Original Article

Continuous barrier range options

Received (in revised form): 14th January 2009

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Q1

ABSTRACT **This article presents a set of exotic barrier options whose appearance/disappearance mechanism lies over a range of values. This family generalizes soft barrier options introduced by Hart and Ross and represents a simple alternative to steps options suggested by Linetsky for reducing knock-out risk. The traditional soft mechanism involves a uniformly distributed process. By contrast, Barrier Range Options use other density functions to account for more complex mechanisms.**

Q2

Journal of Derivatives & Hedge Funds (2009) **0,** 1–8. doi:10.1057/jdhf.2009.13

Keywords: exotic barrier option; soft barrier options; knock-out risk

INTRODUCTION

Straight barrier options are path-dependent options that depend on whether the underlying asset's price reaches a certain threshold. They are widely used nowadays for hedging purposes.[1–9] The discrete nature of the barrier may appear critical for users and writers. Hart and Ross[10] explained, for instance, that Knock Out Straight Barrier Options *carry a significant risk of hedge loss* from a trivial incident. Linetsky[11,12] suspected short-term manipulation (and increased volatility) around popular barrier levels. To circumvent this drawback, Hart and Ross[10] introduced *soft* barrier options (SBOs hereafter), defined as averages of barrier options. For his part, Linetsky[1–9] suggested a set of step options that involve a 'knock rate' and that are closely related to the occupation time derivatives studied by Chesney *et al*[13] and Hugonnier.[14]

A Barrier Range Option (BRO) is an exotic barrier option whose disappearance/appearance mechanism lies over a range of values instead of a single and discrete threshold. The sudden appearance (*resp.* disappearance) of standard barrier options is replaced by a gradual birth (*resp.* death) of the contract. As such, this set of options includes, among others, the Down and In and Down and Out SBOs introduced and priced by Hart and Ross.[10] Originally, SBOs were defined as simple averages of straight barrier options. Quite differently, I interpret this mechanism as a birth/death process that is distributed uniformly over the barrier range. Other distributions can therefore be used to account for more sophisticated devices, with three main consequences. First, BROs make it easy to model and price complex features of real-life financial contracts. Second, the set of BROs is uncountable. Third and finally, BROs are simple to compute.

The rest of this article is organized as follows. The next section reviews the valuation of SBOs and completes the set of pricing formulae provided by Hart and Ross.[10] The section after that presents BROs by noting that SBOs form just a specific set of BROs. Specifications of non-uniform birth/death processes are then discussed in the subequent section. The penultimate section applies the methodology to model risk.

SOFT BARRIER OPTIONS

SBOs are certainly the simplest BROs people use. As a result, one first reviews the valuation of such options with the aim to complete the analytical pricing formulae introduced by Hart and Ross[10] for Down and In and Down and Out Soft Options.

Our analysis adopts the conventional continuous time framework of Black and Scholes.[15] Financial markets are perfect and complete, and trading takes place continuously. The term structure of interest rates is flat, and the continuously compounded risk-free interest rate is denoted r. The risk-neutral price process of the underlying asset S is accurately described by the following stochastic differential equation:

$$d\ln S = (r - \frac{1}{2}\sigma_S^2)dt + \sigma_S dZ^S \qquad (1)$$

where Z^S is a standard Brownian motion and σ_S the constant volatility. In the following, E stands for the strike price of the option. It is known that prices of straight barrier options are linear combinations of the following four terms:

$$A(\phi) = \alpha(E;\phi) \; B(\phi) = \alpha(K;\phi)$$
$$C(\eta) = \beta(E;\eta) \; D(\eta) = \beta(K;\eta)$$

where

$$\phi\alpha(F;\phi) = SN[\phi d(S,F,\lambda\sigma^2)] - Ee^{-rt}N[\phi d(S,F,\mu)]$$

$$\eta\beta(F;\eta) = S\left(\frac{K}{S}\right)^{2\lambda} N[\eta d(K^2, FS, \lambda\sigma^2)] - Ee^{-rt}\left(\frac{K}{S}\right)^{2\lambda-2} N[\eta d(K^2, FS, \mu)]$$

with $d(a,b,c) = (\ln(a/b) + ct)/(\sigma\sqrt{t})$, $\lambda\sigma^2 = r + (1/2)\sigma^2$, $\mu = r - (1/2)\sigma^2$ and $\phi, \eta \in -1, 1$. ϕ is 1 (*resp.* -1) when dealing with a call (*resp.* put); η is 1 (*resp.* -1) if the barrier is down (*resp.* up). For readers' convenience, these linear combinations are reported in Table 1.

