in a similar spirit. Under the weaker assumptions we make in the present paper, a weaker bound may also be obtained as a consequence of our analysis. In any case, once $w^\varepsilon$ is seen to be bounded, it naturally possesses a weak limit $w = \lim w^\varepsilon$ in the appropriate space. The limit $w$ clearly satisfies in a weak sense the equation

$$\left(\frac{1}{2}\Delta_x + n^2(0)\right) w(x) = S(x). \tag{1.11}$$

Unfortunately, equation (1.11) does not specify $w = \lim w^\varepsilon$ in a unique way, and it has to be supplemented with a radiation condition at infinity. In view of the equation (1.1) satisfied by $w^\varepsilon$, it has been **conjectured** in [BCKP] and [CPR] that $\lim w^\varepsilon$ actually satisfies

$$\lim w^\varepsilon = w^{\text{out}},$$

where $w^{\text{out}}$ is the outgoing solution defined before. The present paper answers the conjecture formulated in these works. It also gives geometric conditions for the convergence $\lim w^\varepsilon = w^{\text{out}}$ to hold.

As a final remark, let us mention that our anaylsis is purely time-dependent. We wish to indicate that similar results than those in the present paper were recently and independently obtained by Wang and Zhang [WZ] using a stationary approach. Note that their analysis requires the stronger virial condition.

Our main theorem is the following

**Main Theorem**
*Let $w^\varepsilon$ satisfy* $\quad i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S(x),\quad$ *for some sequence $\alpha_\varepsilon > 0$ such that $\alpha_\varepsilon \to 0^+$ as $\varepsilon \to 0$. Assume that the source term $S$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Suppose also that the index of refraction satisfies the following set of assumptions*

- *(smoothness, decay). There exists an exponent $\rho > 0$, and a positive constant $n_\infty^2 > 0$ such that for any multi-index $\alpha \in \mathbb{N}^d$, there exists a constant $C_\alpha > 0$ with*

$$\left|\partial_x^\alpha \left(n^2(x) - n_\infty^2\right)\right| \leq C_\alpha\, \langle x\rangle^{-\rho-|\alpha|}. \tag{1.12}$$

- *(non-trapping condition). The trajectories associated with the Hamiltonian $\xi^2/2 - n^2(x)$ are not trapped at the zero energy. In other words, any trajectory $(X(t,x,\xi), \Xi(t,x,\xi))$ solution to*

$$\frac{\partial}{\partial t}X(t,x,\xi) = \Xi(t,x,\xi), \qquad X(0,x,\xi) = x,$$
$$\frac{\partial}{\partial t}\Xi(t,x,\xi) = \left(\nabla_x n^2\right)(X(t,x,\xi)), \qquad \Xi(0,x,\xi) = \xi, \tag{1.13}$$

  *with initial datum $(x,\xi)$ such that $\xi^2/2 - n^2(x) = 0$, is assumed to satisfy*

$$|X(t,x,\xi)| \to \infty, \qquad \text{as } |t| \to \infty.$$

- *(tranversality condition). The tranvsersality condition* **(H)** *page 35 (see also (7.23) and (7.24)) on the trajectories starting from the origin $x = 0$, with zero energy $\xi^2/2 = n^2(0)$, is satisfied.*

*Then, we do have the following convergence, weakly, when tested against any function $\phi \in \mathcal{S}(\mathbb{R}^d)$,*

$$w^\varepsilon \to w^{\text{out}}.$$

**First remark**
Still referring to **(H)** page 35, or (7.23), (7.24)) for the precise statements, we readily indicate that the transversality assumption **(H)** essentially requires that the set

$$\{(\eta,\xi,t) \in \mathbb{R}^{2d} \times ]0,\infty[ \text{ s.t. } X(t,0,\xi) = 0,\ \Xi(t,0,\xi) = \eta,\ \xi^2/2 = n^2(0)\}$$

is a smooth submanifold of $\mathbb{R}^{2d+1}$, having a codimension $> d+2$, a generic asssumption. In other words, zero energy trajectories issued from the origin and passing several times through the origin $x = 0$ should be "rare". □

**Second remark**
As we already mentionned, it is easily proved that the virial condition $2n^2(x) + x \cdot \nabla_x n^2(x) \geq c > 0$ implies both the non-trapping and the transversality conditions. This observation relies on the identities $\partial_t \left(X(t,x,\xi)^2/2\right) = X(t,x,\xi) \cdot \Xi(t,x,\xi)$ and $\partial_t \left(X(t,x,\xi) \cdot \Xi(t,x,\xi)\right) = \left[2n^2(x) + x \cdot \nabla_x n^2(x)\right]|_{x=X(t,x,\xi)} \geq c > 0$, where $(X(t,x,\xi), \Xi(t,x,\xi)$ is any trajectory with zero energy (see section 6 for computations in this spirit).

In fact, the virial condition implies even more, namely that trajectories issued from the origin with zero energy *never come back to the origin*. In other words, the set involved in assumption **(H)** page 35 is simply *void*, and **(H)** is trivially true under the virial condition. As the reader may easily check, such a situation allows to considerably simplify the proof we give here: the tools developped in sections 3, 4, 5, 6 are actually enough to make the complete analysis, and one does not need to go into the detailed computations of section 7 in that case.

Last, the above Theorem asserts the convergence of $w^\varepsilon$: note in passing that even the weak boundedness of $w^\varepsilon$ under the sole above assumptions (i.e. without the virial condition) is not a known result. □

The above theorem is not only a local convergence result, valid for test functions $\phi \in \mathcal{S}$. Indeed, by density of smooth functions in weighted $L^2$ spaces, it readily implies the following immediate corollary. It states that, provided $w^\varepsilon$ is bounded in the natural weighted $L^2$ space, the convergence also holds weakly in this space. In other words, the convergence also holds globally.

**Immediate corollary**
*With the notations of the main Theorem, assume that the source term $S$ above satisfies the weaker decay property*

$$\|S\|_B := \sum_{j\in\mathbb{Z}} 2^{j/2}\|S\|_{L^2(C_j)} < \infty, \tag{1.14}$$

*where $C_j$ denotes the annulus $\{2^j \le |x| \le 2^{j+1}\}$ in $\mathbb{R}^d$. Suppose also that the index of refraction satisfies the smoothness condition of the main Theorem, with the non-trapping and transversality assumptions replaced by the stronger*

- *(virial-like condition)* $$2\sum_{j\in\mathbb{Z}} \sup_{x\in C_j} \frac{\big(x\cdot\nabla n^2(x)\big)_-}{n^2(x)} < 1. \tag{1.15}$$

*Then, we do have the convergence $w^\varepsilon \to w^{\text{out}}$, weakly, when tested against any function $\phi$ such that $\|\phi\|_B < \infty$,*

Under the simpler virial condition $2n^2(x)+x\cdot n^2(x) \ge c > 0$, a similar result holds with the space $B$ replaced by the more usual weighted space $L^2\left(\langle x\rangle^{1+\delta}dx\right)$ ($\delta > 0$ arbitrary). Here, we give a version where the decay (1.14) assumed on the source $S$ is the optimal one, and the above weak convergence holds in the optimal space.

It is well known that the resolvent of the Helmholtz operator maps the weighted $L^2$ space $L^2\left(\langle x\rangle^{1+\delta}dx\right)$ to $L^2\left(\langle x\rangle^{-1-\delta}dx\right)$ for any $\delta > 0$ ([Ag], [J], [GM]). Agmon and Hörmander [AH] gave an optimal version in the constant coefficients case: the resolvent of the Helmholtz operator sends the weighted $L^2$ space $B$ defined in (1.14) to the dual weighted space $B^*$ defined by

$$\|u\|_{B^*} := \sup_{j\in\mathbb{Z}} 2^{-j/2}\|u\|_{L^2(C_j)}. \tag{1.16}$$

For non-constant coefficients, that are non-compact perturbations of constants, Perthame and Vega in [PV1] and [PV2] established the optimal estimate in $B$-$B^*$ under assumption (1.15). In our perspective, the assumption (1.15) is of technical nature, and it may be replaced by **any** assumption ensuring that the solution $w^\varepsilon$ to (1.1) satisfies the uniform bound

$$\|w^\varepsilon\|_{B^*} \le C_{d,n^2}\ \|S\|_B, \tag{1.17}$$

for some universal constant $C_{d,n^2}$ that only depends on the dimension $d \ge 3$ and the index $n^2$.

**Proof of the immediate Corollary**
Under the virial-like assumption (1.15), it has been established in [PV1] that estimate (1.17) holds true. Hence, by density of the Schwartz class in the space $B$, one readily reduces the problem to the case when the source $S$ and the test function $\phi$ belong to $\mathcal{S}(\mathbb{R}^d)$. The Main Theorem now allows to conclude. □

Needless to say, the central assumptions needed for the theorem are the non-trapping condition together with the transversality condition. Comments are given below on the very meaning of the transversality condition **(H)** page 35 (i.e. (7.23), (7.24)), to which we refer.

To state the result very briefly, the heart of our proof lies in proving that under the above assumptions, the propagator $\exp\left(i\varepsilon^{-1}t\left(-\varepsilon^2\Delta_x/2 - n^2(x)\right)\right)$, or its rescaled value $\exp\left(it\left(-\Delta_x/2 - n^2(\varepsilon x)\right)\right)$, satisfy "similar" dispersive properties as the free Schrödinger operator $\exp\left(it\left(-\Delta_x/2 - n^2(0)\right)\right)$, *uniformly in* $\varepsilon$. This in turn is proved upon distinguishing between small times, moderate times, and very large times, each case leading to the use of different arguments and techniques.

The remainder part of this paper is devoted to the proof of the main Theorem. The proof being long and using many different tools, we first draw in section 2 an outline of the proof, giving the main ideas and tools. We also define the relevant mathematical objects to be used throughout the paper. The proof itself is performed in the next sections 3 to 8. Examples and counterexamples to the Theorem are also proposed in the last section 9.

The main intermediate results are proposition 1, proposition 2, proposition 3, together with the more difficult proposition 4 (that needs an Egorov Theorem for large times stated in Lemma 5). The key (and most difficult) result is proposition 7. The latter uses the tranversality condition mentioned before.

# 2 Preliminary Analysis: outline of the proof of the Main Theorem

## 2.1 Outline of the proof

Let $w^\varepsilon$ be the solution to $i\varepsilon\alpha_\varepsilon w^\varepsilon + \frac{1}{2}\Delta w^\varepsilon + n^2(\varepsilon x)w^\varepsilon = S\left(x\right)$, with $S \in \mathcal{S}(\mathbb{R}^d)$. According to the statement of our main Theorem, we wish to study the asymptotic behaviour of $w^\varepsilon$ as $\varepsilon \to 0$, in a weak sense. Taking a test function $\phi(x) \in \mathcal{S}(\mathbb{R}^d)$, and defining the duality product

$$\langle w^\varepsilon, \phi\rangle := \int_{\mathbb{R}^d} w^\varepsilon(x)\phi(x)\,dx,$$

we want to prove the convergence

$$\langle w^\varepsilon, \phi\rangle \to \langle w^{\text{out}}, \phi\rangle \quad \text{as} \quad \varepsilon \to 0.$$

where the outgoing solution of the (constant coefficient) Helmholtz equation $w^{\text{out}}$ is defined in (1.5), (1.6) before.

**First step: preliminary reduction - the time dependent approach**

In order to prove the weak convergence $\langle w^\varepsilon, \phi\rangle \to \langle w, \phi\rangle$, we define the rescaled function

$$u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} w^\varepsilon\left(\frac{x}{\varepsilon}\right). \tag{2.1}$$

It satisfies $i\varepsilon\alpha_\varepsilon u^\varepsilon + \varepsilon^2/2\,\Delta u^\varepsilon + n^2(x)u^\varepsilon = 1/\varepsilon^{d/2} S\left(x/\varepsilon\right) =: S_\varepsilon(x)$, where for any function $f(x)$ we use the short-hand notation

$$f_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} f\left(\frac{x}{\varepsilon}\right).$$

Using now the function $u^\varepsilon$ instead of $w^\varepsilon$, we observe the equality

$$\langle w^\varepsilon, \phi \rangle = \langle u^\varepsilon, \phi_\varepsilon \rangle. \tag{2.2}$$

This transforms the original problem into the question of computing the semi-classical limit $\varepsilon \to 0$ in the equation satisfied by $u^\varepsilon$. One sees in (2.2) that this limit needs to be computed *at the semiclassical scale* (i.e. when tested upon a smooth, concentrated function $\phi_\varepsilon$).

In order to do so, we compute $u^\varepsilon$ in terms of the semiclassical resolvent $\big(i\varepsilon\alpha_\varepsilon + (\varepsilon^2/2)\Delta + n^2(x)\big)^{-1}$. It is the integral over the whole time interval $[0, +\infty[$ of the propagator of the Schrödinger operator associated with $\varepsilon^2\Delta/2 + n^2(x)$. In other words we write

$$\begin{aligned} u^\varepsilon &= \left( i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta + n^2(x) \right)^{-1} S_\varepsilon \\ &= i \int_0^{+\infty} \exp\left( it \left( i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta + n^2(x) \right) \right) S_\varepsilon \, dt. \end{aligned} \tag{2.3}$$

Now, defining the semi-classical propagator

$$U_\varepsilon(t) := \exp\left( i\frac{t}{\varepsilon} \left( \frac{\varepsilon^2}{2}\Delta + n^2(x) \right) \right) = \exp\left( -i\frac{t}{\varepsilon} H_\varepsilon \right), \tag{2.4}$$

associated with the semi-classical Schrödinger operator

$$H_\varepsilon := -\frac{\varepsilon^2}{2}\Delta - n^2(x), \tag{2.5}$$

we arrive at the final formula

$$\langle w^\varepsilon, \phi \rangle = \langle u^\varepsilon, \phi_\varepsilon \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \, \langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle \, dt. \tag{2.6}$$

Our strategy is to pass to the limit in this very integral.

**Second step: passing to the limit in the time integral (2.6)**

In order to pass to the limit $\varepsilon \to 0$ in (2.6), we need to analyze the contributions of various time scales in the corresponding time integral. More precisely, we choose for the whole subsequent analysis two (large) cutoff parameters in time, denoted by $T_0$ and $T_1$, and we analyze the contributions to the time integral (2.6) that are due to the three regions

$$0 \le t \le T_0\,\varepsilon, \; T_0\,\varepsilon \le t \le T_1\,, \quad \text{and} \quad t \ge T_1\,.$$

We also choose a (small) exponent $\kappa > 0$, and we occasionally treat separately the contributions of very large times

$$t \ge \varepsilon^{-\kappa}.$$

Associated with these truncations, we take once and for all a smooth cutoff function $\chi$ defined on $\mathbb{R}$, such that

$$\begin{gathered} \chi(z) \equiv 1 \;\text{ when }\; |z| \le 1/2, \;\; \chi(z) \equiv 0 \;\text{ when }\; |z| \ge 1, \\ \chi(z) \ge 0 \;\text{ for any } z. \end{gathered} \tag{2.7}$$

To be complete, there remains to finally choose a (small) cutoff parameter in energy $\delta > 0$. Accordingly we distinguish in the $L^2$ scalar product $\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle$ between energies close to (or far from) the zero energy, which is critical for our problem. In other words, we set the self-adjoint operator

$$\chi_\delta\left(H_\varepsilon\right) := \chi\left(\frac{H_\varepsilon}{\delta}\right).$$

This object is perfectly well defined using standard functional calculus for self-adjoint operators. We decompose

$$\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle = \Big\langle U_\varepsilon(t)\,\chi_\delta(H_\varepsilon)S_\varepsilon, \phi_\varepsilon\Big\rangle + \Big\langle U_\varepsilon(t)\;(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle.$$

Following the above described decomposition of times and energies, we study each of the subsequent terms:

• **The contribution of small times** is

$$\frac{1}{\varepsilon}\int_0^{2T_0\,\varepsilon} \chi\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\ dt.$$

We prove in section 3 that this term actually gives the dominant contribution in (2.6), provided the cutoff parameter $T_0$ is taken large enough. This (easy) analysis essentially boils down to manipulations on the time dependent Schrödinger operator $i\partial_t + \Delta_x/2 + n^2(\varepsilon x)$, for *finite times* $t$ of the order $t \sim T_0$ at most.

• **The contribution of moderate and large times, away from the zero energy**, is

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{+\infty} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\,(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt.$$

We prove in section 4 below that this term has a vanishing contribution, provided $T_0$ is large enough. This easy result relies on a non-stationnary phase argument in time, recalling that $U_\varepsilon(t) = \exp(-itH_\varepsilon/\varepsilon)$ and the energy $H_\varepsilon$ is larger than $\delta > 0$.

• **The contribution of very large times, close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{\varepsilon^{-\kappa}}^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt.$$

We prove in section 5 that this term has a vanishing contribution as $\varepsilon \to 0$. To do so, we use results proved by X.P. Wang [Wa]: these essentially assert that the operator $\langle x\rangle^{-s}\ U_\varepsilon(t)\chi_\delta(H_\varepsilon)\ \langle x\rangle^{-s}$ has the natural size $\langle t\rangle^{-s}$ as time goes to infinity, provided the critical zero energy is non-trapping. Roughly, the semiclassical operator $U_\varepsilon(t)\chi_\delta(H_\varepsilon)$ sends rays initially close to the origin, at a distance of the order $t$ from the origin, when the energy is non trapping. Hence the above scalar product involves both a function $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ that is localized at a distance $t$ from the origin, and a function $\phi_\varepsilon$ that is localized at the origin. This makes the corresponding contribution vanish.

