Market Volatility Index and Implicit Maximum Likelihood Estimation of Stochastic Volatility Models (1)

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INTRODUCTION

Practical use of stochastic volatility models requires a preliminary estimation of the parameters of the unobservable latent volatility process. Two kinds of studies have covered this issue.

First, as pointed out by Nelson-Foster [1994], numerous authors have suggested to make inference from the observed asset prices through an approximation of the structural stochastic volatility model, typically an Euler or ARCH type discretization. Estimation of such models have then been conducted in many different ways including (2) simple Method of Moment (MM) by Taylor [1986], Generalized Methods of Moments (GMM) by Melino-Turnbull [1990] Anderson-Sørensen [1993], various Simulated Method of Moment procedures (SMM) by Duffie-Singleton [1989] ; Quasi Maximum Likelihood Estimation (QMLE) by Harvey-Ruiz-Shephard [1992], Simulated Maximum Likelihood Estimation (SMLE) by Danielson-Richard [1993] and Danielson [1994] ; Indirect Inference by Gouriéroux-Montfort-Renault [1993], Moment Matching Approach by Gallant-Tauchen [1992] ; Bayesian Markov Chain Monte-Carlo Analysis (MCMC) by Jacquier-Polson-Rossi [1994] and Kim-Shephard [1994]. Apart from MM, GMM and QML these approaches are computationally intensive.

Second and more recently, some works have proposed to use option implied volatilities as convenient data for recovering the unobserved volatility. Observed assets are, in that case, considered as exogenous. Assuming the observed prices are obtained with the Hull and White [1987] formula, Renault-Touzi [1996] have then remarked that, due to the increasing feature of the Black-Scholes formula, we get a precise definition from the Black-Scholes' implied volatility. Since the derivative with respect to the volatility is positive, Black-Scholes implied volatility is a one to one function of the

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unobservable volatility. The observation of a Black-Scholes implied volatility is thus equivalent to the observation of a realization of the volatility process. In this case, direct maximum likelihood statistical inference, as Pearson-Sun [1994] and Duan [1994] among others have implemented on Bond Market, may be applied. However even if, from a strict theoretical viewpoint, this procedure is easily and directly available, it requires cumbersome charged CPU time ; the transformation between Black-Scholes implied volatility and instantaneous volatility is indeed non linear and non analytical. It is important to recall that Pearsun-Sun [1994] and Duan [1994] have succeeded thanks to the convenient properties of the *exponential affine* interest rate models ([1]). In this special case, the transformation between instantaneous interest rate and yield to maturity is quite simple. To obtain an estimation of the volatility process Renault-Touzi [1996] have proposed an iterative procedure in the optimization of the log-likelihood function. The key point is that this estimation procedure provides simultaneously Hull and White's implicit volatilities and consistent estimators of the volatility process parameters. In the special case where at-the-money Black-Scholes implied volatilities are available, Renault-Touzi [1996] precise that this iterative procedure is an *EM* (expectation-maximization) algorithm, introduced by Dempster *et al.* [1977]. Nevertheless they noticed that the general properties of *EM* algorithms do not apply since the support of the latent variables given the observable ones depends on the unknown parameters. They finally argued that for a large enough sample size the algorithm converges *almost surely* towards the true value of the parameters. Moreover, this procedure can be seen as correcting the approximating bias of the method used by Heynen-Kemna-Vorst [1991] who considered near-the-money, short maturity Black-Scholes implicit volatilities as proxies. In fact, these latter methodologies exploit the fact that option markets are considered as volatility markets and this position is largely confirmed by the growing number of market volatility index. For example, following the CBOE (Chicago Board of Exchange) Market Volatility Index (*VIX*), the MONEP (Marché des Options Négociables de Paris) created, on the 8th October 1997, two volatility indexes (*VX1* and *VX6*), based on implied volatilities of around at-the-money CAC40 Index option (*PX1*). *VX1* is an average of four CAC 40 call option implied volatilities. Moreover, on the basis of the evidences reported in the studies of Fleming-Ostdiek-Whaley [1995] for the VIX and of Moraux-Navatte-Villa [1999] for the VX1, market volatility indexes appear to be useful proxies for expected markets volatility.

In our sample, Renault-Touzi procedure does not permit us to obtain a whole implied volatility series and consequently fails to provide estimates of parameters. We therefore suggest to exploit Feinstein [1992]'s research. In a stochastic volatility model *à la* Hull-White [1987], he has demonstrated that the implied volatility approximates the market expectation of the average volatility over the life of the option. More precisely, a new methodology based on Feinstein's definition of theoretical implied volatilities is implemented in order to estimate the unobservable volatility process parameters. This is first applied in an Hull-White [1987] setting and second in a more general framework allowing the so-called leverage effect. Based on a Maximum Likelihood type estimation, the procedure is applied on a *VX1* time series to recover implicit instantaneous volatility. This methodology and its result are compared both with that of Heynen-Kemna-

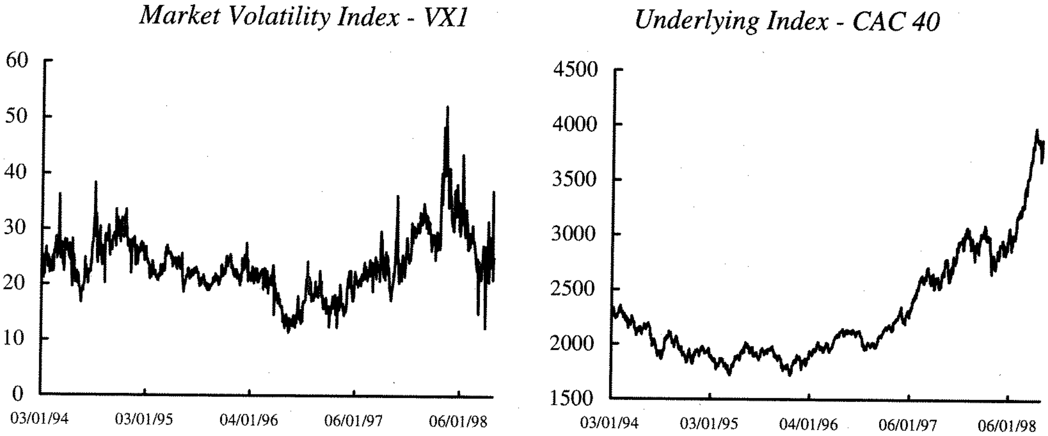
Vorst [1994], who used Black-Scholes implicit volatilities as proxies and with Renault-Touzi's [1996] statistical iterative procedure of filtering of the latent volatility process and estimation of its parameters. To implement this methodology, the variance is supposed to follow a mean-reverting "square-root" diffusion process. Among other things, it is well known that this process is analytically tractable. For example, Cox-Ingersoll-Ross [1985] implicitly solve for the moment-generating function of the average of this process in the derivation of their formula for the pricing of a discount bond. Ball-Roma [1994] use this result to derive a simple closed-form expression for the expected value of average future volatility.