An SBO is similar to a standard barrier option, except that the barrier is no longer a constant threshold K. Rather, it involves a range of values delimited by an upper level U and a lower level L. Hart and Ross[10] explain that as the price process goes through $[L, U]$, the value of an SBO declines or grows *proportionaly* (depending on whether the option is knock out or knock in). They argue then that SBOs are averages of straight barrier options over the Barrier Range $[L, U]$ and they can derive the following pricing formula:

$$SBO = \frac{1}{U-L}\int_L^U BO(H)dH \qquad (2)$$

where $BO(H)$ is the standard Barrier Option with barrier H. And this is clearly a refinement of the sudden death/birth mechanism of standard barrier options.

Table 1: Vanilla barrier options

		Kick IN	*Knock IN*
Call	Up	B(+1)−C(−1)+D(−1)	A(+1)
Call	Down	A(+1)−B(+1)+D(+1)	C(+1)
Put	Up	A(−1)−B(−1)−D(−1)	−C(−1)
Put	Down	B(−1)+C(+1)−D(+1)	A(−1)

Given the linear property of the integral, Table 1 still applies, meaning that it is necessary and sufficient to integrate the four previous terms. For ease of exposition, let us introduce the following difference operator $[f(H)]_l^u = f(u) - f(l)$. After tedious calculations (involving integration by parts), one finds:

$$\phi \int_U^L B(\phi)(H)dH = S\Upsilon_1 - Ee^{-rt}\Upsilon_2$$

$$\int_U^L C(\eta)(H)dH = \frac{S^{1-2\lambda}}{2\lambda+1}\Gamma_1 - Ee^{-rt}\frac{S^{2-2\lambda}}{2\lambda-1}\Gamma_2$$

$$\int_U^L D(\eta)(H)dH = \frac{S^{1-2\lambda}}{2\lambda+1}\Delta_1 - Ee^{-rt}\frac{S^{2-2\lambda}}{2\lambda-1}\Delta_2$$

where :

$$\Upsilon_1 = [HN(\phi d(S,H,\lambda\sigma^2))]_L^U - Se^{-(\lambda+\frac{1}{2})\sigma^2 t}[N(\phi d(S,H,(\lambda+1)\sigma^2))]_L^U$$

$$\Upsilon_2 = [HN(\phi d(S,H,(\lambda-1)\sigma^2))]_L^U - Se^{-(\lambda-\frac{1}{2})\sigma^2 t}[N(\phi d(S,H,\lambda\sigma^2))]_L^U$$

$$\Gamma_1 = \Psi_E - \eta(ES)^{\lambda+\frac{1}{2}}e^{-\frac{1}{2}(\lambda+\frac{1}{2})(\lambda-\frac{1}{2})\sigma^2 t}[N(\eta d(H^2,ES,-\sigma^2/2))]_L^U$$

$$\Gamma_2 = \Xi_E - \eta(ES)^{\lambda-\frac{1}{2}}e^{-\frac{1}{2}(\lambda-\frac{1}{2})(\lambda-\frac{3}{2})\sigma^2 t}[N(\eta d(H^2,ES,-\sigma^2/2))]_L^U$$

$$\Delta_1 = \Psi_H - \eta S^{(2\lambda+1)}e^{\frac{1}{2}(2\lambda+1)\sigma^2 t}[N(\eta d(H,S,-(\lambda+1)\sigma^2))]_L^U$$

$$\Delta_2 = \Xi_H - \eta S^{(2\lambda-1)}e^{\frac{1}{2}(2\lambda-1)\sigma^2 t}[N(\eta d(H,S,-\lambda\sigma^2))]_L^U$$

with:

$$\Psi_F = [H^{2\lambda+1}N(\eta d(H^2,FS,\lambda\sigma^2))]_L^U$$

$$\Xi_F = [H^{2\lambda-1}N(\eta d(H^2,FS,(\lambda-1)\sigma^2))]_L^U$$

The first term A is not a function of the barrier, and thus it is invariant by integration. This is important for the existence of the following parity relation: $iSBO_{IN} + iSBO_{OUT} = A(\phi)$. In other words, a portfolio made of two SBOs, identical except in their appearance/disappearance feature, duplicates the plain vanilla option. All types of SBO contracts can therefore be priced with very few moduli.