The most difficult terms are the last two that we describe now.

• **The contribution of large times, close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} \mathrm{e}^{-\alpha_\varepsilon t}\, \Big\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt.$$

The treatment of this term is performed in section 6. It is similar in spirit to (though much harder than) the analysis performed in the previous term: using only information on the localization properties of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ and $\phi_\varepsilon$, we prove that this term has a vanishing contribution, provided $T_1$ is large enough. To do so, we use ideas of Bouzouina and Robert [BR], to establish a version of the Egorov theorem that holds true for *polynomially large times* in $\varepsilon$. We deduce that for any time $T_1 \leq t \leq \varepsilon^{-\kappa}$, the term $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ is localized close to the value at time $t$ of a trajectory shot from the origin. The non-trapping assumption then says that for $T_1$ large enough, $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ is localized away from the origin. This makes the scalar product $\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\rangle$ vanish asymptotically.

• **The contribution of moderate times close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\, \mathrm{e}^{-\alpha_\varepsilon t}\, \Big\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt.$$

This is the most difficult term: contrary to all preceding terms, it cannot be analyzed using only geometric informations on the microlocal support of the relevant functions. Indeed, keeping in mind that the function $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ is localized on a trajectory initially shot from the origin, whereas $\phi_\varepsilon$ stays at the origin, it is clear that for times $T_0\,\varepsilon \leq t \leq T_1$, the support of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ and $\phi_\varepsilon$ *may intersect*, due to trajectories passing *several times* at the origin. This might create a dangerous accumulation of energy at this point. For that reason, we need a precise evaluation of the semi-classical propagator $U_\varepsilon(t)$, for times up to the order $t \sim T_1$. This is done using the elegant wave-packet approach of M. Combescure and D. Robert [CRo] (see also [Ro], and the nice lecture [Ro2]): projecting $S_\varepsilon$ over the standard gaussian wave packets, we can compute $U_\varepsilon(t)S_\varepsilon$ in a quite explicit fashion, with the help of classical quantities like, typically, the linearized flow of the Hamiltonian $\xi^2/2 - n^2(x)$. This gives us an integral representation with a complex valued phase function. Then, one needs to insert a last (small) cutoff parameter in time, denoted $\theta > 0$. For small times, using the above mentioned representation formula, we first prove that the term

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{\theta} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\, \mathrm{e}^{-\alpha_\varepsilon t}\, \Big\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt,$$

vanishes asymptotically, provided $\theta$ is small, and $T_0$ is large enough. To do so, we use that for small enough $\theta$, the propagator $U_\varepsilon(t)$ acting on $S_\varepsilon$ resembles the free Schrödinger operator $\exp\big(it[\Delta_x/2 + n^2(0)]\big)$. In terms of trajectories, on this time scale, we use that $U_\varepsilon(t)S_\varepsilon$ is localized around a ray that leaves the origin *at speed* $n(0)$. Then, for later times, we prove that the remaining contribution

$$\frac{1}{\varepsilon}\int_{\theta}^{T_1} \mathrm{e}^{-\alpha_\varepsilon t}\, \Big\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt,$$

is small. This uses stationary phase formulae in the spirit of [CRR], and this is where the transversality assumption **(H)** page 35 enters: trajectories passing several times at the origin do not accumulate to much energy at this point.

We end up this sketch of proof with a figure illustrating the typical trajectory (and the associated cutoffs in time) that our analysis has to deal with.

## 2.2 Notations used in the proof

Throughout this article, we will make use of the following notations.

**• Semi-classical quantities**
The semi-classical Hamiltonian $H_\varepsilon$ and its associated propagator $U_\varepsilon(t)$ have already been defined. We also need to use the Weyl quantization. For a symbol $a(x,\xi)$ defined on $\mathbb{R}^{2d}$, its Weyl quantization is

$$\left(\mathrm{Op}_\varepsilon^w(a)f\right)(x) := \frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{i\frac{(x-y)\cdot\xi}{\varepsilon}}\; a\left(\frac{x+y}{2},\xi\right)\; f(y)\;dy\;d\xi.$$

Throughout the paper, we use the standard semi-classical symbolic calculus, and refer, e.g., to [DS] or [Ma]. In particular, for a weight $m(x,\xi)$, we use symbols $a(x,\xi)$ in the class $S(m)$, i.e. symbols such that for any multi-index $\alpha$, there exists a constant $C_\alpha$ so that

$$|\partial^\alpha a(x,\xi)| \leq C_\alpha m(x,\xi),\ \forall (x,\xi)\in\mathbb{R}^{2d}.$$

The notation $a \sim \sum \varepsilon^k a_k$ means that for any $N$ and any $\alpha$, there exists a constant $C_{N,\alpha}$ such that

$$\left|\partial^\alpha\left(a(x,\xi)-\sum_{k=0}^N \varepsilon^k a_k(x,\xi)\right)\right| \leq C_{N,\alpha}\;\varepsilon^{N+1} m(x,\xi),\ \forall (x,\xi)\in\mathbb{R}^{2d}.$$

**• Classical quantitities**
Associated with the Hamiltonian $H(x,\xi)=\xi^2/2-n^2(x)$, we denote the Hamiltonian flow

$$\Phi(t,x,\xi) = (X(t,x,\xi)\,,\,\Xi(t,x,\xi)),$$

defined as the solution of the Hamilton equations

$$\begin{aligned}
&\frac{\partial}{\partial t}X(t,x,\xi) = \Xi(t,x,\xi), && X(0,x,\xi)=x,\\
&\frac{\partial}{\partial t}\Xi(t,x,\xi) = \left(\nabla_x n^2\right)(X(t,x,\xi))\,, && \Xi(0,x,\xi)=\xi.
\end{aligned}\tag{2.8}$$

These may be written shortly

$$\frac{\partial}{\partial t}\Phi(t,x,\xi) = J\,\frac{DH}{D(x,\xi)}\left(\Phi(t,x,\xi)\right),\tag{2.9}$$

where $J$ is the standard symplectic matrix

$$J=\begin{pmatrix} 0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}.\tag{2.10}$$

The linearized flow of $\Phi$ is denoted by

$$F(t,x,\xi) := \frac{D\Phi(t,x,\xi)}{D(x,\xi)}.\tag{2.11}$$

It may be decomposed into

$$F(t,x,\xi) = \begin{pmatrix} A(t,x,\xi) & B(t,x,\xi) \\ C(t,x,\xi) & D(t,x,\xi) \end{pmatrix}, \tag{2.12}$$

where the matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are, by definition

$$A(t,x,\xi) = \frac{DX(t,x,\xi)}{Dx}, \quad B(t,x,\xi) = \frac{DX(t,x,\xi)}{D\xi},$$
$$C(t,x,\xi) = \frac{D\Xi(t,x,\xi)}{Dx}, \quad D(t,x,\xi) = \frac{D\Xi(t,x,\xi)}{D\xi}.$$

Upon linearizing (2.8), the matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ clearly satisfy the differential system

$$\begin{aligned} &\frac{\partial}{\partial t}A(t,x,\xi) = C(t,x,\xi), && A(0,x,\xi) = \mathrm{Id}, \\ &\frac{\partial}{\partial t}C(t,x,\xi) = \frac{D^2 n^2}{Dx^2}\left(X(t,x,\xi)\right)\ A(t,x,\xi), && C(0,x,\xi) = 0, \end{aligned} \tag{2.13}$$

together with

$$\begin{aligned} &\frac{\partial}{\partial t}B(t,x,\xi) = D(t,x,\xi), && B(0,x,\xi) = 0, \\ &\frac{\partial}{\partial t}D(t,x,\xi) = \frac{D^2 n^2}{Dx^2}\left(X(t,x,\xi)\right)\ B(t,x,\xi), && D(0,x,\xi) = \mathrm{Id}. \end{aligned} \tag{2.14}$$

In short, one may write as well

$$\frac{\partial}{\partial t}F(t,x,\xi) = J\ \frac{D^2 H}{D(x,\xi)^2}\left(\Phi(t,x,\xi)\right)\ F(t,x,\xi). \tag{2.15}$$

A last remark is in order. Indeed, it is a standard fact to observe that the matrix $F(t,x,\xi)$ is a symplectic matrix, in that

$$F(t,x,\xi)^{\mathrm{T}} J F(t,x,\xi) = J, \tag{2.16}$$

for any $(t,x,\xi)$. Here, the exponent T denotes transposition. Decomposing $F(t)$ as in (2.12), this gives the relations

$$\begin{aligned} &A(t)^{\mathrm{T}}C(t) = C(t)^{\mathrm{T}}A(t),\ B(t)^{\mathrm{T}}D(t) = D(t)^{\mathrm{T}}B(t), \\ &A(t)^{\mathrm{T}}D(t) - C(t)^{\mathrm{T}}B(t) = \mathrm{Id}. \end{aligned} \tag{2.17}$$

These can be put in the following useful form

$$\begin{aligned} &\left(A(t)+iB(t)\right)^{\mathrm{T}}\left(C(t)+iD(t)\right) = \left(C(t)+iD(t)\right)^{\mathrm{T}}\left(A(t)+iB(t)\right) \\ &\left(C(t)+iD(t)\right)^{\mathrm{T}}\left(A(t)-iB(t)\right) \\ &\qquad - \left(A(t)+iB(t)\right)^{\mathrm{T}}\left(C(t)-iD(t)\right) = 2i\mathrm{Id}. \end{aligned} \tag{2.18}$$

These relations will be used in section 7.

## 3 Small time contribution: the case $0 \le t \le T_0\,\varepsilon$

In this section, we prove the following

**Proposition 1.** *We use the notations of section 2. The refraction index $n^2$ is assumed bounded and continuous. The data $S$ and $\phi$ are supposed to belong to $\mathcal{S}(\mathbb{R}^d)$. Then, the following holds:*
*(i) for any fixed value of $T_0$ , we have the asymptotics*

$$\frac{i}{\varepsilon}\int_0^{2T_0\,\varepsilon} \chi\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \langle U_\varepsilon(t)S_\varepsilon,\phi_\varepsilon\rangle\ dt$$
$$\underset{\varepsilon\to 0}{\longrightarrow} i\int_0^{2T_0} \chi\left(\frac{t}{T_0}\right)\ \langle \exp\left(it(\Delta_x/2+n^2(0))\right)\ S,\phi\rangle\ dt. \tag{3.1}$$

*(ii) Besides, there exists a universal constant $C_d$ depending only on the dimension, such that the right-hand-side of (3.1) satisfies*

$$\left| i\int_0^{2T_0} \chi\left(\frac{t}{T_0}\right)\ \langle \exp\left(it(\Delta_x/2+n^2(0))\right)\ S,\phi\rangle\ dt - \langle w^{\rm out},\phi\rangle \right|$$
$$\le C_d\,{T_0}^{-d/2+1} \underset{T_0\to\infty}{\longrightarrow} 0. \tag{3.2}$$

**Proof of proposition 1**
*Part (i)*
In order to recover the limiting value announced in (3.1), we first perform the inverse scaling that leads from $w^\varepsilon$ to $u^\varepsilon$ (see (2.1)). We rescale time $t$ by a factor $\varepsilon$ as well. This gives

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^{+\infty} \chi\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \langle U_\varepsilon(t)S_\varepsilon,\phi_\varepsilon\rangle\ dt\\
&\quad=\int_0^{+\infty} \chi\left(\frac{t}{T_0}\right)\ \mathrm{e}^{-\varepsilon\alpha_\varepsilon t}\ \langle U_\varepsilon(\varepsilon\,t)S_\varepsilon,\phi_\varepsilon\rangle\ dt\\
&\quad=\int_0^{+\infty} \chi\left(\frac{t}{T_0}\right)\ \mathrm{e}^{-\varepsilon\alpha_\varepsilon t}\ \langle \exp\left(it\left(\Delta/2+n^2(\varepsilon x)\right)\right)S,\phi\rangle\ dt.
\end{aligned}$$

We now let

$$\mathbf{w}^\varepsilon(t,x) := \exp\left(it\left(\Delta/2+n^2(\varepsilon x)\right)\right)S(x).$$

The function $\mathbf{w}^\varepsilon(t,x)$ is bounded in $L^\infty\left(\mathbb{R};L^2\left(\mathbb{R}^d\right)\right)$, and it satisfies in the distribution sense

$$i\partial_t\mathbf{w}^\varepsilon(t,x) = -\frac{1}{2}\Delta_x\mathbf{w}^\varepsilon(t,x) - n^2(\varepsilon x)\mathbf{w}^\varepsilon,\quad \mathbf{w}^\varepsilon(0,x)=S(x).$$

These informations are enough to deduce that there exists a function $\mathbf{w}(t,x)\in L^\infty\left(\mathbb{R};L^2\left(\mathbb{R}^d\right)\right)$ such that a subsequence of $\mathbf{w}^\varepsilon(t,x)$ goes, as $\varepsilon\to 0$, to $\mathbf{w}(t,x)$ in $L^\infty\left(\mathbb{R};L^2\left(\mathbb{R}^d\right)\right)$ - weak$*$. On the more, the limit $\mathbf{w}(t,x)$ obviously satisfies in the distribution sense

$$i\partial_t\mathbf{w}(t,x) = -\frac{1}{2}\Delta_x\mathbf{w}(t,x) - n^2(0)\mathbf{w},\quad \mathbf{w}(0,x)=S(x).$$

In other words

$$\mathbf{w}(t) = \exp\left(it\left(\Delta/2 + n^2(0)\right)\right) S(x).$$

Hence, by uniqueness of the limit, the whole sequence $\mathbf{w}^\varepsilon(t,x)$ goes to $\mathbf{w}(t,x)$ in $L^\infty\left(\mathbb{R}; L^2\left(\mathbb{R}^d\right)\right)$-weak$*$. This proves (3.1) and part (i) of the proposition.

*Part (ii)*
This part is easy and relies on the standard dispersive properties of the free Schrödinger equation. Indeed, we have

$$\begin{aligned}
&\left|\langle \exp\left(it\left(\Delta_x/2 + n^2(0)\right)\right) S, \phi\rangle\right| \\
&\qquad \leq \left\| \exp\left(it\left(\Delta_x/2 + n^2(0)\right)\right) S \right\|_{L^\infty} \|\phi\|_{L^1} \\
&\qquad \leq C_d\, t^{-d/2}\, \|S\|_{L^1}\, \|\phi\|_{L^1},
\end{aligned}$$

(recall that $S$ and $\phi$ are assumed smooth enough to have finite $L^1$ norm), for some constant $C_d > 0$ that only depends upon the dimension $d$. This, together with the integrability of the function $t^{-d/2}$ at infinity when $d \geq 3$, ends the proof of (3.2). □

# 4 Contribution of moderate and large times, away from the zero energy

In this section we prove the (easy)

**Proposition 2.** *We use the notations of section 2. The index $n^2$ is assumed to have the symbolic behaviour (1.12). The data $S$ and $\phi$ are supposed to belong to $L^2(\mathbb{R}^d)$. Then, there exists a constant $C_\delta > 0$, which depends on the cutoff parameter $\delta$, such that for any $\varepsilon \leq 1$, and $T_0 \geq 1$, we have*

$$\begin{aligned}
&\left| \frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{+\infty} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right) \mathrm{e}^{-\alpha_\varepsilon t} \left\langle (1-\chi_\delta\left(H_\varepsilon\right)) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \right\rangle dt \right| \\
&\qquad\qquad \leq C_\delta \left(\frac{1}{T_0} + \alpha_\varepsilon^2\right).
\end{aligned} \tag{4.1}$$

**Proof of proposition 2**
The proof relies on a simple non-stationary phase argument. Indeed, this term has the value

$$\frac{1}{\varepsilon}\int_0^{+\infty} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right) \mathrm{e}^{-\alpha_\varepsilon t} \left\langle (1-\chi_\delta\left(H_\varepsilon\right)) \exp\left(-i\frac{t}{\varepsilon}H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt.$$

Hence, making the natural integrations by parts in time, we recover the value

$$\begin{aligned}
&\varepsilon^2 \int_0^{+\infty} \frac{\partial^3}{\partial t^3}\left((1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\mathrm{e}^{-\alpha_\varepsilon t}\right) \\
&\qquad\qquad \left\langle \frac{(1-\chi_\delta\left(H_\varepsilon\right))}{(-iH_\varepsilon)^3} \exp\left(-i\frac{t}{\varepsilon}H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt.
\end{aligned}$$

A direct inspection shows that this is bounded by

$$C\,\varepsilon^2\,\delta^{-3}\,\|S\|_{L^2}\,\|\phi\|_{L^2}\int_0^{+\infty}\left|\frac{\partial^3}{\partial t^3}\left((1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\mathrm{e}^{-\alpha_\varepsilon t}\right)\right|dt$$
$$\le C\,\varepsilon^2\,\delta^{-3}\;\|\chi\|_{W^{3,\infty}}\;\left(\frac{1}{{T_0}^2\varepsilon^2}+\frac{1}{T_0\,\varepsilon}+\alpha_\varepsilon^2+\alpha_\varepsilon^2\right).$$

□

# 5 Contribution of large times, close to the zero energy: the case $t \geq \varepsilon^{-\kappa}$

In this section we prove the following

**Proposition 3.** *We use the notations of section 2. The index $n^2$ is assumed to have the symbolic behaviour (1.12). The Hamiltonian flow associated with $\xi^2/2 - n^2(x)$ is assumed non-trapping at the zero energy level. Finally, the data $S$ and $\phi$ are supposed to belong to $\mathcal{S}(\mathbb{R}^d)$. Then, for any $\delta > 0$ small enough, and for any $\kappa > 0$, there exists a constant $C_{\kappa,\delta}$ depending on $\kappa$ and $\delta$, so that*

$$\left|\frac{1}{\varepsilon}\int_{\varepsilon^{-\kappa}}^{+\infty}\mathrm{e}^{-\alpha_\varepsilon t}\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon,\phi_\varepsilon\right\rangle dt\right| \le C_{\kappa,\delta}\,\varepsilon. \tag{5.1}$$

The proof relies on the dispersive properties of the semi-classical propagator $U_\varepsilon(t)$, inherited from the ones of the classical flow $\Phi(t)$. More quantitatively, we use in this section a Theorem by X.P. Wang [Wa], that we now state. Our index of refraction $n^2(x)$ is such that $n^2(x)$ lies in $C^\infty(\mathbb{R}^d)$, and it has the symbolic behaviour

$$n^2(x) = n_\infty^2 - V(x), \quad \text{with } |\partial^\alpha V(x)| \le \langle x\rangle^{-\rho-|\alpha|}$$

(the case $0 < \rho \le 1$ is the long-range case, and the case $\rho > 1$ is the short-range case, in the terminology of quantum scattering). On the more, the trajectories of the classical flow at the zero energy (i.e. on the set $\{(x,\xi)\in\mathbb{R}^{2d}$ s.t. $\xi^2/2 - n^2(x) = 0\}$) are assumed non-trapped. It is known [DG] that this non-trapping behaviour is actually an open property, in that

there exists a $\delta_0 > 0$ such that for any energy $E$
satisfying $|E| \le \delta_0$, the trajectories of the classical flow (5.2)
at the energy $E$ are non-trapping as well.