The rest of the paper is organized as follows. Section 1 explores the statistical properties of the French Market Volatility Index, *VX1*. Section 2 is devoted to the Hull-White [1987] stochastic volatility model estimation. Section 3 highlights the so-called leverage effect in the procedure.

1. Statistical properties of *VX1*

The method used by the MONEP ([2]) to compute the *VX1* and the *VX6* indexes is based on observing a quasi-linear relationship between the premium and the volatility of the series around the at-the-money benchmark, i.e. the most liquid series. The method used by the MONEP includes five stages. Let S_t be the price of the CAC 40 at date t and n be the number of days used in the calculation ($n = 31$ for the short-term index, *VX1*, and $n = 185$ for the long-term index, *VX6*). The aim of the calculations is to establish, at t, the implied volatility of a « virtual » at-the-money contract (*i.e.* the strike price is equal to the index S_t) with a constant time to maturity of n days. Since strike prices are set at the standard 25-point intervals, options are almost never at the money. Consequently, linear interpolation is used to estimate the data. The first stage consists in identifying the two nearest expiry months, being one on each side of the calculation period n. Let τ_1 and τ_2 be the residual times to maturity (in days) corresponding to these two expiries. The next stage consists in enclosing the last price of the CAC 40 index by two strike prices, which are written K_a and K_b. Based on these two expiries τ_1 and τ_2, and the two strike prices K_a and K_b, the following four options series: (K_a, τ_1), (K_b, τ_1), (K_a, τ_2) and (K_b, τ_2) are obtained. Stage three consists in computing the value of two synthetic options with a residual life t and strike prices K_a and K_b : $C^*(K_a, n)$ and $C^*(K_b, n)$. By interpolating these synthetic values the MONEP then calculates the final value $C^{**}(S_t, n)$. The final calculation gives the price of an at-the-money option with a maturity n. It is used to obtain implied volatility. The volatility index is simply the implied volatility of the synthetic value C^{**}. To solve for implied volatility the MONEP suggests using the binomial model adjusted from the daily dividends for each option contract with τ periods to the expiration date from time t. Since the implied volatilities of the *PX1* option series used in computing *VX1* are stated in calendar days (rather than in trading days), the return variance over a weekend should be three times greater than it is over any other pair of trading days. However, on an empirical evidence, weekend volatility is approximately the same as the volatility during trading days. For this reason, each *VX1* day is adjusted to a trading day basis by multiplying the ratio of the square root of the number of calendar days, 31, to the square root of the number of trading days, 22.

FIGURE 1



Historical data is available from the MONEP WEB-site on *VX1* since the beginning of 1994 through April 1998. Figure 1 plots the daily closing Volatility Index levels versus the CAC 40 Index levels. Over the sample period studied, the Market Volatility did not drift in one direction or another. Moreover, during this period spikes in the MONEP Market Volatility Index, *VX1*, are usually accompanied by large movements, up or down, in the stock Index level. The December 1997 Asian crisis is accompanied by more than a 50% level of Market Volatility Index.

Following Fleming-Ostdiek-Whaley [1995] the empirical analysis of the Volatility Index in this paper is done on volatility changes. *Table* 1 summarizes the properties of daily *VX1* changes. The mean volatility change over the entire sample period is 0.00175. The standard deviation of the volatility changes seems high, i.e. 2.039. *Table* 1 also provides the auto-correlation structure of the Volatility Index, based on volatility changes from one through three lags. The first and second order coefficients, respectively -0.270 and -0.112 reveal a significant negative auto-correlation. This degree of correlation is similar to the auto-correlation reported by Harvey-Whaley [1991] for individual S&P100 options. They reported for daily volatility changes implied by the nearby at-the-money call (respectively put) an auto-correlation structure of -0.33 and -0.13 (respectively -0.33 and -0.09). However, this degree of correlation is much higher than the auto-correlation reported by Fleming-Ostdiek-Whaley [1995] for CBOE Market Volatility Index changes, -0.073 and -0.104. This higher auto-correlation for *VX1* relative to *VIX* may be attributed to the difference in how the indexes are constructed. *Indeed, VX1 is a weighted average of the volatilities implied by only call PX1 option prices, whereas VIX is a weighted average of four call and four put option prices.* In fact, a call (put) implied volatility computed from the reported index level during a rising market can be upward or downward biased. Since the upward (downward) bias of the call implied volatility is approximately equal to the downward (upward) bias of the put implied volatility, the effect of infrequent trading of index stocks on the level of *VIX* is mitigated. Consequently, *VIX* Index construction reduces the oriented overreaction. Most notably the auto-correlation for one through three lags is negligible for daily CAC 40 Index returns.