BARRIER RANGE OPTIONS

BROs are characterized by the way their value *gradually* declines or grows as the underlying price goes through the interval $[L, U]$. Following Hart and Ross,[10] the SBO was defined in the previous section as an average of barrier options over a range of thresholds. Alternatively, one may view the seminal *soft* mechanism as a uniformly distributed birth/death process over the range $[L, U]$. To formalize this ‘probabilistic’ interpretation, one can rewrite the equation (2):

$$iSBO = \frac{1}{U-L}\int_{U}^{L} iBO(H)dH$$
$$= \int_{R} iBO(H)u(H)dH$$

where the function u (such that $u(H) = (1/U{-}L)1_{[L,\,U]}(H)$) is the probability density function of the uniform distribution over the range $[L, U]$. Clearly, there can be as many extensions of the seminal SBOs as relevant distributions to define the appearance/disappearance process. This richer class may be termed 'continuous BROs', even if the considered distribution does not charge the range $[L, U]$ in a continuous way (see below). The cardinal of such a class of options is potentially infinite. In particular, continuous barrier contracts are far from being restricted to soft ones.

BROs are related to a distribution that charges the interval $[L, U]$ or, more formally, to a probability density function f or, equivalently, to the corresponding cumulative density function. The function f is the birth/death or appearance/disappearance density function. In addition, BROs will be analyzed with the help of a companion function Γ. The price of the option will be given by:

$$iBRO = \int_{R} iBO(K)f(H)dH \qquad (3)$$

where f is a probability density function charging the interval $[L, U]$. Note that various appearance/disappearance density functions may induce the same price. To describe the appearance/disappearance mechanism further, it is useful to consider a companion function, say, Γ, which is a cumulative gain/loss function over the range $[L, U]$. For Up BROs, Γ is defined, over the range $[L, U]$, by $\Gamma(H) = \int_L^H f(v)dv$ and, for any $H \leqslant L$, $\Gamma(H) = 0$, while for any $H \geqslant U$, $\Gamma(H) = 1$. The cumulative gain/loss function therefore increases so that it can be identified to the cumulative density function. Note that this is not the case for Down BROs because their cumulative gain/loss functions decrease.

Various birth/death processes can now be discussed through different specifications of f and Γ. In what follows, I focus on Up BROs and identify, without loss of generality, the upper barrier U to the threshold K.

Example 1: Straight barrier options are 'Single Dirac'-BRO. Straight barrier options arise when the distribution charges the range $[L, U]$ as the Dirac one. Because K is the ultimate threshold level, the density function is given by $f(H) = \delta_K(H)$, where $\delta_K(H)$ equals one if H equals K; otherwise it equals zero. The cumulative gain/loss function is then the step function $1_{\{H \geqslant K\}}(H)$. This is plotted in Figure 1(a), and the appearance/disappearance of the option is viewed as total once the barrier is reached. The corresponding BRO is of course a standard barrier option.

Example 2: SBOs are 'Uniformly distributed' BROs. The probability density function, SBOs consider, is that of the uniform distribution $f(H) = (1/U{-}L)1_{[L,\,U]}(H)$ Over the range $[L, U]$, the companion cumulative gain/loss function is given by: $\Gamma(H) = (H{-}L)/(K{-}L)$. It is worth nothing when H is smaller than L and 1 when H is greater than U. This latter affine function illustrates that the appearance/