Under these circumstances, it has been proved in [Wa] that for any real $s > 0$, and for any $\eta > 0$, the following weighted estimate holds true,

$$\forall t\in\mathbb{R},\quad \|\langle x\rangle^{-s}U_\varepsilon(t)\chi_\delta(H_\varepsilon)f\|_{L^2} \le \frac{C_{\delta,\eta,s}}{\langle t\rangle^{s-\eta}}\;\|\langle x\rangle^s f(x)\|_{L^2}, \tag{5.3}$$

provided the cutoff in energy $\delta$ satisfies $\delta \le \delta_0$, i.e. provided we are only looking at trajectories having a non-trapping energy. This inequality holds for any test function $f$, and for some constant $C_{\delta,\eta,s}$ depending only on $\delta$, $\eta$ and $s$. In the short-range case ($\rho > 1$), one may even take $\eta = 0$ in the above estimate. Note that [Wa] actually proves more: in some sense, the non-trapping behaviour of the classical flow is *equivalent* to the time decay (5.3). We refer to the original article for details. We are now ready to give the

**Proof of proposition 3**
Taking $\delta \le \delta_0$, we estimate, using (5.3),

$$\begin{aligned}
&\frac{1}{\varepsilon}\left| \int_{\varepsilon^{-\kappa}}^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle \chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \right\rangle \, dt \right| \\
&\quad \le \frac{1}{\varepsilon} \int_{\varepsilon^{-\kappa}}^{+\infty} \|\langle x\rangle^{-s} U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon\|_{L^2} \, \|\langle x\rangle^s \phi_\varepsilon\|_{L^2} \, dt \\
&\quad \le \frac{1}{\varepsilon} \, \|\langle x\rangle^s S_\varepsilon(x)\|_{L^2} \, \|\langle x\rangle^s \phi_\varepsilon\|_{L^2} \int_{\varepsilon^{-\kappa}}^{+\infty} \frac{C_{\delta,\eta,s}}{\langle t\rangle^{s-\eta}} \, dt \\
&\quad \le C_{\delta,\eta,s} \, \varepsilon^{\kappa(s-\eta-1)-1} \, \|\langle x\rangle^s S_\varepsilon(x)\|_{L^2} \, \|\langle x\rangle^s \phi_\varepsilon\|_{L^2}
\end{aligned}$$

Hence, taking $s$ large enough, and $\eta$ small enough, e.g. $s = 2 + 2/\kappa$, $\eta = 1$, we obtain an upper bound of the size

$$C_{\kappa,\delta} \, \varepsilon \, \|\langle x\rangle^s S(x)\|_{L^2} \, \|\langle x\rangle^s \phi\|_{L^2}.$$

Here we used the easy fact that $\|\langle x\rangle^s f_\varepsilon(x)\|_{L^2} \le \|\langle x\rangle^s f(x)\|_{L^2}$, when $\varepsilon \le 1$, together with $\|\langle x\rangle^s S(x)\|_{L^2} < \infty$, and similarly for $\phi$. □

# 6 Contribution of large times, close to the zero energy: the case $T_1 \le t \le \varepsilon^{-\kappa}$

To complete the analysis of the contribution of "large times" and "small energies" in (2.6) that we begun in section 5, there remains to estimate the term

$$\frac{1}{\varepsilon} \int_{T_1}^{\varepsilon^{-\kappa}} (1-\chi)\left(\frac{t}{T_1}\right) \, \mathrm{e}^{-\alpha_\varepsilon t} \left\langle \chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \right\rangle \, dt. \tag{6.1}$$

In this section, we prove the

**Proposition 4.** *We use the notations of section 2. The index $n^2$ is assumed to have the symbolic behaviour (1.12) with $n_\infty^2 > 0$[4]. The Hamiltonian flow associated with $\xi^2/2 - n^2(x)$ is assumed non-trapping at the zero energy. Finally, the data $S$ and $\phi$ are supposed to belong to $\mathcal{S}(\mathbb{R}^d)$. Then, for $\delta > 0$ small enough, there exists a $T_1(\delta)$ depending on $\delta$ such that for any $T_1 \ge T_1(\delta)$, we have for $\kappa$ small enough,*

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \int_{T_1}^{\varepsilon^{-\kappa}} (1-\chi)\left(\frac{t}{T_1}\right) \, \mathrm{e}^{-\alpha_\varepsilon t} \left\langle \chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \right\rangle \, dt \right| \\
&\qquad \le C_{\kappa,\delta} \, \varepsilon, \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{6.2}$$

*for some constant $C_{\kappa,\delta}$ that depends upon $\kappa$ and $\delta$.*

[4]The assumption $n_\infty^2$ is crucial, see Lemma 5 below. It ensures that the wave $U_\varepsilon(t)S_\varepsilon$ propagates with a uniformly non-zero speed, at infinity in time $t$.

The idea of proof is the following: the functions $S_\varepsilon$ and $\phi_\varepsilon$ are microlocally supported close to points $(x_0, \xi_0) \in \mathbb{R}^{2d}$ such that $x_0 = 0$ (due to the concentration of both functions close to the origin as $\varepsilon \to 0$). On the more, using the Egorov Theorem, one may think of the time evolved function $U_\varepsilon(t)S_\varepsilon$ as being microlocally supported close to points $(X(t; x_0, \xi_0), \Xi(t; x_0, \xi_0))$ that are trajectories of the classical flow, with initial data $(x_0, \xi_0)$ such that $x_0 = 0$. Using the non-trapping assumption on the classical flow, we see that for large times $t \geq T_1$ with $T_1$ large enough, the trajectory $X(t; x_0, \xi_0)$ with $x_0 = 0$ is far away from the origin. Hence the microlocal support of $U_\varepsilon(t)S_\varepsilon$ and $\phi_\varepsilon$ do not intersect, and the factor (6.1) should be arbitrary small in $\varepsilon$ as $\varepsilon \to 0$.

The difficulty in making this last statement rigorous lies in the fact that we need to use the Egorov Theorem up to (polynomially) large times of the order $t \sim \varepsilon^{-\kappa}$. This difficulty is solved in Lemma 5 below. Indeed, upon adapting a recent result of Bouzouina and Robert [BR] we give remainder estimates in the Egorov Theorem that hold up to polynomially large times (logarithmic times are obtained in the context of [BR]). This is enough to conclude.

## 6.1 Proof of proposition 4

The proof is given in several steps.

**First step***: Preliminary reduction*

In this step we quantify the fact that the functions involved in the scalar product in (6.2) are microlocalized close to the zero energy $\xi^2/2 = n^2(x)$ (in frequency) and close to the origin $x = 0$ (in space). To do so, we simply write, using the fact that $S$ and $\phi$ belong to $\mathcal{S}(\mathbb{R}^d)$,

$$\phi_\varepsilon(x) = \chi_\delta(|x|)\phi_\varepsilon(x) + O_\delta(\varepsilon^\infty) \quad \text{in } L^2(\mathbb{R}^d),$$

and similarly for $S_\varepsilon$. This means that for any integer $N$, there exists a $C_{N,\delta} > 0$ that depends on $N$ and $\delta$, such that $\|\phi_\varepsilon(x) - \chi_\delta(|x|)\phi_\varepsilon(x)\|_{L^2(\mathbb{R}^d)} \leq C_N \varepsilon^N$. As a consequence, we may rewrite the contribution (6.1) we are interested in as

$$\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} (1-\chi)\left(\frac{t}{T_1}\right) \, \mathrm{e}^{-\alpha_\varepsilon t} \, \langle \chi_\delta(|x|)\, \chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t)\chi_\delta(|x|)\, S_\varepsilon, \phi_\varepsilon\rangle \, dt$$

up to an $O_\delta(\varepsilon^\infty)$. There remains to bound the above term by

$$\begin{aligned}
&\leq \|S_\varepsilon\|_{L^2}\, \|\phi_\varepsilon\|_{L^2} \times \frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} \left\|\chi_\delta\left(|x|\right)\chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t)\chi_\delta\left(|x|\right)\right\|_{\mathcal{L}(L^2)} dt \\
&\leq \frac{C}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} \left\|\chi_\delta\left(|x|\right)\chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t)\chi_\delta\left(|x|\right)\right\|_{\mathcal{L}(L^2)} dt\,, \qquad (6.3)
\end{aligned}$$

up to an $O_\delta(\varepsilon^\infty)$. Our strategy is to now evaluate the operator norm under the integral sign. This task is performed in the next two steps.

**Second step:** *symbolic calculus*

In view of (6.3), our analysis boils down to computing, for any $T_1 \leq t \leq \varepsilon^{-\kappa}$, the operator norm

$$\left\|\chi_\delta\left(|x|\right)\chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t)\chi_\delta\left(|x|\right)\right\|^2_{\mathcal{L}(L^2)}.$$

Expanding the square, this norm has the value

$$\left\| \chi_\delta\left(|x|\right) U_\varepsilon^*(t) \chi_\delta\left(H_\varepsilon\right) \chi_\delta^2\left(|x|\right) \chi_\delta\left(H_\varepsilon\right) U_\varepsilon(t) \chi_\delta\left(|x|\right) \right\|_{\mathcal{L}(L^2)}. \tag{6.4}$$

Now, and for later convenience, we rewrite the above localizations in energy and space, as microlocalisations in position and frequency.

Using the functional calculus for pseudodifferential operators of Helffer and Robert [HR] (see also the lecture notes [DS] and [Ma]), there exists a symbol $\mathcal{X}_\delta(x,\xi)$ such that

$$\chi_\delta\left(H_\varepsilon\right) = \mathrm{Op}_\varepsilon^w(\mathcal{X}_\delta) + O(\varepsilon^\infty) \quad \text{in } \mathcal{L}(L^2).$$

The symbol $\mathcal{X}_\delta(x,\xi)$ is given by a formal expansion

$$\mathcal{X}_\delta(x,\xi) \sim \sum_{k\geq 0} \varepsilon^k {\mathcal{X}_\delta}^{(k)}(x,\xi), \tag{6.5}$$

where the expansion (6.5) holds in the class of symbols that are bounded together with all their derivatives. Furthermore, the principal symbol of $\mathcal{X}_\delta$ is computed through the natural equality

$$ {\mathcal{X}_\delta}^{(0)}(x,\xi) = \chi_\delta\left(\frac{\xi^2}{2} - n^2(x)\right).$$

Finally, the explicit formulae in [DS] give at any order $k \geq 0$ the following information on the support of the symbols ${\mathcal{X}_\delta}^{(k)}$,

$$\mathrm{supp}\ {\mathcal{X}_\delta}^{(k)} \subset \{|\xi^2/2 - n^2(x)| \leq \delta\}.$$

Hence (6.4) becomes, using standard symbolic calculus,

$$\left\| \chi_\delta\left(|x|\right) U_\varepsilon^*(t) \left[\mathrm{Op}_\varepsilon^w(\mathcal{X}_\delta(x,\xi) \,\sharp\, \chi_\delta^2\left(|x|\right) \,\sharp\, \mathcal{X}_\delta(x,\xi))\right] U_\varepsilon(t)\, \chi_\delta\left(|x|\right) \right\|_{\mathcal{L}(L^2)}, \tag{6.6}$$

up to an $O_\delta(\varepsilon^\infty)$ (Here we used the uniform bound $\|U_\varepsilon(t)\|_{\mathcal{L}(L^2)} \leq 1$). Let us define for convenience the following short-hand notation for the symbol in brackets in (6.6):

$$b_\delta(x,\xi) := \mathcal{X}_\delta(x,\xi) \,\sharp\, \chi_\delta^2\left(|x|\right) \,\sharp\, \mathcal{X}_\delta(x,\xi).$$

The only information we need in the sequel is that $b_\delta$ admits an asymptotic expansion $b_\delta = \sum_{k\geq 0} \varepsilon^k b_\delta^{(k)}$, where each $b_\delta^{(k)}$ has support

$$\mathrm{supp}\ {b_\delta}^{(k)} \quad \subset \quad \{|x| \leq \delta\} \cap \{|\xi^2/2 - n^2(x)| \leq \delta\} =: E(\delta).$$

This serves as a definition of the (compact) set $E(\delta)$ in phase space. In the sequel, we summarize these informations in the following abuse of notation

$$\mathrm{supp}\ b_\delta \subset E(\delta). \tag{6.7}$$

The remainder part of our analysis is devoted to estimating

$$\left\| \chi_\delta\left(|x|\right) U_\varepsilon^*(t)\, \mathrm{Op}_\varepsilon^w(b_\delta(x,\xi))\, U_\varepsilon(t)\, \chi_\delta\left(|x|\right) \right\|_{\mathcal{L}(L^2)},$$

and the hard part of the proof lies in establishing an "Egorov theorem for large times", to compute the conjugation $U_\varepsilon^*(t)\mathrm{Op}_\varepsilon^w(b_\delta(x,\xi))U_\varepsilon(t)$ in (6.4).

**Third step:** *an Egorov theorem valid for large times - End of the proof*
Now we claim the following

**Lemma 5.** *We assume that the refraction index has the symbolic behaviour (1.12) with* $n_\infty^2 > 0$[5]*. We also assume that the zero energy is non-trapping for the flow. Take the cutoff parameter in energy* $\delta$ *small enough. Then,*

*(i) Let* $\Phi(t,x,\xi)$ *be the classical flow associated with the Hamiltonian* $\xi^2/2 - n^2(x)$*. Let* $F(t,x,\xi)$ *be the linearized flow. For any multi-index* $\alpha$*, and for any (small) parameter* $\eta > 0$*, there exists a constant* $C_{\delta,|\alpha|,\eta}$ *such that for any initial datum* $(x,\xi) \in E(\delta) = \{|x| \le \delta\} \cap \{|\xi^2/2 - n^2(x)| \le \delta\}$*, we have*

$$\forall t \in \mathbb{R}, \qquad \left| \frac{\partial^\alpha F(t,x,\xi)}{\partial (x,\xi)^\alpha} \right| \le C_{\delta,|\alpha|,\eta} \, \langle t \rangle^{(1+\eta)(1+|\alpha|)+2|\alpha|}. \tag{6.8}$$

*In other words, the linearized flow has at most polynomial growth with time.*

*(ii) As a consequence, for any time* $t$*, there exists a time-dependent symbol*

$$\mathbf{b}_\delta(t,x,\xi) \sim \sum_{k\ge 0} \varepsilon^k {\mathbf{b}_\delta}^{(k)}(t,x,\xi),$$

*such that the following holds: there exists a number* $c_\delta > 0$ *such that for any* $N > 0$*, there exists a constant* $C_{\delta,N}$ *such that*

$$\left\| U_\varepsilon^*(t) \mathrm{Op}_\varepsilon^w \left(b_\delta\right) U_\varepsilon(t) - \mathrm{Op}_\varepsilon^w \left( \sum_{k=0}^N \varepsilon^k {\mathbf{b}_\delta}^{(k)} \right) \right\|_{\mathcal{L}(L^2)} \le C_{\delta,N} \, \varepsilon^{N+1} \langle t \rangle^{c_\delta N^2}. \tag{6.9}$$

*Again, the error grows polynomially with time, and we have some control on the dependence of the estimates with the truncation parameter* $N$*.*

*(iii) Moreover, we have the natural formulae*

$${\mathbf{b}_\delta}^{(0)}(t,x,\xi) = b_\delta \left( \Phi(t,x,\xi) \right),$$

*and, for any* $k \ge 0$ *we have the information on the support*

$$\mathrm{supp}\ {\mathbf{b}_\delta}^{(k)}(t,x,\xi) \subset \{(x,\xi) \in \mathbb{R}^{2d} \ \ s.t. \ \ \Phi(t,x,\xi) \in E(\delta)\}.$$

We postpone the proof of Lemma 5 to paragraph 6.2 below. We first draw its consequences in our perspective.