TABLE 1

Statistical Properties of Daily Closing MONEP Market Volatility Index Level Changes and CAC 40 Index Returns

Series	*Statistics*	*Results*
Volatility Index Changes	Mean	0,0017476
	Standard Deviation	2,0391567
	Autocorrelation (ρ=1)	-0,27018145*
	Autocorrelation (ρ=2)	-0,1123194*
	Autocorrelation (ρ=3)	-0,02991091
CAC 40 Index Returns	Mean	0,00048822
	Standard Deviation	0,01127158
	Autocorrelation (ρ=1)	-0,00362296
	Autocorrelation (ρ=2)	0,02580488
	Autocorrelation (ρ=3)	-0,03153352

* identifies correlation significant at the 5% level where the standard error is calculated as $T^{-0.5}$.

2. The Hull-White [1987] general framework

2.1. *The Hull-White [1987] model*

Many stochastic volatility models have been proposed in the literature. In this article we assume that the variance follows a square-root diffusion process. The mean-reverting feature is attractive for several reasons. First, Day-Lewis [1993] empirically show that volatility shocks are persistent and mean-reverting. Second, the relation between the spot volatility and the long-run volatility can be examined directly. Finally, this process is analytically tractable. Cox-Ingersoll-Ross [1985] implicitly solve for the moment-generating function of the average of this process in the derivation of their formula for the price of a discount bond. Ball and Roma [1994] use this result to derive a simple closed-form expression for the expected value of average future volatility.

The data generating process used is defined on a probability space (Ω, F, P) the fundamental space of the underlying asset price process S that is described by :

$$\frac{dS}{S} = \mu(t,S,\sigma)dt + \sigma dW_1(t)$$
$$d\sigma^2 = \kappa(\vartheta - \sigma^2)dt + \gamma\sigma dW_2(t)$$

where $W = (W_1, W_2)$ is a standard bidimensional Brownian motion. We denote by r the instantaneous interest rate supposed to be constant, so that the price of a zero coupon bond maturing at time T is given by $e^{-r(T-t)}$. Let C be the price process of a European call option on the asset S with strike K and maturity T. We introduce the variable $x = \ln\left(S/Ke^{-r(T-t)}\right)$, and then call option is said to be in-the-money if $x > 0$, out-of-the-money if $x < 0$, at-the-money forward if $x = 0$ and at-the-money if $x = r(T-t)$.

Following Hull-White [1987] we impose the assumption of nonsystematic volatility risk and the risk neutral data generating bivariate process is then given by :

$$\frac{dS}{S} = rdt + \sigma d\tilde{W}_1(t)$$
$$d\sigma^2 = \kappa(\vartheta - \sigma^2)dt + \gamma\sigma d\tilde{W}_2(t)$$

where $\tilde{W} = (\tilde{W}^1, \tilde{W}^2)$ is a standard bidimensional Brownian motion under the risk neutral probability with $\tilde{W}_2 = W_2$.

The Hull-White formula is given by :

$$C(S,\sigma^2) = E\left[C^{BS}\left(S, \frac{1}{T-t}\int_t^T \sigma_u^2 du\right)\right] = \int C^{BS}(S, u/T-t) f(u) du$$

where f is the density of the cumulated variance. Ball-Roma [1994] show that when there is no correlation between innovations in security price and volatility, the characteristic function of the average variance of the price process plays a pivotal role. In fact, they noted that this function can be used in two ways : first, to obtain the joint terminal density of the average variance and the future security price ; second to obtain moments of the average variance. The first way allows option pricing through Fourier inversion method as discussed by Stein-Stein [1991] and the second one permits power series expansion methods as introduced by Hull-White [1987]. Moreover, Villa [1998] has shown, thanks to the power theorem, how the characteristic function of the average variance can be used to calculate Hull and White's option prices.

If we suppose that, the observed prices are supposed to be given by this formula, Renault-Touzi [1996] pointed out that, in this context, due to the increasing feature of the Black-Scholes formula, a precise definition of the Black-Scholes' implied volatility can be given as the unique solution to :

$$\sigma_I^2\left(x,\sigma^2;\Theta\right)=h\left(x,\sigma^2;\Theta\right)$$

where : $h=\left(C^{BS}\right)^{-1}\circ C$ and $\Theta=(\kappa,\vartheta,\gamma)$. In the special case of an at-the-money implied volatility, $x_t=r(T-t)$, this equation reduces to :

$$\sigma_I^2\left(\sigma^2;\Theta\right)=h\left(\sigma^2;\Theta\right)$$

The vector Θ is the vector of parameters to be estimated. Since the derivative with respect to the volatility is positive (see Renault-Touzi [1996]) Black-Scholes implied volatility is a one to one function of the unobservable volatility. If we denote for $i=0,1,\ldots,n$ $\sigma_{I,i}^2$ the i-th discrete and sample observation in an available time series of Market Volatility Index, i.e. at-the-money implied volatility and if the conditional density for $\sigma_{I,i}^2$ is known conditionally on previous instant and noted $f\left(\sigma_{I,i}^2\,|F_{i-1};\Theta\right)=f\left(\sigma_{I,i}^2\,|\sigma_{I,i-1}^2;\Theta\right)$, then standard maximum likelihood estimate of Θ is obtained by using the following direct log-likelihood function :

$$\mathcal{L}\left(\sigma_{I,n}^2,\ldots,\sigma_{I,0}^2;\Theta\right)=\sum_{i=1}^{n}\ln f\left(\sigma_{I,i}^2\,|\sigma_{I,i-1}^2;\Theta\right).$$

Since $\sigma_I^2=h\left(\sigma^2;\Theta\right)$ then the log-likelihood can be expressed as (cf Pearsun-Sun [1994], Duan [1994], ...) :

$$\mathcal{L}\left(\sigma_{I,n}^2,\ldots,\sigma_{I,0}^2;\Theta\right)=\sum_{i=1}^{n}\left\{-\ln|J_i|+\ln f\left(\hat\sigma_i^2\,|\hat\sigma_{i-1}^2;\Theta\right)\right\},$$

where

$J_i=\dfrac{\partial\sigma_{I,i}^2}{\partial\sigma_i^2}$ is the Jacobian of the transformation,