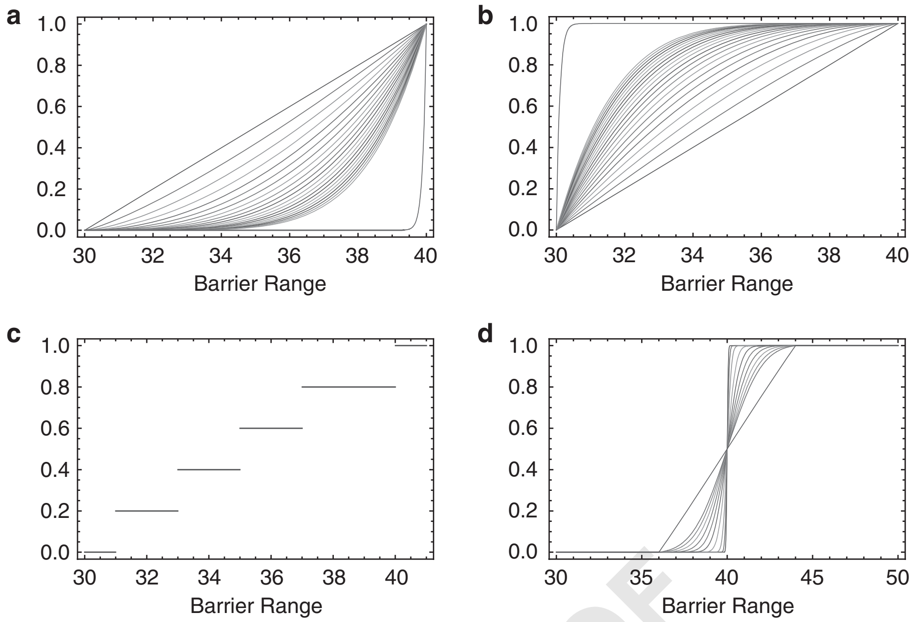

Figure 1: Cumulative Gain/Loss functions for Up Barrier Range Options. These accelerated (**a**), decelerated (**b**), stepwised and (**c**) Gain/Loss birth/death processes are given by equations 2,3 and 4, respectively. Their common range is [30, 40]. The λ parameter (for a, b) lies between 0 and 5 (any 0.25 step) and is finally fixed at 100. Discrete points for (c) are 31, 33, 35, 37, 40. Figure (**d**) depicts gaussian distributed errors around the threshold level on [36, 44]; chosen standard deviations are 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 10.

disappearance features are proportional to the distance potentially covered by the underlying asset's price over $[L, U]$. In SBOs, there is neither an acceleration effect nor a deceleration effect. Let us now consider cases for which the appearance/disappearance is more pronounced as the underlying asset gets closer to the threshold K.

Example 3: BROs with an accelerated birth/death process. Such BROs are characterized by cumulative gain/loss functions verifying: $(\partial^2\Gamma)/(\partial H^2)>0$. For illustration, a very simple birth/death distribution is defined as:

$$\Gamma(H;\lambda) = \left[\frac{H-L}{K-L}\right]^{1+\lambda} \tag{4}$$

where $\lambda \geqslant 0$ is the acceleration parameter. Figure 1(a) draws such a distribution for different values of λ. As expected, when $\lambda = 0$, there is 'no acceleration' and this BRO is an SBO. As λ grows, one tends to a step function and the associated straight *BO* with barrier K.

Example 4: BROs with decelerated birth/death process. Such BROs are characterized by cumulative gain/loss functions verifying: $(\partial^2\Gamma)/(\partial H^2)<0$. For instance, one may consider:

$$\Gamma(H;\lambda) = 1 - \left[\frac{H-K}{L-K}\right]^{1+\lambda} \qquad (5)$$

with $\lambda \geqslant 0$. Figure 1(b) plots such a distribution and, as λ gets larger, the mechanism is (marginally) more pronounced at the beginning of the range. When $\lambda=0$, one recovers the SBO case. For very large λ values, one obtains the step function associated with the standard BO case with barrier L. Nothing prevents the birth/death process from being quite arbitrary, except that Γ must increase for Up BROs and decrease for Down BRO. The process can be, for example, accelerated or decelerated differently over the range (that is $(\partial^2\Gamma \leqslant 0)/(\partial H^2 \geqslant 0)$ over the range). More generally, for any sufficiently smooth function g (in particular integrable), an associated function given by $f(H) = (1/(\int_{Lg(v)dv}^{U}))g(H)1_{[L,\,U]}(H)$ is a density function defined over the range $[L, U]$. The appearance/disappearance process can even be discontinuous, that is, concentrated on certain discrete 'points'. Let us consider, say, an Up and In BRO call option with five specific thresholds $K_i, i=1, \ldots, 5$, the largest one K_5 being the ultimate one. The simplest case arises by assuming that the birth process is equidistributed among these levels. In this case, the cumulative gain function is given by

$$\Gamma(H) = \sum_{i=1}^{5} \alpha_i \Gamma_i(H) \qquad (6)$$

with $\Gamma_i = 1_{\{Ki<H\}}$, $\alpha_i = 1/5$. This cumulative gain function is plotted in Figure 1(c).