Leaving $N$ as a free parameter for the moment, we obtain

$$\begin{aligned}
&\left\| \chi_\delta\left(|x|\right) U_\varepsilon^*(t) \, \mathrm{Op}_\varepsilon^w (b_\delta(x,\xi)) \, U_\varepsilon(t) \, \chi_\delta\left(|x|\right) \right\|_{\mathcal{L}(L^2)} \\
&= \left\| \chi_\delta\left(|x|\right) \mathrm{Op}_\varepsilon^w \left( \sum_{k=0}^N \varepsilon^k {\mathbf{b}_\delta}^{(k)}(t,x,\xi) \right) \chi_\delta\left(|x|\right) \right\|_{\mathcal{L}(L^2)} \\
&\qquad\qquad\qquad + O_\delta\left( \varepsilon^{N+1} \langle t \rangle^{c_\delta N^2} \right) \\
&= \left\| \mathrm{Op}_\varepsilon^w \left( \chi_\delta\left(|x|\right) \sharp \left( \sum_{k=0}^N \varepsilon^k {\mathbf{b}_\delta}^{(k)}(t,x,\xi) \right) \sharp \, \chi_\delta\left(|x|\right) \right) \right\|_{\mathcal{L}(L^2)} \\
&\qquad\qquad\qquad + O_\delta\left( \varepsilon^{N+1} \langle t \rangle^{c_\delta N^2} \right).
\end{aligned}$$

[5]The assumption $n_\infty^2 > 0$ is crucial, see (6.11)

Now, part (iii) of Lemma 5 and standard symbolic calculus indicate that the above symbol has support[6] in

$$\bigcup_{k=0}^{N}\left(\text{supp }\chi_\delta\left(|x|\right)\cap\text{supp }\mathbf{b}_\delta{}^{(k)}(t,x,\xi)\right)$$
$$\subset\{(x,\xi)\text{ s.t. }|x|\le\delta,\text{ and }\Phi(t,x,\xi)\in E(\delta)\}.$$

The non-trapping condition (and more precisely estimate (6.10) below) allows in turn to deduce that this set is void for $t$ large enough. Hence, up to taking a large value of $T_1$ , $T_1\ \ge T_1\,(\delta)$ for some $T_1\,(\delta)$, we eventually obtain in (6.3),

$$\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}}\left\|\chi_\delta\left(|x|\right)\chi_\delta\left(H_\varepsilon\right)U_\varepsilon(t)\chi_\delta\left(|x|\right)\right\|_{\mathcal{L}(L^2)}dt$$
$$\le\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}}O_\delta\left(\varepsilon^{(N+1)/2}\langle t\rangle^{c_\delta N^2/2}\right)dt\le O_\delta\left(\varepsilon^{(N-1)/2-c_\delta\kappa N^2/2}\right)\le O_{\kappa,\delta}(\varepsilon),$$

for $\kappa$ small enough (and $N=4$ will do). This ends the proof of proposition 4.

## 6.2 Proof of Lemma 5: an Egorov theorem for polynomially large times

In view of the above proof, we are left with the task of proving the large time Egorov theorem of Lemma 5. To do so, we follow here closely ideas developped in [BR] in a slightly different context. Part (iii) of the Lemma is proved in [BR], so we will skip this aspect. The implication (i) $\Rightarrow$ (ii) in Lemma 5, which we prove below for completeness, is also essentially proved in [BR]. Our main task in the sequel turns out to be the proof part (i) of the Lemma.

The proof is given in several steps.

**First step:** *estimates on the flow* $\Phi(t,x,\xi)$
In this step, we prove that for small enough a $\delta$, there is a time $T(\delta)$, depending on $\delta$, such that for any initial datum $(x,\xi)$ of phase-space in the set $E(\delta)=\{|x|\le\delta\}\cap\{|\xi^2/2-n^2(x)|\le\delta\}$ (see 6.7)), one has

$$\forall t\ge T(\delta),\quad|X(t,x,\xi)|\ge C_\delta\,t,\tag{6.10}$$

for some constant $C_\delta>0$ that depends on $\delta$, that is however independent of both time $t$ and the initial point $(x,\xi)$ under consideration. The proof is standard and uses the information $n_\infty^2>0$.

First, the non-trapping condition implies that for any large number $R'>0$, and for any initial point $(x,\xi)\in E(\delta)$, there exists a time $T(R',x,\xi)$ such that

$$\forall t\ge T(R',x,\xi),\quad|X(t,x,\xi)|\ge R'.$$

By continuous dependence of the flow $X(t,x,\xi)$ with respect to the initial data $(x,\xi)$, and compactness of the set $E(\delta)$, there is a time $T(R',\delta)$, that now depends upon $R'$ and $\delta$ only, such that for any initial point $(x,\xi)\in E(\delta)$, there holds

$$\forall t\ge T(R'),\quad|X(t,x,\xi)|\ge R'.$$

[6] we make here the same abuse of notation than in (6.7).

In other words, the trajectory $X(t,x,\xi)$ goes to infinity as time goes to infinity, uniformly with respect to the initial datum $(x,\xi) \in E(\delta)$.

Second, we get estimates for the standard "escape function" of quantum and classical scattering, namely the function $X(t) \cdot \Xi(t)$. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\left(X(t,x,\xi)\cdot\Xi(t,x,\xi)\right) &= 2\left(\frac{\Xi^2(t,x,\xi)}{2} - n^2\left(X(t,x,\xi)\right)\right) \\
&\qquad + 2n^2\left(X(t,x,\xi)\right) + X(t,x,\xi)\cdot\nabla n^2\left(X(t,x,\xi)\right) \\
&= 2\left(\frac{\xi^2}{2} - n^2(x)\right) + 2n^2\left(X(t,x,\xi)\right) + X(t,x,\xi)\cdot\nabla n^2\left(X(t,x,\xi)\right) \\
&\qquad \text{(thanks to the conservation of energy)} \\
&\underset{t\to\infty}{\longrightarrow} 2\left(\frac{\xi^2}{2} - n^2(x)\right) + 2n_\infty^2,
\end{aligned}$$

uniformly with respect to the initial datum $(x,\xi) \in E(\delta)$. Hence, using the fact that $n_\infty^2 > 0$, and taking a possibly smaller value of the cutoff parameter $\delta$, we obtain the existence of a constant $C_\delta > 0$, and another time $T(\delta)$, such that

$$\forall t \geq T(\delta), \quad X(t,x,\xi)\cdot\Xi(t,x,\xi) \geq C_\delta\, t. \tag{6.11}$$

Using the fact that $\frac{\partial}{\partial t}\left(\frac{1}{2}\, X^2(t,x,\xi)\right) = X(t,x,\xi)\cdot\Xi(t,x,\xi)$, we deduce the desired lower bound

$$\forall t \geq T(\delta), \quad \frac{1}{2}\left(X^2(t,x,\xi) - X^2(T(\delta),x,\xi)\right) \geq C_\delta\, \frac{t^2}{2}.$$

**Second step:** *estimates on the linearized flow $F(t,x,\xi)$.*
One first proves the estimate (6.8) in the case $\alpha=\beta=0$. By its very definition (2.11), the linearized flow

$$F(t,x,\xi) = \begin{pmatrix} A(t,x,\xi) & B(t,x,\xi) \\ C(t,x,\xi) & D(t,x,\xi) \end{pmatrix}.$$

satisfies (see (2.13), (2.14)) the differential system

$$\begin{aligned}
&\frac{\partial}{\partial t}A(t,x,\xi) = C(t,x,\xi), && A(0,x,\xi) = \mathrm{Id}, \\
&\frac{\partial}{\partial t}C(t,x,\xi) = D^2n^2\left(X(t,x,\xi)\right)\, A(t,x,\xi), && C(0,x,\xi) = 0,
\end{aligned} \tag{6.12}$$

together with

$$\begin{aligned}
&\frac{\partial}{\partial t}B(t,x,\xi) = D(t,x,\xi), && B(0,x,\xi) = 0, \\
&\frac{\partial}{\partial t}D(t,x,\xi) = D^2n^2\left(X(t,x,\xi)\right)\, B(t,x,\xi), && D(0,x,\xi) = \mathrm{Id}.
\end{aligned} \tag{6.13}$$

Here, the notation $D^2n^2(x)$ refers to the Hessian of the function $n^2(x)$ in the variable $x$. Due to the assumption (1.12) on the behaviour of $n^2(x)$ at infinity, we readily have

$$|D^2n^2(x)| \leq C\, \langle x\rangle^{-\rho-2},$$

for some constant $C > 0$, independent of $x$. This, together with the previous bound (6.10) on the behaviour of the flow $X(t, x, \xi)$ at infinity in time, gives the estimate

$$\left| D^2 n^2 \left( X(t, x, \xi) \right) \right| \leq C_0 \, \langle t \rangle^{-\rho - 2}, \tag{6.14}$$

for some constant $C_0 > 0$ which is independent of time $t \geq 0$, and of the point $(x, \xi)$ in phase-space. We are thus in position to estimate $A(t)$ and $C(t)$ using (6.12). Integrating (6.12) in time, and setting

$$\varepsilon(t) := |D^2 n^2 \left( X(t, x, \xi) \right)| \tag{6.15}$$

for convenience, we obtain (dropping the dependence on $(x, \xi)$ of the various functions),

$$|A(t) - \mathrm{Id}| \leq \int_0^t (t - s) \, \varepsilon(s) \, |A(s) - \mathrm{Id}| \, ds + \int_0^t (t - s) \, \varepsilon(s) \, ds, \tag{6.16}$$

$$|C(t)| \leq \int_0^t \varepsilon(s) \, |A(s)| \, ds. \tag{6.17}$$

Choose now a constant $C_*$, and define the time $t_*$ as

$$t_* := \sup\{t \geq 0 \ \text{ s.t. } \ |A(t) - \mathrm{Id}| \leq C_* \langle t \rangle^{1+\eta}\}.$$

We prove that $t_* = +\infty$, provided $C_*$ is large enough. Indeed, for any time $t \leq t_*$, using (6.16) together with the decay (6.14), we have

$$\begin{aligned}
|A(t) - \mathrm{Id}| &\leq C_0 C_* \int_0^t (t - s) \langle s \rangle^{-\rho - 1 + \eta} \, ds \leq C_0 C_* \, t \int_0^t \langle s \rangle^{-\rho - 1 + \eta} \, ds \\
&\leq C_0 C_* C_\eta \, t \\
&\quad \text{(for some constant } C_\eta > 0\text{, provided } \eta > 0 \text{ satisfies } \eta < \rho/2 \text{ )} \\
&< C_* \, \langle t \rangle^{1+\eta} \\
&\quad \text{(provided } t \text{ is large enough, } t \geq T(C_0, C_\eta)\text{, for some } T(C_0, C_\eta) \\
&\quad \text{that only depends on } C_0 \text{ and } C_\eta).
\end{aligned}$$

On the other hand, we certainly have $|A(t) - \mathrm{Id}| \leq C_* \langle t \rangle^{1+\eta}$ for bounded values of time $t \leq T(C_0, C\eta)$, provided $C_*$ is large enough. Hence $t_* = +\infty$. Inserting this upper-bound for $A$ in (6.17) gives

$$|C(t)| \leq C_\eta,$$

for some $C_\eta > 0$, provided $\eta > 0$ is small enough. We may estimate $B(t)$ and $D(t)$ in the similar way. The analysis is the same, and starts with the formulae

$$\begin{aligned}
|B(t)| &\leq t + \int_0^t (t - s) \, \varepsilon(s) \, |B(s)| \, ds, \\
|D(t)| &\leq 1 + \int_0^t \varepsilon(s) \, |B(s)| \, ds.
\end{aligned}$$

We skip the details. At this level, we have obtained the bound

$$|F(t, x, \xi)| \leq C_\eta \, \langle t \rangle^{1+\eta},$$

for any (small enough) $\eta > 0$, and a constant $C_\eta$ independent of $(t, x, \xi)$.

**Third step:** *estimates on the derivatives of the linearized flow*
Let now $\alpha$ be any multi-index. We prove (6.8) by induction on $|\alpha|$. Define, for any $p \geq 1$

$$M_p(t) := \sup_{|\beta|=n} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left| \frac{\partial^\beta \Phi(t,x,\xi)}{\partial(x,\xi)^\beta} \right|,$$

We have proved in the second step above that

$$M_1(t) \leq C_\eta \langle t \rangle^{1+\eta}.$$

Assume that for some integer $p_0$, the estimate

$$M_p(t) \leq C_{p,\eta} \langle t \rangle^{p(1+\eta)+2(p-1)},$$

has been proved for any $p \leq p_0$. We wish to prove the analogous estimate for $M_{p_0+1}$. Take any multi-index $\alpha$ of length $|\alpha| = p_0$. From now on, we systematically omit the dependence of the various functions and derivatives with respect to $(x, \xi)$, and write $\partial^\alpha F(t)$, $\partial^\alpha H$ instead of $\partial^\alpha F(t,x,\xi)/\partial(x,\xi)^\alpha$, $\partial^\alpha H(x,\xi)/\partial(x,\xi)^\alpha$ and so on. Upon differentiating $\alpha$ times the linearized equation (2.15) on $F$, we obtain,

$$\partial_t \left(\partial^\alpha F(t)\right) = J \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta \left(D^2 H\left(\Phi(t)\right)\right) \left(\partial^{\alpha-\beta} F(t)\right). \tag{6.18}$$

In order to make estimates in (6.18), we first need to write the Faà de Bruno formula as

$$\partial^\beta \left(D^2 H \circ \Phi(t)\right) = \beta! \sum_{\gamma,m} \left(\partial^\gamma D^2 H\right) \circ \Phi(t) \times \prod_\zeta \frac{1}{m(\zeta)!} \left(\frac{\partial^\zeta \Phi(t)}{\zeta!}\right)^{m(\zeta)}.$$

Here $\beta \in \mathbb{N}^{2d}$, $\gamma \in \mathbb{N}^{2d}$, and $\zeta \in \mathbb{N}^{2d}$ are multiindices, and $m$ associates to each multi-index $\zeta \in \mathbb{N}^{2d}$, another multi-index $m(\zeta) \in \mathbb{N}^{2d}$. Also, the above sum carries over all values of $\gamma$, $m$, and $\zeta$ such that

$$\sum_\zeta m(\zeta) = \gamma, \; \sum_\zeta \zeta \; |m(\zeta)| = \beta. \tag{6.19}$$

Finally, when $|\beta| \geq 1$, the above sums carries over $\gamma$'s and $\zeta$'s such that $|\gamma| \geq 1$ and $|\zeta| \geq 1$. All this gives in (6.18),

$$\partial_t \left(\partial^\alpha F(t)\right) = J \sum_{\beta\leq\alpha} \beta! \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \sum_{\gamma,m} \left(\partial^\gamma D^2 H\right) \circ \Phi(t)$$
$$\times \prod_\zeta \frac{1}{m(\zeta)!} \left(\frac{\partial^\zeta \Phi(t)}{\zeta!}\right)^{m(\zeta)} \times \; \partial^{\alpha-\beta} F(t).$$

Hence, putting apart the contribution stemming from $\beta = 0$, we recover

$$\partial_t \left(\partial^\alpha F(t)\right) = J \; D^2 H\left(\Phi(t)\right) \; \left(\partial^\alpha F(t)\right) + R_\alpha(t), \tag{6.20}$$

where the remainder term $R_\alpha(t)$ is estimated by

$$
\begin{aligned}
&|R_\alpha(t)| \\
&\le C_{|\alpha|} \sum_{0\neq\beta\le\alpha} \sum_{\gamma,m} \left| \left(\partial^\gamma D^2 H\right) \circ \Phi(t) \right| \prod_\zeta \left(|\partial^\zeta \Phi(t)|\right)^{|m(\zeta)|} |\partial^{\alpha-\beta} F(t)| \\
&\le C_{|\alpha|} \sum_{0\neq\beta\le\alpha} \sum_{\gamma,m} |\partial^{\alpha-\beta} F(t)| \prod_\zeta \left(|\partial^\zeta \Phi(t)|\right)^{|m(\zeta)|}.
\end{aligned}
$$

for some constant $C_{|\alpha|} > 0$ that depends on $|\alpha|$. The last line uses the fact that $\sup_{x,\xi} |\partial^\gamma D^2 H(x,\xi)| \le C_\gamma$ for some constant $C_\gamma$. Using the inductive assumption, we recover

$$
\begin{aligned}
|R_\alpha(t)| &\le C_{|\alpha|,\eta} \sum_{0\neq\beta\le\alpha} \sum_{\gamma,m} \langle t\rangle^{(|\alpha-\beta|+1)(1+\eta)+2|\alpha-\beta|} \\
&\qquad\qquad \times \prod_\zeta \langle t\rangle^{(|\zeta|(1+\eta)+2(|\zeta|-1))\,|m(\zeta)|} \\
&\le C_{|\alpha|,\eta} \sum_{0\neq\beta\le\alpha} \langle t\rangle^{(1+\eta)\left(1+|\alpha-\beta|+\sum_\zeta |\zeta||m(\zeta)|\right)+2\left(|\alpha-\beta|+\sum_\zeta (|\zeta|-1)\,|m(\zeta)|\right)} \\
&= C_{|\alpha|,\eta} \sum_{0\neq\beta\le\alpha} \langle t\rangle^{(1+\eta)(1+|\alpha-\beta|+|\beta|)+2(|\alpha-\beta|+|\beta|-|\gamma|)} \\
&\le C_{|\alpha|,\eta} \, \langle t\rangle^{(1+\eta)(1+|\alpha|)+2(|\alpha|-1)}.
\end{aligned}
$$

Here we used the constraints (6.19) together with the information $|\gamma| \ge 1$. Using Lemma 6 below in equation (6.20) satisfied by $\partial^\alpha F$, we obtain,

$$
|\partial^\alpha F(t)| \le C_{|\alpha|,\eta} \, \langle t\rangle^{(1+\eta)(|\alpha|+1)+2|\alpha|}.
$$

Hence

$$
M_{p_0+1}(t) \le C_{p_0,\eta} \, \langle t\rangle^{(1+\eta)(p_0+1)+2p_0}.
$$

This ends the recursion.