$\hat\sigma^2$ is the implicit spot volatility found as $\sigma_I^2=h\left(\hat\sigma^2;\Theta\right)$. In practice such an Hull and White's implicit volatility can be obtained as a limit of the following Newton-Raphson procedure :

$$\hat\sigma_i^2(p+1)=\hat\sigma_i^2(p)-\left[\frac{\partial h\left(\hat\sigma_i^2(p),\Theta\right)}{\partial\hat\sigma_t^2}\right]^{-1}\left[h\left(\hat\sigma_i^2(p),\Theta\right)-\sigma_{I,i}^2\right]$$

f is the associated transition density. In this case, this is a non central χ^2 distribution :

$$\log f\left(\sigma_i^2\,|\sigma_{i-1}^2,\Delta\right)=\log c-c\left(\sigma_i^2+e^{-k\Delta}\sigma_{i-1}^2\right)$$
$$+\frac{1}{2}q\log\left(\frac{\sigma_i^2}{\sigma_{i-1}^2e^{-k\Delta}}\right)+\log \mathrm{I}_q\left(2c\sqrt{\sigma_i^2\sigma_{i-1}^2e^{-k\Delta}}\right),$$

where : $c = \dfrac{2\kappa}{\gamma^2\left(1-e^{-\kappa\Delta t}\right)}$, $q = \dfrac{2\kappa\vartheta}{\gamma^2} - 1$, $I_q(\cdot)$ is a modified Bessel function of the first kind of order q :

$$I_q(z) = \left(\frac{z}{2}\right)^q \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n}}{n! \cdot \Gamma(q+n+1)}$$

From a theoretical point of view, this procedure can be directly implementable ; however, it requires cumbersome charged CPU time since the transformation-between Black-Scholes implied volatility and instantaneous volatility is non linear and non analytical.

2.2. The estimation procedure proposed by Renault-Touzi [1996]

To obtain $\hat{\Theta}$ Renault-Touzi [1996] proposed an iterative procedure for implementation of the transformation between σ_I^2 and σ^2, $\sigma_I^2 = h\left(\hat{\sigma}^2;\Theta\right)$, in the optimization of the log-likelihood function. The key point is that this estimation procedure provides simultaneously Hull and White's implicit volatilities and consistent estimators of the volatility process parameters. This is repeated until $\hat{\Theta}$ converges. More precisely, they introduced the following iterative procedure :

Step $2p$ $\quad\quad \Theta^{(p)} \rightarrow \sigma_i^{(p+1)}$, $i = 0, 1, \ldots, n$

Step $2p+1$ $\quad\quad \sigma_i^{(p+1)}, i = 0, 1, \ldots, n \rightarrow \Theta^{(p+1)}$.

where step *2p* is performed by solving an Hull and White's implicit volatility and step *2p+1* is the maximum likelihood estimate from data obtained by step *2p*. In the special case where at-the-money Black-Scholes implied volatilities are available, this iterative procedure is shown to be an *EM* (expectation-maximization) algorithm, introduced by Dempster *et al.* [1977], where the step 2p (step *2p+1*) corresponds to step *E* (step *M*) of the *EM* algorithm. However, they noticed that the general properties of *EM* algorithms do not apply since the support of the latent variables given the observable ones depends on the unknown parameters. They finally argued that for a large enough sample size the algorithm converges *almost surely* towards the true value of the parameters. Furthermore as Renault-Touzi pointed out this procedure can be seen as correcting the approximating bias of the method used by Heynen-Kemna-Vorst [1991] who considered near-the-money, short maturity Black-Scholes implicit volatilities as proxies.

This iterative procedure reduces the charged CPU time dramatically because the numerical integration procedure only has to be called a fraction of the times compared with directly maximizing the log-likelihood function where $\hat{\sigma}^2$ changes for every change in Θ. Nevertheless, the drawback of this procedure is that we do not get an explicit estimate of the volatility risk premium. Thanks to Hull-White model who assumed a nonsystematic volatility risk this drawback is not important here. More importantly, Renault and Touzi [1996] pointed out that there are two asymptotic points of view relevant for this iterative procedure. The first one concerns the continuous time limit obtained by letting the time space between observations go to zero (this is related to the near integrated time series). The second one consists of considering an infinite number of observations with fixed time space between observations. As Phillips [1973] showed that, except for the instantaneous variance parameter, the maximum likelihood estimator does not converge to the true value of the parameters when the time space between observations goes to zero, they considered the second asymptotic point of view.

As recalled in section 1, in order to estimate the volatility process parameters, Renault-Touzi [1996] proposed an iterative procedure. The key point is that this estimation procedure provides simultaneously Hull and White's implicit volatilities and consistent estimators of the volatility process parameters. If at-the-money options are available at any time, this iterative procedure is shown to be an *EM* (expectation-maximization) algorithm, associated with the observations of Black-Scholes implied volatility, that converges *almost surely* towards the true value of the parameters. As Renault-Touzi pointed out a natural starting point of the iterative procedure is $\Theta^{(0)} = 0$.

Exhibit 2A reports the two first steps of Renault-Touzi [1996] iterative procedure (p=0). The step one reports the maximum likelihood parameter estimates of the previous volatility process and the corresponding asymptotics standard errors. The sample period is from January 1994 through April 1998.