The following section illustrates that BROs are useful to account for the estimation risk associated with the barrier.

APPLICATION TO BARRIER MODEL RISK

Choosing a straight barrier option is a not-so-straightforward exercise for managers because it critically depends on the way the threshold level is determined. Often, computation of the barrier K is tricky because data are sparse and frameworks are multiple.[16] Statistical, accounting or financial-based methods provide a range of potential values rather than a pointwise and riskless estimate, and they give an error distribution f. If, for instance, the error is equidistributed between a lower boundary, say $\bar{K}$, and an upper one, say $\tilde{K}$, then the error distribution is defined by $f(H) = (1/(\tilde{K}-\bar{K}))1_{[\bar{K},\,\tilde{K}]}(H)$. For normally distributed errors with mean K and volatility σ_K, an *arbitrary* density function may be defined for values higher than $\bar{K}$ and lower than $\tilde{K}$: $f(H) = (n((H-K)/\sigma_K))/(\int_{\bar{K}}^{\tilde{K}} n((L-K)/\sigma_K)dL)$ where n is the gaussian density. The associated cumulative gain/loss function is represented in Figure 1(d). In fact, any kind of error specification can be considered here.

To illustrate the benefit of using BROs, let us consider an investor who needs a Down and Out barrier option. Let us also assume that, because of the way he or she computes the barrier, possible threshold levels stand in a closed interval $[\bar{K}, \tilde{K}]$ with an error function f. This investor may buy a standard barrier option with the average barrier ($K = \int Hf(H)dH$), and this means that she or he accepts a part of the model risk associated with the barrier determination. Otherwise, he or she may buy a BRO with

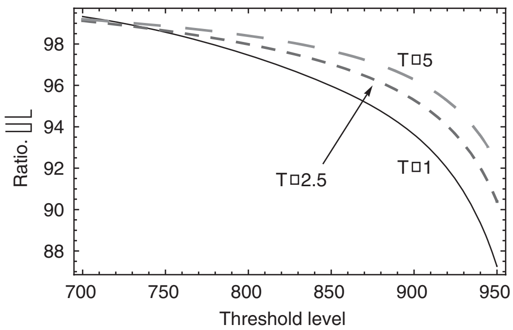

Figure 2: Barrier uncertainty and pricing bias for barrier options. Computed biases result from a potential ± 10 per cent error in the barrier specification for different average threshold levels and time to expiry ($T = 1, 2.5, 5$). Other parameters are $S = 1000$, $E = 800 \times e^{rT}$, $\sigma_S = 30$ per cent and $r = 5$ per cent.

distribution *f*. Assuming that the barrier uncertainty is described by $K \pm 10$ per cent, Figure 2 plots the ratio $(\int iBO(H) f(H) dH)/(iBO(K))$. This graph illustrates that the ratio appears non-negligible as the expiration time gets closer. It is always less than one, meaning that in the considered case the buyer should prefer a BRO to the straight *BO*.

CONCLUDING REMARKS

This article has introduced and analyzed the class of Continuous BROs. These are exotic Barrier Options that 'gradually' disappear or appear. This set generalizes the SBOs introduced by Hart and Ross[10] and represents a simple alternative to the step options of Linetsky.[11,12] According to Hart and Ross,[10] soft contracts *should be a dominant financial instrument (…) because of the considerable improvement in hedging characteristics caused by the gradual rather than instantaneous knock-out*. This study illustrates that BROs can consider complex birth/death mechanism so that they have wide potential for pricing and designing complex structured securities. Of course, motivations for including a continuous birth/death process in Inside Barrier Options also involve Outside Barrier Options that put the activating/deactivating barrier on a second asset. The appendix shows that materials exposed in this article are easy to extend to this class of options.

ACKNOWLEDGEMENTS

Moraux acknowledges financial support from IAE Rennes and CREM (CNRS 6211).