**Fourth step:** *A Gronwall Lemma for solutions to the linearized Hamilton equation*

The preceding step uses the following

**Lemma 6.** *Assume the function $G(t,x,\xi)$ satisfies the differential equation*

$$
\begin{aligned}
&\frac{\partial G(t,x,\xi)}{\partial t} = J \cdot D^2 H\left(\Phi(t,x,\xi)\right) \cdot G(t,x,\xi) + O\left(\langle t\rangle^\lambda\right), \qquad (6.21)\\
&G(0,x,\xi) = 0,
\end{aligned}
$$

*where the $O\left(\langle t\rangle^\lambda\right)$ is uniform in $(x,\xi)$. Then, $G$ satisfies the uniform estimate*

$$
G(t,x,\xi) = O\left(\langle t\rangle^{\lambda+2}\right).
$$

**Proof of Lemma 6**

Decompose $G(t) \equiv G(t,x,\xi)$ as

$$
G(t) = \begin{pmatrix} A_G(t) & B_G(t) \\ C_G(t) & D_G(t) \end{pmatrix}.
$$

Then, equation (6.21) for $G$ writes

$$\frac{\partial}{\partial t} A_G(t) = C_G(t) + O\left(\langle t\rangle^{\lambda}\right), \qquad A_G(0) = 0, \tag{6.22}$$
$$\frac{\partial}{\partial t} C_G(t) = D^2 n^2\left(X(t)\right)\ A_G(t) + O\left(\langle t\rangle^{\lambda}\right), \qquad C_G(0) = 0,$$

together with

$$\frac{\partial}{\partial t} B_G(t) = D_G(t) + O\left(\langle t\rangle^{\lambda}\right), \qquad B_G(0) = 0, \tag{6.23}$$
$$\frac{\partial}{\partial t} D_G(t) = D^2 n^2\left(X(t)\right)\ B_G(t) + O\left(\langle t\rangle^{\lambda}\right), \qquad D_G(0) = 0.$$

Equations (6.22) give rise to the estimates

$$|A_G(t)| \leq C \int_0^t (t-s)\ \left(\ \varepsilon(s)\ |A_G(s)| + \langle s\rangle^{\lambda}\right)\ ds, \tag{6.24}$$

$$|C_G(t)| \leq C \int_0^t \varepsilon(s)\ |A_G(s)|\ ds, \tag{6.25}$$

where the function $\varepsilon(s)$ is defined in (6.15) above. Using $\varepsilon(s) \leq C_0\ \langle s\rangle^{-\rho-2} \leq C_\eta\ \langle s\rangle^{-\eta-2}$ for any small $\eta > 0$ (see (6.14)), gives in equation (6.24),

$$|A_G(t)| \leq C_\eta\ t \int_0^t \langle s\rangle^{-\eta-2}\ |A_G(s)|\ ds + C\ \langle t\rangle^{\lambda+2}\ . \tag{6.26}$$

From this it can be deduced that

$$|A_G(t)| \leq C\ \langle t\rangle^{\lambda+2}.$$

(for a given constant $C_*$, define indeed $t_* = \sup\{t \geq 0 \text{ s.t. } |A_G(t)| \leq C_*\ \langle t\rangle^{\lambda+2}$ - one deduces from (6.26) that $t_* = +\infty$ provided $C_*$ is large enough - see (6.16) and sequel for details). Equation (6.25) then gives

$$|C_G(t)| \leq C_\eta \int_0^t \langle s\rangle^{-\eta-2}\ |A_G(s)|\ ds \leq C_\eta\ \langle t\rangle^{\lambda+1-\eta}.$$

The estimates for $B_G$ and $D_G$ are the same. This ends the proof of the Lemma.

**Fifth step***: adapting the estimates of [BR]*
We now put together the estimates on the linearized flow obtained before, to complete the proof of parts (ii) and (iii) of Lemma 5.

The construction of the symbols ${\mathbf{b}_\delta}^{(k)}(t,x,\xi)$ in Lemma 5 is made in an explicit way in [BR]. Part (iii) of Lemma 5 follows. Also, the remainder estimate (6.9) is a consequence of the above estimates on the linearized flow $F(t,x,\xi)$ and its derivatives, upon adapting the analysis of [BR]. Let us indeed write the rough (but simpler) estimate

$$|\partial^\alpha F(t,x,\xi)| \leq C_\alpha \langle t\rangle^{4|\alpha|+2},$$

corresponding to the special choice $\eta = 1$ in (6.8). Then, Theorem 1.2 - formula (12) of [BR],

$${\mathbf{b}_\delta}^{(0)}(t,x,\xi) = b_\delta\left(\Phi(t,x,\xi)\right),$$

together with the Faá de Bruno formula, give for any multi-index $\alpha$ the estimate

$$|\partial^\alpha \mathbf{b}_\delta{}^{(0)}(t,x,\xi)| \leq C_{|\alpha|}\ \langle t\rangle^{4|\alpha|}.$$

From Theorem 1.2 - formula (14) of [BR], we have for any $k \geq 1$ the explicit value

$$\mathbf{b}_\delta{}^{(k)}(t,x,\xi) = \sum_{\substack{|\alpha|+\ell=k+1 \\ 0\leq \ell \leq k-1}} \Gamma(\alpha) \int_0^t \left[\partial^\alpha H \times \partial^\alpha \mathbf{b}_\delta{}^{(\ell)}\right] \circ \Phi(t-s,x,\xi)\, ds,$$

where $\Gamma(\alpha)$ is a harmless coefficient whose explicit value is given in [BR]. This, together with the Faá de Bruno formula, implies for any $k \geq 1$, the upper-bound

$$|\partial^\alpha \mathbf{b}_\delta{}^{(k)}(t,x,\xi)| \leq C_{|\alpha|,k}\ \langle t\rangle^{c_0(k|\alpha|+k^2+1)},$$

for some fixed number $c_0$, independent of $\alpha$ and $k$. Then, using formulae (51), together with (52), (54), (97) and (99) of [BR] gives the estimate (6.9). This ends the proof of Lemma 5. □

# 7 Contribution of moderate times, close to the zero energy

After the work performed in sections 3 through 6, there only remains to estimate the most difficult term

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\chi_\delta\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt.$$

This is the key point of the present paper.

The main result of the present section is the following

**Proposition 7.** *We use the notations of section 2. The index $n^2$ is assumed to have the symbolic behaviour (1.12). The zero energy is assumed non-trapping for the Hamiltonian $\xi^2/2 - n^2(x)$. Finally, we need the tranversality condition* **(H)** *page 35 on the trajectories $\Phi(t,x,\xi)$ with initial data satisfying $x=0$, $\xi^2/2 = n^2(0)$. Then, the following two estimates hold true,*

*(i) for any* fixed *value of the truncation parameters $\theta$, $T_1$ and $\delta$, we have*

$$\frac{1}{\varepsilon}\int_{\theta}^{T_1} (1-\chi)\left(\frac{t}{\theta}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\chi_\delta\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

*(ii) for $\theta > 0$ small enough, there exists a constant $C_\theta > 0$ such that for any $\varepsilon \leq 1$, we have*

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{2\theta} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\chi\left(\frac{t}{\theta}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\chi_\delta\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt$$
$$\leq C_\theta\ T_0{}^{-d/2+1} \underset{T_0\to+\infty}{\longrightarrow} 0.$$

The remainder part of this paragraph is devoted to the proof of proposition 7. In order to shorten the notations, we define

$$\widetilde{\chi}_\varepsilon(t) := (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}, \tag{7.1}$$

so that the proof of proposition 7 boils down to estimating

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} \widetilde{\chi}_\varepsilon(t)\left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\right\rangle dt. \tag{7.2}$$

The precise value of the cut-off function $\widetilde{\chi}_\varepsilon(t)$ in the analysis of (7.2) will be essentially irrelevant in the sequel.

**Proof of proposition 7**

The proof is given in several steps. As in section 6, we begin with some preliminary reductions, exploiting the informations on the microlocal support of the various functions. Then, we use the elegant wave-packet approach of Combescure and Robert [CRo] to compute the semi-classical propagator $U_\varepsilon(t)$ in (7.2) in a very explicit way - see Theorem 8 below: this gives a representation in terms of a Fourier integral operator *with complex phase*, that is very well suited for our asymptotic analysis (see also [CRR], or the work by Hagedorn and Joye [H1], [H2], [HJ], or by Robinson [Rb], or even the seminal work by Hepp [He] for similar representations - see also Butler [Bt]). This eventually reduces the analysis to stationary phase arguments that are very much in the spirit of [CRR], and where the tranversality assumption **(H)** page 35 turns out to play a crucial role.

**First Step:** *Preliminary reduction, projection over the gaussian wave packets*
As in section 6 (see (6.3), (6.5), (6.7)), we may first build up a symbol $a_0(x,\xi) \in C_c^\infty(\mathbb{R}^{2d})$ such that

$$\operatorname{supp} a_0 \subset \{|x| \le \delta\} \cap \{|\xi^2/2 - n^2(x)| \le \delta\}, \tag{7.3}$$

and

$$\left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\right\rangle = \left\langle \mathrm{Op}_\varepsilon^w\left(a_0(x,\xi)\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\right\rangle + O_\delta(\varepsilon^\infty).$$

With the notation (6.5), we actually have the value $a_0(x,\xi) = \mathcal{X}_\delta(x,\xi)\sharp\chi_\delta\left(|x|\right)$. Therefore, the asymptotic analysis of (7.2) reduces to that of the expression

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} \widetilde{\chi}_\varepsilon(t)\left\langle \mathrm{Op}_\varepsilon^w(a_0) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\right\rangle dt. \tag{7.4}$$

Now, to be able to use the wave-packet approach of [CRo], we need to decompose the above scalar product on the basis of the Gaussian wave packets

$$\varphi_{q,p}^\varepsilon(x,\xi) := (\pi\varepsilon)^{-d/4}\ \exp\left(\frac{i}{\varepsilon}p\cdot\left(x-\frac{q}{2}\right)\right)\ \exp\left(-\frac{(x-q)^2}{2\varepsilon}\right).$$

Each function $\varphi_{q,p}^\varepsilon$ is microlocally supported near the point $(q,p)$ in phase-space. Using the well-known orthogonality properties of these states, i.e.

$$\langle u, v\rangle = (2\pi\varepsilon)^{-d}\int_{\mathbb{R}^{2d}} dqdp\ \langle u, \varphi_{q,p}^\varepsilon\rangle\ \langle\varphi_{q,p}^\varepsilon, v\rangle,$$

for any $u(x)$ and $v(x)$ in the space $L^2(\mathbb{R}^d)$, and forgetting the normalizing factors like $\pi$, etc., we obtain in (7.4)

$$\frac{1}{\varepsilon^{d+1}}\int_{T_0\,\varepsilon}^{T_1}\int_{\mathbb{R}^{2d}} dtdqdp\,\widetilde{\chi}_\varepsilon(t)\Big\langle \mathrm{Op}_\varepsilon^w(a_0)S_\varepsilon,\varphi_{q,p}^\varepsilon\Big\rangle\,\Big\langle \varphi_{q,p}^\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\Big\rangle$$
$$=\frac{1}{\varepsilon^{d+1}}\int_{T_0\,\varepsilon}^{T_1}\int_{\mathbb{R}^{2d}} dtdqdp\,\widetilde{\chi}_\varepsilon(t)\Big\langle S_\varepsilon, \mathrm{Op}_\varepsilon^w(a_0)\varphi_{q,p}^\varepsilon\Big\rangle\,\Big\langle U_\varepsilon(t)\varphi_{q,p}^\varepsilon, \phi_\varepsilon\Big\rangle. \tag{7.5}$$

Before going further, and in order to prepare for the use of the stationary phase theorem below, we make the simple observation that the integral $dqdp$ over $\mathbb{R}^{2d}$ in (7.5) may be carried over the compact set $\{|x|\le 2\delta\}\cap\{|\xi^2/2-n^2(x)|\le 2\delta\}$, up to a negligible error $O_\delta(\varepsilon^\infty)$. For that purpose, take a function $\chi_0(q,p)\in C_c^\infty(\mathbb{R}^{2d})$ such that

$$\begin{aligned}&\mathrm{supp}\,\chi_0(q,p)\subset\{|x|\le 2\delta\}\cap\{|\xi^2/2-n^2(x)|\le 2\delta\}\\ &\chi_0(q,p)\equiv 1 \;\text{ on }\; \big\{|x|\le 3\delta/2\}\cap\{|\xi^2/2-n^2(x)|\le 3\delta/2\big\}\,.\end{aligned}\tag{7.6}$$

We claim the following estimate holds true:

$$\int_{\mathbb{R}^{2d}} dqdp\;(1-\chi_0(q,p))\;\Big\|\mathrm{Op}_\varepsilon^w(a_0)\varphi_{q,p}^\varepsilon\Big\|_{L^2(\mathbb{R}^d)}^2=O_\delta(\varepsilon^\infty). \tag{7.7}$$

Indeed, we have the following simple computation:

$$\begin{aligned}\big\|\mathrm{Op}_\varepsilon^w(a_0)\varphi_{q,p}^\varepsilon\big\|_{L^2(\mathbb{R}^d)}^2&=\langle \mathrm{Op}_\varepsilon^w(a_0\sharp a_0)\varphi_{q,p}^\varepsilon,\varphi_{q,p}^\varepsilon\rangle\\ &=\int_{\mathbb{R}^{2d}} dxd\xi\,(a_0\sharp a_0)(x,\xi)\;W(\varphi_{q,p}^\varepsilon)(x,\xi)\\ &\qquad\text{(where } W(\varphi_{q,p}^\varepsilon) \text{ denotes the Wigner transform of } \varphi_{q,p}^\varepsilon)\\ &=\varepsilon^{-d}\int_{\mathbb{R}^{2d}} dxd\xi\,(a_0\sharp a_0)(x,\xi)\;\exp\left(-\frac{|q-x|^2+|p-\xi|^2}{\varepsilon}\right),\end{aligned}$$

and the last line uses the fact that the Wigner transform of $\varphi_{q,p}^\varepsilon$ is a Gaussian. Now, using $\mathrm{supp}\,(a_0\sharp a_0)\subset\{|x|\le\delta\}\cap\{|\xi^2/2-n^2(x)|\le\delta\}$, together with (7.6), establishes (7.7).

Using this estimate (7.7), and replacing back the factor $\mathrm{Op}_\varepsilon^w(a_0)$ by the identity in (7.5), we arrive at the conclusion

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1}\widetilde{\chi}_\varepsilon(t)\,\Big\langle \chi_\delta\left(H_\varepsilon\right)S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\Big\rangle\,dt=O_{T_1\,,\delta}\left(\varepsilon^\infty\right)+$$
$$\frac{1}{\varepsilon^{d+1}}\int_{T_0\,\varepsilon}^{T_1}\int_{\mathbb{R}^{2d}} dtdqdp\,\widetilde{\chi}_\varepsilon(t)\,\chi_0(q,p)\,\Big\langle S_\varepsilon,\varphi_{q,p}^\varepsilon\Big\rangle\,\Big\langle U_\varepsilon(t)\varphi_{q,p}^\varepsilon,\phi_\varepsilon\Big\rangle\,.$$

Our strategy is to now pass to the limit in the term

$$\frac{1}{\varepsilon^{d+1}}\int_{T_0\,\varepsilon}^{T_1}\int_{\mathbb{R}^{2d}} dtdqdp\,\widetilde{\chi}_\varepsilon(t)\,\chi_0(q,p)\,\Big\langle S_\varepsilon,\varphi_{q,p}^\varepsilon\Big\rangle\,\Big\langle U_\varepsilon(t)\varphi_{q,p}^\varepsilon,\phi_\varepsilon\Big\rangle. \tag{7.8}$$

In order to do so, we need to compute the time evolved gaussian wave packet $U_\varepsilon(t)\varphi_{q,p}^\varepsilon$ in an accurate way.