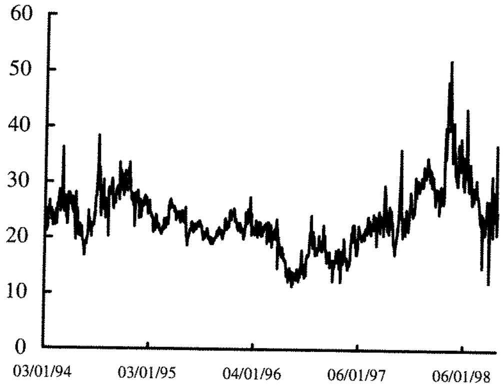
Since $\Theta^{(0)} = 0$, it is clear that the first step filtered Hull-White's implicit volatilities equal to the market volatility index, VX1. Furthermore as Renault-Touzi pointed out this first iteration (p=0) corresponding to the step 0 and 1 can be identified to the method used by Heynen-Kemna-Vorst [1991] who considered near-the-money, short maturity Black-Scholes implicit volatilities as proxies. The estimate for κ, i.e. the adjustment speed for σ^2, is 21.52, which implies a very fast mean-reversion. To get a feeling for the adjustment speed we can use the following conditional expectation :

$$E\left[\sigma_s^2 \middle| \sigma_t^2\right] = \sigma_t^2 e^{-\kappa(s-t)} + \vartheta\left(1 - e^{-\kappa(s-t)}\right).$$

For example, the half-life of the process , the time when the variance is expected to have a value halfway between the current level and the long-run mean, is $\ln(2)/\kappa = 0.0322$ or about one week. It is worthwhile to note that if we had used Heynen-Kemna-Vorst's methodology, this result for adjustment speed would have been too quick relative to previous studies (see for example Bates [1996]). Finally note that all parameter estimates are significant.

EXHIBIT 2A. — *The two first steps of Renault-Touzi [1996] procedure* ($\Theta^{(0)} = 0$)

Step Zero (Step E)



Step One (Step M)

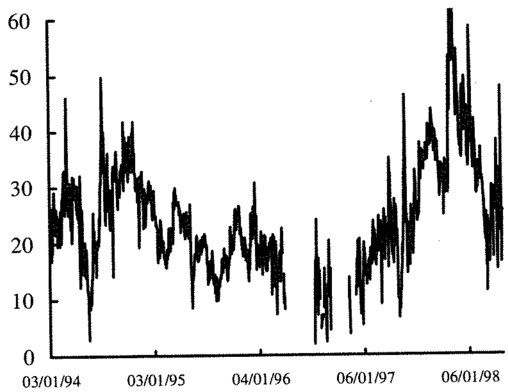
$$d\sigma^2 = \kappa\left(\vartheta - \sigma^2\right)dt + \gamma\sigma dW_2(t)$$

	Value	t - *stat*
κ	21.5234	8.90338
ϑ	0.059076	15.5108
γ	0.676479	112.510
$\mathcal{L}$	3529.81	

Exhibit 2B reports the two second steps of Renault-Touzi [1996] iterative procedure (p=1). Step Two is obtained by solving an Hull and White's implicit volatility with the previous parameters. Step Three estimate the parameters from data obtained by step $2p$ using a Maximum Likelihood procedure.

EXHIBIT 2B. — *The two following steps of Renault-Touzi [1996] procedure*
($\Theta^{(1)} = \{21.5234, 0.059076, 0.676479\}$)

Step Two (Step E)



Step One (Step M)

$$d\sigma^2 = \kappa\left(\vartheta - \sigma^2\right)dt + \gamma\sigma dW_2(t)$$

	Value	t - *stat*
κ	------	------
ϑ	------	------
γ	------	------
$\mathcal{L}$	------	

Surprisingly, Renault-Touzi procedure does not permit to obtain a whole implied volatility series and obviously fails to provide parameters estimate. Nevertheless, we suggest to exploit Feinstein [1992]'s research, who demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of the option.

2.3. *A new simple procedure to correct maturity bias*

Following Ball-Roma [1994], it can be shown that :

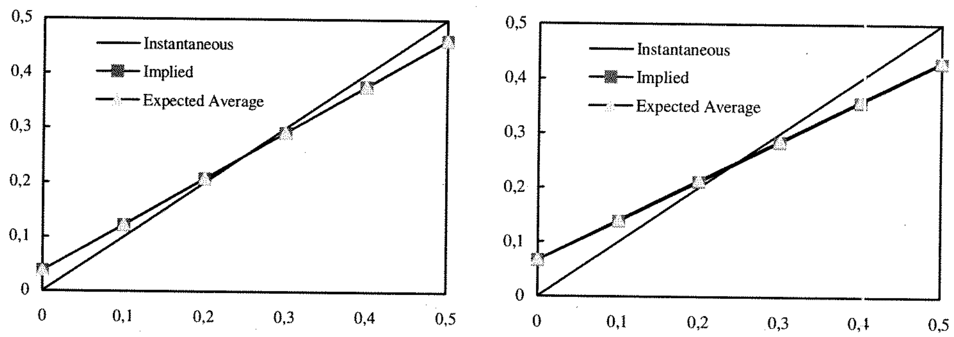
$$\sigma_I^2 \approx E\left[\frac{1}{\tau}\int_0^{\tau}\sigma_u^2 du\right] = \vartheta + \left(\sigma^2 - \vartheta\right)\frac{1-e^{-\kappa\tau}}{\kappa\tau}.$$

From Exhibit 3 it can be inferred, first, that Theoretical Implied Variance and Expected Average Variance seem to be very closed which justified Feinstein [1992]'s approximation. Second, since Feinstein's definition assumed implicitly in this model that Theoretical Implied Variance is an affine function of the Instantaneous one, it can be verified from Exhibit 1 that it seems to be true. Finally, it can also be noticed that the relationship between Theoretical Implied Variance and Instantaneous Variance seems to be justified only when its value is near its long term level (the intersection point of Theoretical Implied Variance and the 45 % line).