NOTES AND REFERENCES

1 For brevity, readers are assumed to be familiar with barrier options. Others can consult for a description of exotic options the book of Clewlow and Strickland[2].
2 Clewlow, L. and Strickland, C. (1997) *Exotic Options: The State of the Art*. International Thomson Business Press. Q3
3 For the pricing of straight barrier options, see the article of Reiner and Rubinstein[4].
4 Reiner, E. and Rubinstein, M. (1991) Breaking down the barriers. *Risk* 4(8): 6–14.
5 And, finally, for hedging issues see both the articles of Rich, Rich, D. (1994) The mathematical foundations of barrier option-pricing theory. *Advances in Futures and Operations Research* 7: 267–311, or that of Carr and Chou[6].
6 Carr, P. and Chou, A. (1997) Breaking barriers. *Risk* 10(9): 139–145.
7 Straight barrier options are very useful for designing and pricing many other complex financial products. But, often, real-life contracts involve knock in/knock out mechanisms far more complex than the standard one. Binary double barrier options are useful for analyzing structured range notes (see Hui[8]).
8 Hui, C. (1996) One-touch double barrier binary option values. *Applied Financial Economics* 6: 343–346.
9 In the present article, I won't elaborate on this further, but BROs appear very useful for modeling covenants embedded in corporate liabilities.
10 Hart, I. and Ross, M. (1994) Striking continuity. *Risk* 7(6): 46–51.

11 Linetsky, V. (1998) Step to the barriers. *Risk* 11(4): 62–65.

12 Linetsky, V. (1999) Step options. *Mathematical Finance* 9(1): 55–96.

13 Chesney, M., Jeanblanc, M. and Yor, M. (1997) Brownian excursion and Parisian barrier options. *Advances in Applied Probability* 29: 165–184.

14 Hugonnier, J. (1999) The Feynman-Kac formula and pricing occupation times derivatives. *International Journal of Theoretical and Applied Finance* 2(2): 153–178.

15 Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy* 81: 637–654.

16 For instance, computations of value-at-risk or short fall provide different thresholds.

17 Heynen, R. and Kat, H. (1994) Crossing barriers. *Risk* 7(6): 51–56.

APPENDIX

OUTSIDE BROs

Heynen and Kat[17] were the first to study straight outside Barrier Options. They derive analytical pricing formulae under the assumption that the price process of the outside asset is described by a standard (but possibly correlated) geometric Brownian Motion. Let us denote by $V = (V_t)_t$ the 'outside' price process, by σ_V the associated volatility and by ρ the linear correlation between geometric Brownian motions. Closed form formulae for outside Barrier Options involve cumulative distribution functions of standard bivariate normal distribution. Prices of knock-out options, for instance, are given by:

$$\begin{aligned} oBO_{OUT}(\phi) = & \phi S[N[\phi d_1, \eta e_1, \phi\eta\rho] \\ & - \left(\frac{K}{V}\right)^{2(\mu_V \sigma_V^{-2} + \rho)} N\left[\phi d_1', \eta e_1', \phi\eta\rho\right]\right] \\ & - \phi E e^{-rt}[N[\phi d_2, \eta e_2, \phi\eta\rho] \\ & - \left(\frac{K}{V}\right)^{2\mu_V \sigma_V^{-2}} N\left[\phi d_2', \eta e_2', \phi\eta\rho\right]\right] \end{aligned}$$

where: $d_1 = d_{\sigma_S}(S, E, \mu_S + \sigma_S^2)$, $e_1 = d_{\sigma_V}(V, K, \mu_V + \rho\sigma_S\sigma_V)$, $d_2 = d_{\sigma_S}(S, E, \mu_S)$, $e_2 = d_{\sigma_V}(V, K, \mu_V)$, $d_i' = d_i - 2\rho d_{\sigma_S}(V, K, 0)$, $e_i' = e_i - 2d_{\sigma_V}(V, K, 0)$ $i = 1, 2$, with $\mu = r - (1/2)\sigma^2$. ϕ is 1 (*resp.* -1) when dealing with a call (*resp.* put); η is 1 (*resp.* -1) is if the barrier is down (*resp.* up). Other oBO_{IN} are obtained by the standard parity relation.

The analysis by Heynen and Kat[17] may be easily extended to account for a barrier range instead of a discrete barrier. Outside BROs over the range $[L, U]$ may be defined by:

$$oBRO = \int oBO(H) f(H) dH$$

where f is a probability density function charging $[L, U]$. Note that the presence of the cumulative bivariate standard normal distribution in the pricing formulae fully justifies the use of numerical integration techniques (even for Outside SBOs, for which $oSBO = 1/(U-L)\int_L^U oBO(H) dH$).