**Second Step:** *Computation of $U_\varepsilon(t)\varphi_{q,p}^\varepsilon$ - reducing the problem to a stationary phase formula*

The following theorem is proved in [CRo] (see also [Ro], [Ro2])

**Theorem 8. ([CRo], [Ro])** *We use the notations of section 2. Under assumption (1.12) on the refraction index $n^2(x)$, there exists a family of functions $\{p_{k,j}(t,q,p,x)\}_{(k,j)\in\mathbb{N}^2}$, that are polynomials of degree at most $k$ in the variable $x \in \mathbb{R}^d$, with coefficients depending on $t$, $q$, $p$, such that for any $\varepsilon \leq 1$, the following estimate holds true: for any given value of $T_1$, and any given integer $N$, we have, for any time $t \in [0, T_1]$,*

$$\Bigg\| U_\varepsilon(t)\varphi^\varepsilon_{q,p} - \exp\left(\frac{i}{\varepsilon}\delta(t,q,p)\right)\mathcal{T}_\varepsilon(q_t,p_t)\Lambda_\varepsilon Q_N(t,q,p,x) \\ \mathcal{M}(F(t,q,p))\left(\pi^{-d/4}\exp\left(-x^2/2\right)\right)\Bigg\|_{L^2(\mathbb{R}^d)} \leq C_{N,T_1}\ \varepsilon^N, \tag{7.9}$$

*where*

$$Q_N(t,q,p,x) := 1 + \sum_{(k,j)\in I_N} \varepsilon^{\frac{k}{2}-j} p_{k,j}(t,q,p,x)\,,$$
$$I_N := \{1 \leq j \leq 2N-1,\ 1 \leq k-2j \leq 2N-1,\ k \geq 3j\}\,.$$

*Here, the following quantities are defined:*

- $\Lambda_\varepsilon$ *is the dilation operator*

$$(\Lambda_\varepsilon u)(x) := \varepsilon^{-d/4}\, u\left(\frac{x}{\sqrt{\varepsilon}}\right), \tag{7.10}$$

- $\mathcal{T}_\varepsilon(q_t,p_t)$ *is the translation (in phase-space) operator*

$$(\mathcal{T}_\varepsilon(q_t,p_t)u)(x) := \exp\left(\frac{i}{\varepsilon}p_t\cdot\left(x-\frac{q_t}{2}\right)\right)\ u(x-q_t), \tag{7.11}$$

- $(q_t,p_t)$ *denotes the trajectory*

$$(q_t,p_t) := (X(t,q,p), \Xi(t,q,p))\,, \tag{7.12}$$

- $\delta(t,q,p)$ *denotes quantity*

$$\delta(t,q,p) = \int_0^t \left(\frac{p_s^2}{2} + n^2(q_s)\right)\ ds - \frac{q_t\cdot p_t - q\cdot p}{2}, \tag{7.13}$$

- $\mathcal{M}(F(t,q,p))$ *is the metaplectic operator associated with the symplectic matrix $F(t,q,p)$. It acts on the gaussian as*

$$\mathcal{M}(F(t,q,p))\left(\exp\left(-\frac{x^2}{2}\right)\right) = \\ \det(A(t,q,p)+iB(t,q,p))_{\rm c}^{-1/2}\ \exp\left(i\frac{\Gamma(t,q,p)x\cdot x}{2}\right). \tag{7.14}$$

*Here, the square root $\det(A(t,q,p)+iB(t,q,p))_{\rm c}^{-1/2}$ is defined by continuously (hence the index* c*) following the argument of the complex number $\det(A(t,q,p)+iB(t,q,p))$ starting from its value 1 at time $t=0$. Also, the complex matrix $\Gamma(t,q,p)$ is defined as*

$$\Gamma(t,q,p) = (C(t,q,p)+iD(t,q,p))\ (A(t,q,p)+iB(t,q,p))^{-1}. \tag{7.15}$$

**Remark**
If the refraction index $n^2(x)$ is *quadratic* in $x$, then formula (7.9) is *exact*, and the whole family $\{p_{k,j}\}$ vanishes. This is essentially a consequence of the Mehler formula. We refer to [Fo] for a very complete discussion about the propagators of pseudo-differential operators with *quadratic* symbols.

In the case when $n^2(x)$ is a general function, the polynomials $p_{k,j}$ are obtained in [CRo] using perturbative expansions "around the quadratic case". We refer to [Ro] for a very clear and elegant derivation of these polynomials. Let us quote that similar formulae are derived and used in [HJ]. The idea of considering such perturbations "around the quadratic case" traces back to [He], see also [H1], [H2], [Rb].

The fact that the matrix $A(t) + iB(t)$ is invertible, and $\Gamma(t)$ is well defined, is proved in [Fo], see also [Ro2]. It is a consequence of the symplecticity of $F(t)$ (see the relations (2.17)). We refer to the sequel for an explicit use of these important relations. □

In the next lines, we apply the above theorem, and transform formula (7.8) accordingly.

On the one hand, we use the Parseval formula in (7.8) to compute the two scalar products. Forgetting the normalizing factors like $\pi$, etc., it gives, e.g. for the first scalar product,

$$\begin{aligned}\langle S_\varepsilon, \varphi^\varepsilon_{q,p}\rangle &= \varepsilon^{-d/2} \int_{\mathbb{R}^d} dx d\xi \, \exp(ix\cdot\xi/\varepsilon)\, \widehat{S}(\xi)\, \varphi^\varepsilon_{q,p}(x) \\ &= \varepsilon^{-d/2} \int_{\mathbb{R}^d} dx d\xi \, \exp(ix\cdot\xi/\varepsilon)\, \chi_1(x)\, \widehat{S}(\xi)\, \varphi^\varepsilon_{q,p}(x) + O(\varepsilon^\infty),\end{aligned}$$

for any truncation function $\chi_1$ being $\equiv 1$ close to the origin. On the other hand, we use formula (7.9) to compute $U_\varepsilon(t)\varphi^\varepsilon_{q,p}$ in (7.8), using the short-hand notation

$$P_N(t,q,p,x) := \pi^{-d/4} \det(A(t,q,p) + iB(t,q,p))_{\rm c}^{-1/2}\, Q_N(t,q,p,x).$$

These two tasks being done, we eventually obtain in (7.8), upon computing the relevant phase factors explicitly,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} \widetilde{\chi}_\varepsilon(t) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dt = O_{T_1,\delta}\left(\varepsilon^\infty\right) + \\ &\quad \frac{1}{\varepsilon^{(5d+2)/2}} \int_{T_0\,\varepsilon}^{T_1} \int_{\mathbb{R}^{6d}} dt dq dp d\xi d\eta dx dy\, \widetilde{\chi}_\varepsilon(t) \exp\left(\frac{i}{\varepsilon}\Psi(x,y,\xi,\eta,q,p,t)\right) \\ &\qquad \widehat{S}(\xi)\widehat{\phi}^*(\eta)\chi_0(q,p)\chi_1(x,y) P_N\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right). \qquad (7.16)\end{aligned}$$

where $\chi_1 \in C_c^\infty$ is $\equiv 1$ close to $(0,0)$. Here, the crucial (complex) phase factor has the value

$$\begin{aligned}\Psi(x,y,\xi,\eta,q,p,t) &= \int_0^t \left(\frac{p_s^2}{2} + n^2(q_s)\right) ds - p\cdot(x-q) + p_t\cdot(y-q_t) \\ &\quad + x\cdot\xi - y\cdot\eta + i\frac{(x-q)^2}{2} + \frac{\Gamma(t)(y-q_t)\cdot(y-q_t)}{2} \qquad (7.17)\end{aligned}$$

Our goal is now to apply the stationary phase formula to estimate (7.17). Obviously, the cutoff in time away from $t = 0$ in (7.16) prevents one to use directly the stationary phase formula close to $t = 0$. This is the reason why times close to 0 are treated apart in the sequel (see steps four and five below - see also the outline of proof given in section 2).

**Third Step:** *computing the first and second order derivatives of the phase* $\Psi$
First, it is an easy exercice, using the symplecticity relations (2.17), to prove that the matrix $\Gamma(t)$ is symmetric and it has positive imaginary part. The relation

$$\mathrm{Im}\,(\Gamma(t)(y-q_t)\cdot(y-q_t)) = \left|\,(A(t)+iB(t))^{-1}\;(y-q_t)\right|^2,$$

implies indeed

$$\mathrm{Im}\,\Psi = |x-q|^2 + \left|\,(A(t)+iB(t))^{-1}\;(y-q_t)\right|^2.$$

Hence we recover the equivalence

$$\mathrm{Im}\,\Psi = 0 \;\text{ iff }\; y = q_t \;\text{ and }\; x = q. \tag{7.18}$$

Second, using the differential system (2.13), (2.14) satisfied by the matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$, we prove

$$\nabla_{q,p}\left(\int_0^t \left(\frac{p_s^2}{2} + n^2(q_s)\right) ds\right) = \begin{pmatrix} A(t)^{\mathrm{T}} p_t - p \\ B(t)^{\mathrm{T}} p_t \end{pmatrix}.$$

This gives the value of the gradient of $\Psi$

$$\nabla_{x,y,\xi,\eta,q,p,t}\Psi(x,y,\xi,\eta,q,p,t) = \begin{pmatrix} -p+\xi+i(x-q) \\ p_t-\eta+\Gamma(t)(y-q_t) \\ x \\ -y \\ C(t)^{\mathrm{T}}(y-q_t)+i(q-x)+A(t)^{\mathrm{T}}\Gamma(t)(q_t-y) \\ -(x-q)+D(t)^{\mathrm{T}}(y-q_t)+B(t)^{\mathrm{T}}\Gamma(t)(q_t-y) \\ -\dfrac{p_t^2}{2}+n^2(q_t)+\nabla n^2(q_t)\cdot(y-q_t)+p_t\cdot\Gamma(t)(q_t-y) \end{pmatrix}. \tag{7.19}$$

This computation is done up to irrelevant $O\left((y-q_t)^2+(x-q)^2\right)$ terms.

These observations allow to compute the stationary set, defined as

$$M := \{(x,y,\xi,\eta,q,p,t) \in \mathbb{R}^{6d}\times]0,+\infty[ \\ \text{s.t. } \mathrm{Im}\,\Psi = 0 \;\text{ and }\; \nabla_{x,y,\xi,\eta,q,p}\Psi = 0\}. \tag{7.20}$$

Note (see above) that we exclude the original time $t = 0$ in the definition of $M$. In view of (7.18), (7.19), the set $M$ has the value

$$M = \{(x,y,\xi,q) \;\text{ s.t. }\; x = y = q = 0,\; \xi = p\} \\ \cap \left\{(p,\eta,t) \;\text{ s.t. }\; \frac{\eta^2}{2} = n^2(0),\; q_t = 0,\; p_t = \eta\right\}. \tag{7.21}$$

Note that the second set reads also, by definition,

$$\left\{(p,\eta,t) \;\text{ s.t. }\; \frac{\eta^2}{2} = n^2(0),\; X(t,0,p) = 0,\; \Xi(t,0,p) = \eta\right\}.$$

Last, there remains to compute the Hessian of $\Psi$ at the stationary points. A simple but tedious computation gives, for any point $(x, y, \xi, \eta, q, p, t) \in M$, the value

$$D^2_{x,y,\xi,\eta,q,p,t}\Psi\Big|_{(x,y,\xi,\eta,q,p,t)\in M} =$$

$$\begin{pmatrix} i\mathrm{Id} & 0 & \mathrm{Id} & 0 & -i\mathrm{Id} & -\mathrm{Id} & 0 \\ 0 & \Gamma_t & 0 & -\mathrm{Id} & C_t-\Gamma_t A_t & D_t-\Gamma_t B_t & \nabla n^2(0) -\Gamma_t\eta \\ \mathrm{Id} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id} & 0 & 0 & 0 & 0 & 0 \\ -i\mathrm{Id} & C_t^{\mathrm{T}}-A_t^{\mathrm{T}}\Gamma_t & 0 & 0 & -C_t^{\mathrm{T}}A_t+i\mathrm{Id} +A_t^{\mathrm{T}}\Gamma_t A_t & -C_t^{\mathrm{T}}B_t +A_t^{\mathrm{T}}\Gamma_t B_t & -C_t^{\mathrm{T}}\eta +A_t^{\mathrm{T}}\Gamma_t\eta \\ -\mathrm{Id} & D_t^{\mathrm{T}}-B_t^{\mathrm{T}}\Gamma_t & 0 & 0 & \mathrm{Id}-D_t^{\mathrm{T}}A_t +B_t^{\mathrm{T}}\Gamma_t A_t & -D_t^{\mathrm{T}}B_t +B_t^{\mathrm{T}}\Gamma_t B_t & -D_t^{\mathrm{T}}\eta +B_t^{\mathrm{T}}\Gamma_t\eta \\ 0 & \nabla n^2(0)^{\mathrm{T}} -\eta^{\mathrm{T}}\Gamma_t & 0 & 0 & -\eta^{\mathrm{T}}C_t +\eta^{\mathrm{T}}\Gamma_t A_t & -\eta^{\mathrm{T}}D_t +\eta^{\mathrm{T}}\Gamma_t B_t & -\eta^{\mathrm{T}}\nabla n^2(0) +\eta^{\mathrm{T}}\Gamma_t\eta \end{pmatrix}.$$

Here we wrote systematically $A_t$, $B_t$, etc. instead of $A(t)$, $B(t)$, etc. The above matrix is symmetric, due to the relation (2.18). The very last computation we need is that of Ker $D^2\Psi$ at stationary points. The value of $D^2\Psi|_M$ clearly shows that

$$\begin{aligned} \mathrm{Ker}\ \left(D^2\Psi|_M\right) \quad = \quad & \Big\{(X, Y, \Xi, H, Q, P, T) \ \text{ s.t. } \ X = Y = Q = 0, \ \Xi = P, \\ & -H + (D_t - \Gamma_t B_t)P + T(\nabla n^2(0) - \Gamma_t\eta) = 0, \\ & (-C_t^{\mathrm{T}} + A_t^{\mathrm{T}}\Gamma_t)B_t P + T(-C_t^{\mathrm{T}} + A_t^{\mathrm{T}}\Gamma_t)\eta = 0, \\ & (-D_t^{\mathrm{T}} + B_t^{\mathrm{T}}\Gamma_t)B_t P + T(-D_t^{\mathrm{T}} + B_t^{\mathrm{T}}\Gamma_t)\eta = 0, \\ & \eta^{\mathrm{T}}(-D_t + \Gamma_t B_t)P + T\eta^{\mathrm{T}}(-\nabla n^2(0) + \Gamma_t\eta) = 0\Big\}. \end{aligned}$$

Hence, using $D_t^{\mathrm{T}} - B_t^{\mathrm{T}}\Gamma_t = (A_t + iB_t)^{-1}$, together with $C_t^{\mathrm{T}} - A_t^{\mathrm{T}}\Gamma_t = -i(A_t + iB_t)^{-1}$, and $(A_t + iB_t)^{-1,\mathrm{T}} + \Gamma_t B_t = D_t$ (see (2.18)), we obtain

$$\begin{aligned} \mathrm{Ker}\ \left(D^2\Psi|_M\right) = \Big\{&(X, Y, \Xi, H, Q, P, T) \ \text{ s.t. } \ X = Y = Q = 0, \ \Xi = P, \\ &\text{and } \eta^{\mathrm{T}}H = 0, \ B_t P + T\eta = 0, H = D_t P + T\nabla n^2(0) = 0\Big\}. \end{aligned} \tag{7.22}$$

**Fourth Step:** *Application of the stationary phase Theorem - proof of part (i) of proposition 7*

In this step, we formulate the main geometric assumption on the flow $\Phi(t, x, \xi)$, that allows for the proof that the contribution in (7.16) vanishes asymptotically.

**(H) Transversality assumption on the flow**

*We suppose that the stationary set*

$$M = \{x = y = q = 0, \ \xi = p\} \cap \left\{\frac{\eta^2}{2} = n^2(0), \ X(t, 0, p) = 0, \ \Xi(t, 0, p) = \eta\right\}$$

*is a* smooth submanifold *of* $\mathbb{R}^{6d} \times ]0, +\infty[$, *satisfying the additional constraint*

$$k := \mathrm{codim} M > 5d + 2. \tag{7.23}$$

*We also assume that at each point* $m = (x, y, \xi, \eta, q, p, t) \in M$, *the tangent space of* $M$ *at* $m$ *is*

$$\mathrm{T}_m M = \Big\{(X, Y, \Xi, H, Q, P, T) \;\; \textit{s.t.} \;\; X = Y = Q = 0, \; \Xi = P, \\ \textit{and } \eta^{\mathrm{T}} H = 0, \; B_t P + T\eta = 0, -H + D_t P + T\nabla n^2(0) = 0\Big\}. \tag{7.24}$$

*In other words, we assume that* $\mathrm{T}_m M$ *is precisely given by linearizing the equations defining* $M$.