EXHIBIT 3. — *Expected Average Variance and Instantaneous Variance Versus Theoretical ATM Implied Variance*

$\vartheta = 0.25, r = 5\%, T - t = 1$ month

$\kappa = 4, \gamma = 0.4$ $\kappa = 8, \gamma = 0.8$



Instantaneous Variance Instantaneous Variance

In this case,

$$\sigma_{I,i}^2 = \vartheta + \left(\sigma_i^2 - \vartheta\right)\frac{1-e^{-\tilde{\kappa}\tau}}{\tilde{\kappa}\tau} \Leftrightarrow \sigma_i^2 = \vartheta + \left(\sigma_{I,i}^2 - \vartheta\right)\frac{\tilde{\kappa}\tau}{1-e^{-\tilde{\kappa}\tau}}.$$

σ_t^2 is therefore an affine function of the implied volatility, and the log-likelihood function is :

$$\mathcal{L}\left(\sigma_{I,n}^2,\ldots,\sigma_{I,0}^2;\Theta\right)=\sum_{i=1}^{n}\left\{-\ln\left|J_i\right|+\ln f\left(\hat{\sigma}_i^2\left|\hat{\sigma}_{i-1}^2;\Theta\right.\right)\right\},$$

where $J_i=\dfrac{1-e^{-\kappa\tau}}{\kappa\tau}$. Note that if $\kappa(T-t)$ is low, then $1-e^{-\kappa(T-t)}\approx\kappa(T-t)$ and therefore $\sigma_I^2\approx\sigma_t^2$ i.e. this methodology can be identified to the one used by Heynen-Kemna-Vorst [1991] who considered near-the-money, short maturity Black-Scholes implicit volatilities as proxies. In fact, like Renault-Touzi's iterative procedure, this can be seen as correcting the approximating bias of the method used by Heynen-Kemna-Vorst [1991].

TABLE 2. — *Estimators based on Market Volatility Index MONEP-VX1*

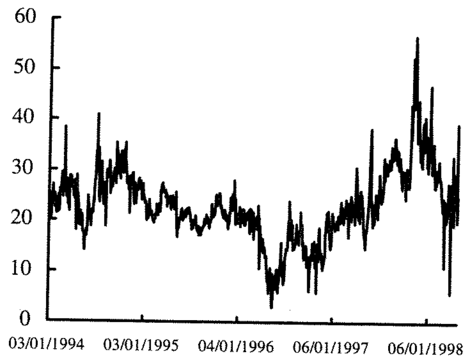
$$d\sigma^2=\kappa\left(\vartheta-\sigma^2\right)dt+\gamma\sigma dW_2(t) \text{ and } \sigma_I^2\approx E\left[\tfrac{1}{T-t}\int_t^T\sigma_s^2ds\right]$$

	Value	t − *stat*
κ	5.41814	3.68011
ϑ	0.063271	4.33034
γ	0.848799	30.0029
$\mathcal{L}$	3537.44	

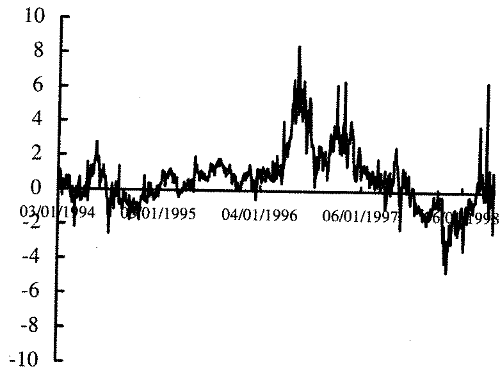
The estimate for κ, i.e. the adjustment speed for σ^2, is 5.42, which implies a less fast mean-reversion than found in Exhibit 2A. The half-life of the process is about one month and a half. Fleming-Ostdiek-Whaley [1995] pointed out that the first step of the analysis of a market volatility index as a forecast of stock market volatility is the consideration of the bias in volatility index as a forecast, that is, the degree to which the volatility index is below or above the subsequent realizations. Although Fleming-Ostdiek-Whaley [1995] found a strong upward forecast bias of the CBOE market volatility index, they argued that it is not problematic if the bias is constant and/or its magnitude is known. More precisely they said : « To the extent the VIX forecast bias is relatively constant, *a naive adjustment* based on a rolling average of past forecast errors may sufficiently correct the bias ». Some factors suggest that the VIX forecast may be well behaved. However, the misspecification of the option pricing model may contribute to the forecast bias of the volatility index, the more natural one is that the option pricing model may be misspecified. From Exhibit 4, it can be argued that the market volatility index is upward forecast bias (as previously stated Fleming-Ostdiek-Whaley [1995]) if a stochastic volatility is taken in account. Moreover, the bias appears not constant through time.

EXHIBIT 4

Implicit Instantaneous Volatility from Market Volatility Index VX1



Market Volatility Index minus Implicit Instantaneous Volatility



Nevertheless, VX1 movements are inversely related to the contemporaneous Stock Market return. As can be seen in table 1, the estimated contemporaneous correlation between changes in expected volatility and CAC 40 returns is relatively large and negative across the sample, -0.3527. The next point will consider this feature.

3. The leverage effect

In this point, the data generating process used, defined on a probability space (Ω, F, P) of the underlying asset price process S is described by :

$$\frac{dS}{S} = \mu(t,S,\sigma)dt + \sigma\sqrt{1-\rho^2}dW_1(t) + \sigma\rho dW_2(t),$$
$$d\sigma^2 = \kappa(\vartheta - \sigma^2)dt + \gamma\sigma dW_2(t),$$

where $W = (W_1, W_2)$ is a standard bidimensional Brownian motion. Following Bates [1996] and Heston [1993] we assumed that the risk neutral data generating process is given by :

$$\frac{dS}{S} = rdt + \sigma\sqrt{1-\rho^2}d\tilde{W}_1(t) + \sigma\rho d\tilde{W}_2(t),$$
$$d\sigma^2 = (\kappa\vartheta - \tilde{\kappa}\sigma^2)dt + \gamma\sigma d\tilde{W}_2(t),$$

where $\tilde{\kappa} = \kappa + \nu$, and $\tilde{W} = \left(\tilde{W}^1, \tilde{W}^2\right)$ is a standard bidimensional Brownian motion under the risk neutral probability. Volatility risk premia is therefore assumed to be proportional to the volatility level. Indeed we suppose a square-root diffusion for the volatility, and we choose Bates [1996] risk premium specification in the spirit of Cox-Ingersoll-Ross [1985] one. In a same context, Romano and Touzi [1997] (Proposition 4.1., p. 408) stated that European call option price is given by :

$$C\left(S,\sigma^2\right) = \tilde{E}\left[C^{\mathcal{BS}}\left(Se^Z; \tfrac{1}{T-t}V^2\right)\right],$$

where $C^{\mathcal{BS}}\left(S,\sigma^2\right)$ is the Black-Scholes formula and :

$$V^2 = \left(1-\rho^2\right)\int_t^T \sigma_u^2 du,$$

$$Z = \rho\int_t^T \sigma_u d\tilde{W}_2(u) - \frac{1}{2}\rho^2\int_t^T \sigma_u^2 du.$$