**First Remark**
We show below examples of flows satisfying the above assumption. It is a natural, and generic, assumption. Note in particular that the assumption on the codimension is natural, in that the equations defining $M$ give (roughly) $4d$ constraints on $(x, y, q, \xi)$, one constraint on $\eta$, and again $2d$ constraints on the momentum $p$, the solid angle $\eta/|\eta|$, and time $t$. Hence one has typically $k = 6d + 1$. □

**Second remark**
Equivalently, the above assumption may be formulated as follows. The set

$$\mathcal{M} := \{(p, \eta, t) \text{ s.t. } \frac{\eta^2}{2} = n^2(0), \; X(t, 0, p) = 0, \; \Xi(t, 0, p) = \eta\}$$

is assumed to be a smooth submanifold of $\mathbb{R}^{2d+1}$, satisfying the additional constraint $\mathrm{codim}\, \mathcal{M} > d + 2$, and whose tangent space is given by

$$\{(P, H, T) \text{ s.t. } \eta^{\mathrm{T}} H = 0, \; B_t P + T\eta = 0, \; D_t P + T\nabla n^2(0) - H = 0\}.$$

Note in passing that the conservation of energy allows to replace the requirement $\eta^2/2 = n^2(0)$ by the equivalent $p^2/2 = n^2(0)$ in the definition of $\mathcal{M}$. □

**Third Remark**
Provided $M$ is a smooth submanifold with tangent space given upon linearizing the constraints, its codimension anyhow satisfies

$$\mathrm{codim}\, M \geq 5d + 2.$$

Equivalently, provided $\mathcal{M}$ is a smooth submanifold with the natural tangent space, its codimension anyhow satisfies

$$\mathrm{codim}\, \mathcal{M} \geq d + 2.$$

As a consequence, the analysis given below (see (7.27)) establishes that $\langle w^\varepsilon, \phi \rangle$ is uniformly bounded in $\varepsilon$. This fact is not known in the literature. □

Under assumption **(H)**, we are ready to use the stationary phase Theorem in (7.16), at least for large enough times $t$ (recall that the very point $t = 0$ is excluded from the definition of $M$ above). Indeed, assumption **(H)** precisely asserts the equality

$$\mathrm{T}_m M = \mathrm{Ker}\, \left(D^2 \Psi|_M\right),$$

so that the Hessian $D^2\Psi|_M$ is non-degenerate on the normal space $(\mathrm{T}_m M)^\perp$. This is exactly the non-degeneracy that we need in order to apply the stationary phase Theorem.

To perform the claimed stationary phase argument, we first take a (small) parameter

$$\theta > 0.$$

We use a cutoff in time $\chi(t/\theta)$ with $\chi$ as in (2.7), and evaluate the contribution

$$\frac{1}{\varepsilon}\int_\theta^{T_1} \widetilde{\chi}_\varepsilon(t)\left(1-\chi\left(\frac{t}{\theta}\right)\right)\left\langle \chi_\delta\left(H_\varepsilon\right)S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon\right\rangle dt = O_{T_1,\delta}\left(\varepsilon^\infty\right) +$$

$$\frac{1}{\varepsilon^{(5d+2)/2}}\int_\theta^{T_1}\int_{\mathbb{R}^{6d}} \widetilde{\chi}_\varepsilon(t)\left(1-\chi\left(\frac{t}{\theta}\right)\right)\exp\left(\frac{i}{\varepsilon}\Psi(x,y,\xi,\eta,q,p,t)\right)$$
$$\widehat{S}(\xi)\widehat{\phi}^*(\eta)\chi_0(q,p)\chi_1(x,y)P_N\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right) dtdxdyd\xi d\eta dqdp.$$

When the point $(x,y,\xi,\eta,q,p,t)$ is far from the stationary set $M$, the integral is $O(\varepsilon^\infty)$. Close to the stationary set $M$, using the fact that the integral carries over a compact support, we may use a partition of unity close to $M$, and on each piece we may use straightened coordinates $(\alpha,\beta)\in\mathbb{R}^{6d+1-k}\times\mathbb{R}^k$ such that

$$(x,y,\xi,\eta,q,p,t)=\gamma(\alpha,\beta),\quad \text{where } \gamma \text{ is a local diffeomorphism, with}$$
$$(x,y,\xi,\eta,q,p,t)\in M \Longleftrightarrow \alpha=0.$$

Using such coordinates, we recover a finite sum of terms of the form

$$\frac{1}{\varepsilon^{(5d+2)/2}}\int_\Omega dxdyd\xi d\eta dqdp\, \exp\left(\frac{i}{\varepsilon}\Psi(x,y,\xi,\eta,q,p,t)\right)$$
$$\widehat{S}(\xi)\widehat{\phi}^*(\eta)P_N\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right)\chi_2(x,y,\xi,\eta,q,p,t)$$
$$=\frac{1}{\varepsilon^{(5d+2)/2}}\int_{\Omega'\times\Omega''} d\alpha\, d\beta\, \exp\left(\frac{i}{\varepsilon}\Psi\circ\gamma(\alpha,\beta)\right)$$
$$\left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma(\alpha,\beta)\;\chi_3(\alpha,\beta), \tag{7.25}$$

where $\Omega$, $\Omega'$, $\Omega''$ are bounded, open subsets, and $\chi_2$, $\chi_3$ are cutoff functions. Thanks to the non-degeneracy of the Hessian $D^2\Psi$ in the normal direction to $M$, for any $\beta$, we have

$$\left(\det\frac{D^2\Psi\circ\gamma}{D\alpha^2}\right)(0,\beta)\neq 0.$$

Hence, by the standard stationary phase Theorem, for any integer $J$, the above integral has the asymptotic expansion to order $J$

$$\varepsilon^{(k-5d-2)/2}\int_{\Omega''} d\beta\,\exp\left(\frac{i}{\varepsilon}\Psi\circ\gamma(0,\beta)\right)$$
$$\times\sum_{j=0}^{J}\varepsilon^j\, Q_{2j}(\partial_\alpha,\partial_\beta)\left(\left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma\;\chi_3\right)(0,\beta) \tag{7.26}$$
$$+\,\varepsilon^{(k-5d-2)/2}O\left(\varepsilon^{J+1}\sup_{k\le 2J+d+3}\left\|\partial^k_{(\alpha,\beta)}\left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\chi_3\right)\right\|\right),$$

where the $Q_{2j}$'s are differential operators of order $2j$. Now, we anyhow have

$$\forall j \in \mathbb{N} \qquad \varepsilon^j \partial_y^{2j} P_N \left( ., ., ., \frac{y}{\sqrt{\varepsilon}} \right) = O(1).$$

On the more, $P_N$ is a *polynomial* of degree $\leq 4N$ in its last argument. This implies that the $\varepsilon^{(k-5d-2)/2} O(\dots)$ in (7.26) has at most the size

$$O\left(\varepsilon^{J+1+(k-5d-2)/2-2N}\right).$$

Hence, taking $J$ large enough ($J \geq 2N$ will do), we eventually obtain in (7.26), using the assumption **(H)** on the codimension $k$ ($k > 5d+2$),

$$\frac{1}{\varepsilon^{(5d+2)/2}} \int_\theta^{T_1} \widetilde{\chi}_\varepsilon(t) \left(1 - \chi\left(\frac{t}{\theta}\right)\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dt$$
$$= O_{\theta, T_1, \delta}\left(\varepsilon^{(k-5d-2)/2}\right) \underset{\varepsilon \to 0}{\longrightarrow} 0. \qquad (7.27)$$

**Fifth Step:** *Elimination of times such that $T_0\,\varepsilon \leq t \leq \theta$ - proof of part (ii) of proposition 7*
The previous step leaves us with the task of estimating

$$\frac{1}{\varepsilon} \int_{T_0\,\varepsilon}^{2\theta} \widetilde{\chi}_\varepsilon(t)\, \chi\left(\frac{t}{\theta}\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dt.$$

The idea is to now come back to the semiclassical scale, and write

$$\frac{1}{\varepsilon} \int_{T_0\,\varepsilon}^{2\theta} \widetilde{\chi}_\varepsilon(t)\, \chi\left(\frac{t}{\theta}\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dt$$
$$= \int_{T_0}^{2\theta/\varepsilon} \chi\left(\frac{\varepsilon t}{\theta}\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \exp\left(-it\left(\varepsilon^2 \Delta + n^2(x)\right)\right) \phi_\varepsilon \right\rangle dt\,. \qquad (7.28)$$

This term is expected to be small, provided $T_0$ is large enough. Indeed, the propagator $\exp\left(-it\left(\varepsilon^2 \Delta + n^2(x)\right)\right)$ acting on $\phi_\varepsilon$ is expected to be close to the free propagator $\exp\left(-it\left(\varepsilon^2 \Delta + n^2(0)\right)\right)$ on the time-scale we consider. Hence the propagator should have size $O(t^{-d/2})$ for large values of time, and the above time integral should be $O({T_0}^{-d/2+1}) \to 0$ as $T_0 \to \infty$.

We give below a quantitative proof of this rough statement, based on the exact computation of the propagator $\exp\left(-it\left(\varepsilon^2 \Delta + n^2(x)\right)\right)$ obtained in Theorem 8. The proof given below could easily be replaced by a slightly simpler one, upon writing the propagator as a Fourier Integral Operator with *real* phase. We do not detail this aspect, since we anyhow had to use in the previous steps the more precise expansion of the propagator given by Theorem 8: this theorem has indeed the great advantage to give a representation of the propagator that is valid *for all times.*

From the second step above (see (7.16)), we know

$$\int_{T_0}^{2\theta/\varepsilon} \chi\left(\frac{\varepsilon t}{\theta}\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \exp\left(-it\left(\varepsilon^2 \Delta + n^2(x)\right)\right) \phi_\varepsilon \right\rangle dt$$
$$= O_{T_1, \delta}\left(\varepsilon^\infty\right) + \int_{T_0}^{2\theta/\varepsilon} \chi\left(\frac{t}{\theta}\right) \times \varepsilon^{-\frac{5d}{2}} \int_{\mathbb{R}^{6d}} \exp\left(i\Psi(\varepsilon t)/\varepsilon\right)$$
$$\widehat{S}(\xi) \widehat{\phi}^*(\eta) \chi_0(q,p) \chi_1(x,y) P_N\left(t, q, p, \frac{y - q_{\varepsilon t}}{\sqrt{\varepsilon}}\right) dx dy d\xi d\eta dq dp, \qquad (7.29)$$

where we drop the dependence of the phase $\Psi$ in $(x, y, \xi, \eta, q, p)$. To estimate this term, we now concentrate our attention on the space integral

$$f_\varepsilon(t) := \varepsilon^{-\frac{5d}{2}} \int_{\mathbb{R}^{6d}} \exp\left(i \frac{\Psi(\varepsilon t)}{\varepsilon}\right) \widehat{S}(\xi) \widehat{\phi}^*(\eta)$$
$$\chi_0(q,p)\chi_1(x,y) P_N\left(t, q, p, \frac{y - q_{\varepsilon t}}{\sqrt{\varepsilon}}\right) dx dy d\xi d\eta dq dp. \tag{7.30}$$

We claim we have the following dispersion estimate, *uniformly in* $\varepsilon$,

$$\left|f_\varepsilon(t)\right| \leq C_\theta \, t^{-d/2}, \quad \text{for some } C_\theta > 0, \text{ provided } \; T_0 \leq t \leq 2\theta/\varepsilon. \tag{7.31}$$

Assuming (7.31) is proved, equation (7.29) shows that

$$\frac{1}{\varepsilon}\left| \int_{T_0\,\varepsilon}^{2\theta} \widetilde{\chi}_\varepsilon(t)\, \chi\left(\frac{t}{\theta}\right) \left\langle \chi_\delta\left(H_\varepsilon\right) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dt \right| \leq C_\theta \, T_0^{\;-\frac{d}{2}+1} \underset{T_0 \to \infty}{\longrightarrow} 0\,, \tag{7.32}$$

in any dimension $d \geq 3$, which is enough for our purposes. It is thus sufficient to prove (7.31).

We have in mind that the integral (7.30) defining $f_\varepsilon(t)$ should concentrate on the set $x = y = q = 0$, $q_t = 0$, $p_t = \eta$, $p = \xi$. Also, the present case should be close to the "free" case where the refraction index $n^2(x)$ has frozen coefficients at the origin $n^2(x) \approx n^2(0)$. For that reason, we perform in (7.30) the changes of variables

$$(x - q)/\sqrt{\varepsilon} \to x, \; (y - q_{\varepsilon t})/\sqrt{\varepsilon} \to y, \; q \to \sqrt{\varepsilon} q,$$
$$\xi \to p + \sqrt{\varepsilon}\xi, \; \eta \to \Xi(\varepsilon t, \sqrt{\varepsilon} q, p) + \sqrt{\varepsilon}\eta.$$

We also put apart the important phase factors in the obtained formula. This gives

$$f_\varepsilon(t) = \int_{\mathbb{R}^{4d}} dq dp d\eta \, \exp\left(it \, \widetilde{\Psi}(p, \varepsilon t, \sqrt{\varepsilon} q, \sqrt{\varepsilon}\eta)\right) G(q, p, \eta, \varepsilon t, \sqrt{\varepsilon} q, \sqrt{\varepsilon}\eta), \tag{7.33}$$

up to introducing the phase

$$\widetilde{\Psi}(p, \varepsilon t, \sqrt{\varepsilon} q, \sqrt{\varepsilon}\eta) := \frac{1}{\varepsilon t} \int_0^{\varepsilon t} \left( \frac{\Xi(s, \sqrt{\varepsilon} q, p)^2}{2} + n^2\left(X(s, \sqrt{\varepsilon} q, p)\right)\right) ds$$
$$+ \frac{\sqrt{\varepsilon} p \cdot q - \Xi(\varepsilon t, \sqrt{\varepsilon} q, p) \cdot X(\varepsilon t, \sqrt{\varepsilon} q, p)}{\varepsilon t}$$
$$+ \sqrt{\varepsilon}\eta \cdot \frac{\sqrt{\varepsilon} q - X(\varepsilon t, \sqrt{\varepsilon} q, p)}{\varepsilon t}\,,$$

together with the amplitude ($C^\infty$, and compactly supported in $p$, $\sqrt{\varepsilon} q$)

$$G(q, p, \eta, \varepsilon t, \sqrt{\varepsilon} q, \sqrt{\varepsilon}\eta) := \int_{\mathbb{R}^{3d}} dx dy d\xi \, \exp\left(i\xi \cdot (q + x) - i\eta \cdot (y + q)\right)$$
$$\exp\left(-\frac{x^2}{2} + i\frac{\Gamma(\varepsilon t, \sqrt{\varepsilon} q, p) y \cdot y}{2}\right)$$
$$\widehat{S}(p + \sqrt{\varepsilon}\xi) \widehat{\phi}^*(\Xi(\varepsilon t, \sqrt{\varepsilon} q, p) + \sqrt{\varepsilon}\eta) \chi_0\left(\sqrt{\varepsilon} q, p\right)$$
$$\chi_1(\sqrt{\varepsilon}(q + x), X(\varepsilon t, \sqrt{\varepsilon} q, p) + \sqrt{\varepsilon} y) \; P_N\left(t, \sqrt{\varepsilon} q, p, y\right). \tag{7.34}$$

Now, the idea is to use the stationary phase formula in the $p$ variable in (7.33), where $t$ plays the role of the large parameter. We wish indeed to recognize in (7.33) a formula of the form

$$\int dp \, \exp\left(-it\frac{p^2}{2}\right) \times \text{smooth}(p),$$

to recover the claimed decaying factor $t^{-d/2}$ in (7.31). In other words, we wish to get the same dispersive properties as for the free Schrödinger equation. This is very much reminiscent of the dispersive effects proved for *small times* in [Dsf] for wave equations with variable coefficients, and relies on the fact that $\widetilde{\Psi} \approx -p^2/2$ as $\varepsilon t \leq \theta$ is small enough.

In order to do so, we need to get further informations both on the phase $\widetilde{\Psi}$ and the amplitude $G$.

Firstly, the smooth amplitude $G$ is defined in (7.34). It clearly is compactly supported in $p$ and $\sqrt{\varepsilon}q$. Also, the gaussian $\exp(-x^2/2 + i\Gamma(\varepsilon t, \sqrt{\varepsilon}q, p)y \cdot y/2)$ belongs to the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ in the variables $x$ and $y$ (recall indeed that $\operatorname{Im}\Gamma(\varepsilon t) > 0$, and $\varepsilon t$ belongs to a compact set), uniformly in the compactly supported parameters $\varepsilon t$, $\sqrt{\varepsilon}q$, and $p$. From this it follows that the amplitude $G(q, p, \eta, \varepsilon t, \sqrt{\varepsilon}q, \sqrt{\varepsilon}\eta)$ belongs to the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ in the first and third variables $q$ and $\eta$, it is $C_c^\infty(\mathbb{R}^d)$ in the second variable $p$, and these informations are uniform with respect to the compactly supported parameters $\varepsilon t$, $\sqrt{\varepsilon}q$, together with the (non-compact) parameter $\sqrt{\varepsilon}\eta$.