We refer to the authors for a proof. As Romano and Touzi [1997] noticed, the European price is the expectation of the Black-Scholes formula where the underlying asset price is replaced by Se^Z and the variance parameter is replaced by $\frac{1}{T-t}V^2$. Denoting $H\left(x,\sigma^2\right) = C\left(S,\sigma^2\right)/S$ and $H^{\mathcal{BS}}\left(x,\sigma^2\right) = C^{\mathcal{BS}}\left(S,\sigma^2\right)/S$, the option pricing formula is reduced to :

$$H\left(x,\sigma^2\right) = \tilde{E}\left[e^Z H^{\mathcal{BS}}\left(x+Z, \tfrac{1}{T-t}V^2\right)\right];$$

we can separate the two variables within the expectation brackets by a change of probability measure. Conditionally on volatility path and the information set up to time *t*, we can define an new equivalent measure characterized by its density $\exp Z$. Then we can rewrite the formula as :

$$H\left(x,\sigma^2\right) = \hat{E}\left[e^Z\right]\cdot\tilde{E}\left[H^{\mathcal{BS}}\left(x+Z, \tfrac{1}{T-t}V^2\right)\right] = \hat{E}\left[H^{\mathrm{BS}}\left(x+Z, \tfrac{1}{T-t}V^2\right)\right].$$

Under this new measure, we notice that the European price is the expectation of the Black-Scholes formula where x is replaced by $x+\frac{1}{T-t}Z$ and the variance parameter is replaced by $\frac{1}{T-t}V^2$. We finally find in the same previous way dynamics of the volatility process :

$$d\sigma^2 = \left(\kappa\vartheta - \hat{\kappa}\sigma^2\right)dt + \gamma\sigma d\hat{W}_2(t),$$

where : $\hat{W}_2(t) = \tilde{W}_2(t) - \rho\int_0^t \sigma_u du$ and $\hat{\kappa} = \tilde{\kappa} - \rho\gamma$.

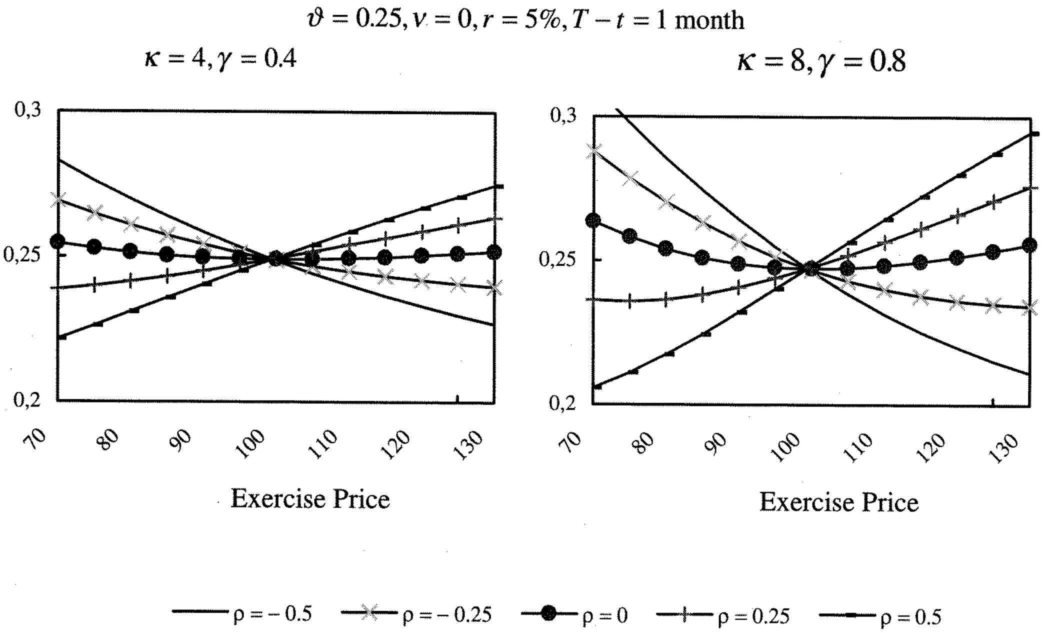
The likelihood of the full data generating process is a bivariate one. Of course, by reducing the likelihood to the volatility one, we generally bear a loss of information due to the underlying asset price data. In the previous case (in the Hull-White [1987] settings) the correlation is zero, therefore instantaneous volatility process's parameters are the only ones to be estimated. Thus, it might be reasonable not to take account of the information conveyed by the underlying asset price data. Even if, in our case, correlation induces a relation between parameters and underlying asset price data, as Patilea-Ravoteur-Renault [1996] we neglect this information in order to have an inference procedure robust with respect to a possible misspecification of the drift of the underlying asset price.

Following Renault-Touzi [1996], due to the increasing feature of the Black-Scholes formula, a precise definition of the Black-Scholes's implied volatility can be given as the unique solution to :

$$\sigma_I^2\left(x,\sigma^2;\Theta\right)=h\left(x,\sigma^2;\Theta\right)$$

where : $h=\left(H^{BS}\right)^{-1}\circ H$ and $\Theta=(\kappa,\vartheta,\gamma,\rho)$. Different shapes of the volatility smile are consistent with different distributions of the underlying asset. For instance, a symmetric volatility smile is consistent with leptokurtosis or « fat tails » in the distribution, i.e. higher probabilities, as compared with the normal distribution, of larger positive or negative changes, as would result from returns with stochastic volatility ([3]). At times, the probability of future asset or index return realizations is not symmetrically distributed around at the money strike price. When this asymetry is present, the smile can be transformed into a « smirk », with the option's implied volatility rising more sharply for strike prices on one side of the asset return than on the other. This smile shape is consistent with a non zero correlation (see exhibit 1).