Secondly, the smooth phase $\widetilde{\Psi}$ depends upon the small parameter $\varepsilon t \in [0, 2\theta]$, together with the two position/velocity variables $\sqrt{\varepsilon}q$ and $p$. All of them belong to a compact set. It also depends upon the variable $\sqrt{\varepsilon}\eta$, which is not in a compact set. On the more, we have the easy first order expansion in the (small) parameter $\varepsilon t \leq 2\theta$,

$$\widetilde{\Psi}(p, \varepsilon t, \sqrt{\varepsilon}q, \sqrt{\varepsilon}\eta) = \\ -\frac{p^2}{2} + n^2(\sqrt{\varepsilon}q) - \sqrt{\varepsilon}q \cdot \nabla_x n^2\left(\sqrt{\varepsilon}q\right) - \sqrt{\varepsilon}\eta \cdot (p + O(\theta)) + O\left(\theta^2\right).$$

Here the remainder terms $O(\theta)$, $O(\theta^2)$, only depend upon the compactly supported parameters $\varepsilon t \leq 2\theta$ and $p$, $\sqrt{\varepsilon}q$ (they do not depend upon $\sqrt{\varepsilon}\eta$), and they are uniform with respect to these variables. Hence, the stationary points of the phase (in the $p$ variable) are given by

$$-p - \sqrt{\varepsilon}\eta(1 + O(\theta)) + O\left(\theta^2\right) = 0. \tag{7.35}$$

Finally, there remains to observe that the Hessian of the phase in $p$ is

$$\frac{D^2\widetilde{\Psi}}{Dp^2} = -\mathrm{Id} + O(\theta). \tag{7.36}$$

Upon taking $\theta$ small enough, all these informations allow us to make use of the standard stationary phase estimate in $p$. More precisely, we write,

$$f_\varepsilon(t) = \int_{\mathbb{R}^{2d}} \frac{dqd\eta}{\langle q\rangle^{2d}\,\langle \eta\rangle^{2d}} \int_{\mathbb{R}^d} dp \, \exp\left(i\, t\, \widetilde{\Psi}(p, \varepsilon t, \sqrt{\varepsilon}q, \sqrt{\varepsilon}\eta)\right) \\ \langle q\rangle^{2d}\,\langle \eta\rangle^{2d}\, G(q, p, \eta, \varepsilon t, \sqrt{\varepsilon}q, \sqrt{\varepsilon}\eta). \tag{7.37}$$

For each given values of $q$ and $\eta$, we analyze the integral over $p$ in (7.37). If $\sqrt{\varepsilon}\eta$ is outside some compact set around the support of $G$ in $p$, integrations by parts in $p$ together with the information (7.35), allow to prove that the integral over $p$ in (7.37) is bounded, for any integer $N$, by $C_{N,\theta}t^{-N}$ for some $C_{N,\theta} > 0$ independent of $q$ and $\eta$. Hence the corresponding contribution to $f_\varepsilon$ is bounded by $C_{N,\theta}t^{-N}$ as well. Now, for $\sqrt{\varepsilon}\eta$ in some compact set around the support of $G$ in $p$, we may use the information (7.36): this, together with the stationary phase Theorem with the parameters $\varepsilon t$, $\sqrt{\varepsilon}q$, $\sqrt{\varepsilon}\eta$ in a compact set, establishes that the integral over $p$ in (7.37) is bounded by $C_\theta t^{-d/2}$ for some $C_\theta > 0$, and $C_\theta$ turns out to be independent of $q$ and $\eta$. Hence the corresponding contribuition to $f_\varepsilon$ in (7.37) is bounded by $Ct^{-d/2}$ as well.

All this gives the claimed estimate

$$|f_\varepsilon(t)| \leq C_\theta t^{-d/2}.$$

The proof of proposition 7 is complete.

# 8 Conclusion: Proof of the main Theorem

We want to prove the convergence

$$\langle w^\varepsilon, \phi\rangle \longrightarrow \langle w^{\text{out}}, \phi\rangle,$$

when the source $S$ and the test function $\phi$ are Schwartz class. Therefore, one needs to prove

$$\frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\,\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\, dt \to \langle w^{\text{out}}, \phi\rangle \quad \text{as } \varepsilon \to 0.$$

Proposition 1 asserts

$$\frac{i}{\varepsilon}\int_0^{2T_0\,\varepsilon} \chi\left(\frac{t}{T_0\,\varepsilon}\right)\, \mathrm{e}^{-\alpha_\varepsilon t}\,\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\, dt = \langle w^{\text{out}}, \phi\rangle + O_{T_0}\left(\varepsilon^0\right) + O\left(\frac{1}{T_0^{d/2-1}}\right),$$

where the notation $O(\varepsilon^0)$ denotes a term going to zero with $\varepsilon$, and $O_{T_0}\left(\varepsilon^0\right)$ emphasizes the fact that the convergence depends a priori on the value of $T_0$. On the other hand Proposition 2 asserts

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{+\infty} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\, \mathrm{e}^{-\alpha_\varepsilon t}\,\Big\langle U_\varepsilon(t)\,(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt = O\left(\frac{1}{T_0}\right) + O(\varepsilon^0).$$

Now, for very large times and almost zero energies, Proposition 3 shows, for $\delta$ small enough, and any $\kappa$,

$$\frac{1}{\varepsilon}\int_{\varepsilon^{-\kappa}}^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\,\Big\langle U_\varepsilon(t)\chi_\delta\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\, dt = O_{\kappa,\delta}(\varepsilon).$$

As for large times and almost zero energies, Proposition 4 shows that, for $\delta$ small enough, $\kappa$ small enough, and $T_1$ large enough,

$$\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} \mathrm{e}^{-\alpha_\varepsilon t}\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon,\phi_\varepsilon\right\rangle dt = O_{\kappa,\delta}(\varepsilon)$$

Finally, for moderate times and almost zero energies, one has the following two informations. First, for $\theta$ small enough, and uniformly in $\varepsilon$, we have

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{2\theta}(1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\chi\left(\frac{t}{\theta}\right)\mathrm{e}^{-\alpha_\varepsilon t}\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon,\phi_\varepsilon\right\rangle dt$$
$$= O_\theta\left(\frac{1}{T_0{}^{d/2-1}}\right).$$

Second, for any fixed value of $\theta>0$, and $T_1$,

$$\frac{1}{\varepsilon}\int_{\theta}^{T_1}(1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\mathrm{e}^{-\alpha_\varepsilon t}\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon,\phi_\varepsilon\right\rangle dt$$
$$= O_{\theta,T_1,\delta}\left(\varepsilon^0\right).$$

All these informations show our main Theorem, upon conveniently choosing the cutoff parameters $\theta$, $T_0$, $T_1$ (in time), $\delta$ (in energy), and the exponent $\kappa$ (in time). This ends our proof.

# 9 Examples and counterexamples

## 9.1 The harmonic oscillator

Given an appropriate potential $V(x)$, and defining the semi-classical Schrödinger operator

$$H_\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x + V(x),$$

our main Theorem proves

$$\frac{1}{\varepsilon}\int_0^{+\infty}\mathrm{e}^{-\alpha_\varepsilon t}\left\langle \exp\left(-i\frac{t}{\varepsilon}H_\varepsilon\right)S_\varepsilon,\phi_\varepsilon\right\rangle dt \underset{\varepsilon\to 0}{\longrightarrow}$$
$$\int_0^{+\infty}\left\langle \exp\left(-it\left[-\Delta_x/2+V(0)\right]\right)S,\phi\right\rangle dt. \tag{9.1}$$

Though we used in many places that our analysis requires a potential of the form

$$V(x) = -n^2(x) = -n_\infty^2 + O(\langle x\rangle^{-\rho}),$$

it seems interesting to investigate the validity of (9.1) when the potential is harmonic

$$V(x) = V(0) + \sum_{j=1}^{d}\frac{\omega_j^2}{2}x_j^2, \tag{9.2}$$

for some frequencies $\omega_j\in\mathbb{R}$, and a given value $V(0)<0$. Such a potential does not enter our analysis since it is confining. However, it is easily proved that for *pairwise rationally independent* values of the frequencies $\omega_j$, the transversality assumption **(H)** page 35 is true for this potential, whereas in the extreme case where all $\omega_j$'s are equal, this assumption fails. On the other hand, one may use the Mehler formula [Ho] (see [C] for a use of these formulae in the nonlinear context) to compute the propagator

$$\exp\left(-i\frac{t}{\varepsilon}\left[-\varepsilon^2\Delta_x/2+\sum_{j=1}^d \omega_j^2 x_j^2/2\right]\right) = \tag{9.3}$$

$$\prod_{j=1}^d \left(\frac{\omega_j}{2i\pi\varepsilon\sin(\omega_j t)}\right)^{1/2} \exp\left(\frac{i\omega_j}{2\varepsilon\sin(\omega_j t)}\left[(x_j^2+y_j^2)\cos(\omega_j t)-2x_j y_j\right]\right)$$

(Here we identified the propagator and its integral kernel).

Surprinsingly enough, using the Mehler formula to compute the limit on the left-hand-side of (9.1), we may prove that for *rationally independent* $\omega_j$'s, the convergence result (9.1) is *locally true* in this case, for dimensions $d \geq 4$, i.e (9.1) is true with the upper bounded $+\infty$ replaced by $T$, for any value of $T > 0$.

We do not give the easy computations leading to this result. The idea is the following: at each time $k\pi/\omega_j$ ($k \in \mathbb{Z}$), the trajectory of the harmonic oscillator shows periodicity in the direction $j$. However, due to rational independence, at times $k\pi/\omega_j$, the trajectory does not show periodicity in any of the $d-1$ other directions. Hence one gets enough local dispersion from these directions to show that the corresponding contribution to the time integral on the left-hand-side of (9.1) is roughly

$$O\left(\int_{(-1+k\pi/\omega_j)/\varepsilon}^{(1+k\pi/\omega_j)/\varepsilon} t^{-(d-1)/2}\,dt\right) = O\left(\varepsilon^{(d-1)/2-1}\right) \to 0,$$

as long as $d-1>2$, i.e. $d \geq 4$.

Needless to say, in the extreme case where all $\omega_j$'s are equal, the result in (9.1) is *false*, even locally: in this case, periodicity creates a disastrous accumulation of energy at the origin (*all* rays periodically hit the origin at times $k\pi/\omega$, $k \in \mathbb{Z}$).

To our mind, this simple example indicates that our main Theorem probably holds true for less stringent assumptions on the refraction index. For instance, a uniform (in time) version of our transversality assumption is probably enough to get the result (without assuming neither decay at infinity of the refraction index, nor assuming the non-trapping condition).

## 9.2 Examples of flows satisfying the transversality condition

We already observed that the harmonic oscillator with rationally independent frequencies does satisfy the transversality assumption **(H)**. One actually has the value $k = 6d+1$ (see (7.23)) of the codimension in that case.

It is also easily verified that the flow of a particle in a constant electric field, i.e. the case of a potential

$$V(x) = x_1,$$

does satisfy **(H)** as well, with $k = 6d+1$.

Coupling the two flows, it is also verified that the potential

$$V(x) = x_1 + \sum_{j=1}^{d} \omega_j^2 x_j^2/2,$$

does satisfy **(H)** as well, with $k = 6d + 1$.

Clearly, these examples are satisfactory, in that we may assume that the potential has the above mentioned values *close* to the origin, and we may truncate outside some neighbourhood of the origin so as to build up a potential that satisfies the global assumptions we met in our main Theorem.

**Acknowledgements:**
The author wishes to thank B. Perthame and F. Nier for numerous discussions on the subject. Also, he wishes to thank D. Robert for his help concerning the wave-packet approach, and for giving him the written notes [Ro2], as well as X.P. Wang for discussions and for pointing out reference [Wa]. Finally, he wishes to thank R. Carles for a careful reading of a first version of this manuscript.

This work has been partially supported by the "ACI Jeunes Chercheurs - Méthodes haute fréquence pour les Equations différentielles ordinaires, et aux dérivées partielles. Applications", by the GDR "Amplitude Equations and Qualitative Properties" (GDR CNRS 2103 : EAPQ) and the European Program 'Improving the Human Potential' in the framework of the 'HYKE' network HPRN-CT-2002-00282.

# References

[Ag] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa, Vol. 2, N. 4, pp. 151-218 (1975).

[AH] S. Agmon, L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Anal. Math. Vol. 30, pp. 1-37 (1976).

[BCKP] J.D. Benamou, F. Castella, Th. Katsaounis, B. Perthame, *High frequency limit in the Helmholtz equation*, Rev. Mat. Iberoamericana, Vol. 18, N. 1, pp. 187-209 (2002), and, Séminaire E.D.P., École Polytechnique, exposé N. V, 27 pp., 1999-2000.

[BR] A. Bouzouina, D. Robert, *Uniform semiclassical estimates for the propagation of quantum observables*, Duke Math. J., Vol. 111, no. 2, pp. 223-252 (2002).

[Bu] N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not., no. 5, pp. 221-241 (2002).

[Bt] J. Butler, *Global H Fourier Integral operators with complex valued phase functions*, preprint Bologna (2001).

[C] R. Carles, *Remarks on nonlinear Schrdinger equations with harmonic potential*, Ann. Henri Poincaré, Vol. 3, N. 4, pp 757-772 (2002).

[CPR] F. Castella, B. Perthame, O. Runborg, *High frequency limit of the Helmholtz equation II: source on a general smooth manifold*, Comm. P.D.E., Vol. 27, N. 3-4, pp. 607-651 (2002).

[CRu] F. Castella, O. Runborg, In preparation.

[CRo] M. Combescure, D. Robert, *Semiclassical spreading of quantum wave packets and applications near unstable fixed points of the classical flow*, Asymptot. Anal., Vol. 14, no. 4, pp. 377-404 (1997).

[CRR] M. Combescure, J. Ralston, D. Robert, *A proof of the Gutzwiller semiclassical trace formula using coherent states decomposition*, Comm. Math. Phys., Vol. 202, no. 2, pp. 463-480 (1999).

[DG] J. Dereziński, C. Gérard, **Scattering theory of classical and quantum $N$-particle systems**, Texts and Monographs in Physics, Springer-Verlag, Berlin (1997).

[DS] M. Dimassi, J. Sjöstrand, **Spectral Asymptotics in the Semiclassical Limit**, London Math. Soc. Lecture Notes series 268, Cambridge Universuty Press (1999).

[Dsf] D. Dos Santos Ferreira, *Inégalités de Carleman $L^p$ pour des indices critiques*, Thesis, Université de Rennes 1 (2002).

[Fo] G.B. Folland, **Harmonic analysis in phase space**, Annals of Mathematics Studies, Vol. 122. Princeton University Press, Princeton, NJ (1989).

[Fou] E. Fouassier, Thesis, University Rennes 1, in preparation.

[GM] C. Gérard, A. Martinez, *Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris Sér. I Math., Vol. 306, no. 3, pp. 121-123 (1988).

[HJ] G.A. Hagedorn, A. Joye, *Semiclassical dynamics with exponentially small error estimates*, Comm. Math. Phys., Vol. 207, pp. 439-465 (1999).

[H1] G.A. Hagedorn, *Semiclassical quantum mechanics III*, Ann. Phys., Vol. 135, pp. 58-70 (1981).

[H2] G.A. Hagedorn, *Semiclassical quantum mechanics IV*, Ann. IHP, Vol. 42, pp. 363-374 (1985).

[HR] B. Helffer, D. Robert, *Caclcul fonctionnel par la transformation de Mellin et opérateurs admissibles*, J. Funct. Anal., Vol. 53, N. 3, pp. 246-268 (1983).

[He] K. Hepp, *The classical limit of quantum mechanical correlation functions*, Comm. Math. Phys., Vol. 35, pp. 265-277 (1974).

[Ho] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math. Z. Vol. 219, N. 3, pp. 413-449 (1995).

[J] T. Jecko, *From classical to semi-classical non-trapping behaviour: a new proof*, Preprint University of Rennes 1 (2002).

[Ma] A. Martinez, **An introduction to Semicassical and Microlocal Analysis**, Universitext, Springer-Verlag, New York (2002).

[PV1] B. Perthame, L. Vega, *Morrey-Campanato estimates for Helmholtz Equation*, J. Funct. Anal., Vol. 164, N. 2, pp. 340-355 (1999).

[PV2] B. Perthame, L. Vega, *Sommerfeld radiation condition for Helmholtz equation with variable index at infinity*, Preprint (2002).

[Ro] D. Robert, *Remarks on asymptotic solutions for time dependent Schrödinger equations*, to appear.

[Ro2] D. Robert, notes on lectures given at the University of Nantes (1999-2000).

[Rb] S. Robinson, *Semiclassical mechanics for time-dependent Wigner functions*, J. Math. Phys., Vol. 34, pp. 2150-2205 (1993).

[Wa] X.P. Wang, *Time decay of scattering solutions and resolvent estimates for Semiclassical Schrödinger operators*, J. Diff. Eq., Vol. 71, pp. 348-395 (1988).

[WZ] X.P. Wang, P. Zhang, *High frequency limit of the Helmholtz equation with variable refraction index*, Preprint (2004).