EXHIBIT 5. — *Expected Average Variance and Instantaneous Variance Versus Theoretical ATM Implied Variance*



From Exhibit 1, it can be noticed that an important feature of an at-the-money implied volatility is its relative independence of the correlation. It is worthy to note that, in a Gram-Sharmlier series expansion framework, Navatte-Villa [1999] have already shown that an important feature of an at-the-money implied volatility is its relative independance with the skewness that one can identify to the correlation in our particular framework. Therefore, it seems reasonable to think that the procedure applied in the previous section can be used in this special case.

In the Hull-White [1987] settings the expected of the average variance is taken under the historical probability. In this special case we have at least two choices : we compare at-the-money implied variances of the theoretical option prices (calculated via Heston's Fourier Transform using two sets of parameters) with the two corresponding expected average variances, $\tilde{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$ and $\hat{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$. The at-the-money implied variances and the two corresponding expected average variances are listed in Table 3. From Table 3, it can be inferred that there is a close resemblance between these three variances, although, it can be noticed that for a negative correlation,

$\hat{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$ is better than $\tilde{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$. In the case of a positive correlation, the reverse seems true. More simulations not presented here come to the same conclusion.

TABLE 3. — *Expected Average Variance Versus Theoretical ATM Implied Variance*

$\sigma^2 = 0.25, \vartheta = 0.25, \nu = 0, r = 5\%, T-t = 1$ month

ρ	$\kappa = 4, \gamma = 0.4$			$\kappa = 8, \gamma = 0.8$		
	$\hat{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$	σ_I^2	$\tilde{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$	$\hat{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$	σ_I^2	$\tilde{E}\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$
-0.5	0.24814	0.248602	0.25	0.246657	0.246266	0.25
-0.25	0.249067	0.248856	0.25	0.24832	0.246744	0.25
0	0.25	0.249126	0.25	0.25	0.247233	0.25
0.25	0.250937	0.249412	0.25	0.251697	0.247733	0.25
0.5	0.25188	0.249716	0.25	0.25341	0.248248	0.25

In this case,

$$\sigma_{I,i}^2 = \hat{\vartheta} + \left(\sigma_i^2 - \hat{\vartheta}\right)\frac{1-e^{-\hat{\kappa}\tau}}{\hat{\kappa}\tau} \Leftrightarrow \sigma_i^2 = \hat{\vartheta} + \left(\sigma_{I,i}^2 - \hat{\vartheta}\right)\frac{\hat{\kappa}\tau}{1-e^{-\hat{\kappa}\tau}},$$

where $\hat{\kappa} = \tilde{\kappa} - \rho\gamma, \hat{\vartheta} = \frac{\kappa\vartheta}{\hat{\kappa}}$. σ_t^2 is therefore an affine function of the implied volatility, and the log-likelihood function is :

$$\mathcal{L}\left(\sigma_{I,n}^2,\ldots,\sigma_{I,0}^2;\Theta\right) = \sum_{i=1}^{n}\left\{-\ln\left|J_i\right| + \ln f\left(\hat{\sigma}_i^2 \left|\hat{\sigma}_{i-1}^2;\Theta\right.\right)\right\},$$

where $J_i = \frac{1-e^{-\hat{\kappa}\tau}}{\hat{\kappa}\tau}$.

TABLE 4. — Estimators based on Market Volatility Index MONEP-VX1
$\rho = -0.3527$

	Value	t - *stat*
κ	5.35169	3.60719
ϑ	0.064342	4.23741
γ	0.855741	29.6810
$\mathcal{L}$	3537.38	

CONCLUSION

The MONEP Market Volatility Index (*VX1*) is an average of CAC 40 option (*PX1*) implied volatilities. On the basis of findings reported in the study of Fleming-Ostdiek-Whaley [1995] for the CBOE Market Volatility Index (*VIX*) and Moraux-Navatte-Villa [1999] for the MONEP one, a maximum likelihood estimation procedure on a *VX1* time series is applied to estimate the volatility process of stochastic volatility models. While the Renault-Touzi's [1996] statistical iterative procedure of filtering (of the latent volatility process) and estimation (of its parameters) failed to provide estimates of the parameters of the unobservable latent volatility process, we exploited Feinstein's [1992] research which demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of the option. In that case, since implied volatility is used in the same spirit as yield to maturity on the bond market, we applied direct maximum likelihood statistical inference on our analysis as Pearson-Sun [1994] and Duan [1994] had done in the case of interest rates. Finally, the so-called leverage effect have been taken into account.

NOTES

(1) The authors thank all the participants of the International Conference of the French Finance Association (AFFI) at Lille-July 1998, in particular Patrice Poncet, Serge Darolles, Vincent Lacoste and Jean Luc Prigent. This paper has also benefitted of helpful comments from the participants of the CREST seminar and more precisely Christian Gourieroux and Jean Paul Laurent. We are also grateful for suggestion form to an anonymous referee.

(3) See Duffie-Kan [1996] for more details.

(4) A complete and simple description is available from the MONEP WEB-site :
http://issy.integra.fr:8080/monep/top_frames.htx?d=http://www.monep.fr/monep1_navig_111.htm.

(5) Renault and Touzi (1996) showed that, when volatility was stochastic but uncorrelated with changes in the spot price, Black and Scholes implied volatility was lowest for at-the-money forward (strike price equal the forward underlying asset price or $x = 0$), increasing for both in-the-money (call's strike lower than the forward underlying

asset price or $x > 0$) and out-of-the-money (call's strike greater than the forward underlying asset price or $x < 0$) options. This pattern has been referred to the « volatility smile », so named for the appearance of a graph with implied volatility on the vertical axis and the option strike price on the horizontal axis.

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