On Perpetual American Strangles

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This article analyzes perpetual American strangles with no recourse to advanced numerical techniques. Our analytical approach rests on an analogy with asymmetric rebates of double knock-out barrier options. The optimal exercise policy is modeled by a couple of boundaries that simultaneously solve a system of two nonlinear equations. Numerical investigations then highlight salient features of American strangles and compare them with portfolios of options that may be used as proxies. Overall, results show that these latter are significantly upward biased in terms of prices and that, more dramatically, they lead the holder to exercise inappropriately.

Strangles and straddles are classical ways to build volatility strategies. Compared to other options positions, they are even really popular. Chaput and Ederington [2005] document, for instance, that they account for about 80% of all volatility trades on the Eurodollar options market of the CME. Strangles are formed by holding simultaneously a long position in a European put option and a long position in a European call option. Both options of this portfolio have the same underlying asset and the same expiration date, but strike prices are most often different (that of the put option being the smaller one); otherwise, strangles are termed straddles.

Traditional strangles are European style positions that can only be exercised at expiration but Chiarella and Ziogas [2005] consider that "*there exists a market for option portfolios comprised of American options.*" They defend, however, another form of American strangles that can be exercised before expiration (in the best interest of holders) *and* that collapse once the decision to exercise is made. Such contracts only allow a single exercise, which terminates the contract. They are very similar to their European counterparts except in their expected expiration date.

The economic rationale behind their suggestion is not much discussed in their article but one argues that a "two-in-one" American contract is preferable to a straight portfolio for (at least) a couple of reasons. First of all, the self-closing mechanism prevents potential sellers from being exposed to the risk of two successive and unfavorable exercises. And, second, these American strangles are cheaper than their synthetic counterparts. Both features are clearly of importance from a market perspective. Beyond that point, American strangles also extend the set of real options available to explain and model corporate decisions to invest or to finance and all decisions that imply a corridor of inaction.

American options are sometimes considered more challenging to price than European contracts because they require solving simultaneously both a valuation problem and the optimal exercise issue. American strangles are, in this view, rather representative because they involve a couple of early exercise boundaries that depend on one another. A simple way to

capture the American feature of strangles consists in building a portfolio made of two individual American options. Of course, this is inappropriate because the obtained synthetic position is not self-closing as American strangles are supposed to be. Portfolios of American individuals are actually biased proxies of American strangles. They are more expensive and cannot model the optimal exercise policy either.

American strangles have mainly been analyzed with the help of advanced numerical techniques. Typically, Alobaidi and Mallier [2002] examine straddles with Laplace transforms and some techniques described in Alobaidi and Mallier [2000]. These authors succeed in finding quasi-closed form solutions that cannot, however, be inverted analytically. Their approach rests on the performance of numerical inversion methods, but they provide no guidelines on the technique to use. Chiarella and Ziogas [2005] derive expressions for American strangles with finite expiration by exploiting Fourier transforms and the decomposition technique suggested by Kim [1990], Jacka [1991], and Carr, Jarrow, and Myneni [1992]. These expressions depend on two nonconstant exercise boundaries that simultaneously obey a system of two equations. Chiarella and Ziogas [2005] demonstrate that this system can be resolved with the help of a numerical scheme similar to that used for solving Volterra integral equations.

While completing this article, I found other connections with Gerber and Shiu [1994a] and Gerber and Shiu [1994b]. Gerber and Shiu [1994a] study the pricing of various perpetual American options by means of the Esscher transform and the optional sampling (stopping) theorem, while Gerber and Shiu [1994b] bridge the gap between perpetual American strangles and Russian options. Gerber and Shiu [1994a] were actually the first to derive a pricing formula for American strangles, but they provide neither insights nor comments on the price properties, the optimal exercise policy (and related exercise boundaries), or the hedging problem. By contrast, Chiarella and Ziogas [2005] investigate properties of short-term American strangles numerically and offer comparisons with portfolios of individual American options.

This article analyzes perpetual American strangles and studies both their price properties and exercise boundaries. It completes the work of Chiarella and Ziogas [2005] by considering the long end of the maturity spectrum. As a good starting point for the article, one can note that prices of perpetual American strangles do not suffer from time-value decay (because of their infinite expiration). Perpetuity implies that the optimal exercise policy is time-independent or, equivalently, that exercise boundaries are constant through time. Assuming that such boundaries are known, holders of strangles receive, at the first hitting time, a known and deterministic pay-off. The main idea of this article is that perpetual American strangles may be viewed as asymmetric rebates of double knock-out barrier options with constant barriers, infinite maturity, and specific payoffs (just as standard perpetual American call or put options can be regarded as rebates of perpetual knock-out barrier options). The first time the underlying asset reaches one of the early exercise boundaries is critical for the analysis. The reason is that the Laplace transform of its distribution provides an expression for the discount factor to use. However, these two early exercise boundaries are endogenous and characterized by a system of two nonlinear equations. Computing them simultaneously is the main challenge of the approach.

One can then derive various analytical formulas useful for pricing, hedging, and managing perpetual American strangles. Closed-form formulas are provided for the delta, the gamma, the minimum price, and the expected exercise time of perpetual American strangles. Numerical investigations emphasize salient features of these parameters. Following Chiarella and Ziogas [2005], properties are contrasted with those of portfolios made of individual American vanilla options. Essentially, the analysis reveals that using proxy portfolios can be misleading. They are biased upward in terms of prices and, more dramatically, can lead to suboptimal decisions to exercise. One even highlights circumstances where investors decide to exercise while American strangles are worth their minimum value.

THE FRAMEWORK AND MOTIVATION

This article considers a financial market à la Black, Scholes, and Merton, where both a risky asset (the stock) and a zero-coupon bond are traded in continuous time. The market is supposed perfect and complete and the riskless interest rate is constant. There exists an unique risk-neutral probability measure $\mathbb{Q}$ such that any discounted price process is a martingale. Under this unique subjective probability, the stock price process is assumed to be correctly described by:

$$dS_t = (r - \delta)S_t dt + \sigma S_t dB_t \qquad (1)$$

$$S_0 = S_0$$

where r is the instantaneous risk-free rate, δ the dividend rate, σ the volatility, and B a standard Brownian motion. It is well-known that prices of perpetual American options, denoted by G, solve the standard (time-independent) PDE:

$$(r-\delta)S\frac{\partial G}{\partial S}+\frac{1}{2}\sigma^2 S\frac{\partial^2 G}{\partial S^2}-rG=0 \qquad (2)$$

General solutions of Equation (2) may be written $G(S)=A_1S^{\lambda_1}+A_2S^{\lambda_2}$ where

$$\lambda_i=\frac{1}{\sigma^2}\left(-\left(r-\delta-\frac{1}{2}\sigma^2\right)-(-1)^i\sqrt{\left(r-\delta-\frac{1}{2}\sigma^2\right)^2+2r\sigma^2}\right),\quad i=1,2 \qquad (3)$$

are such that $\lambda_2<0<\lambda_1$. Both A_1 and A_2 are obtained with limit conditions that characterize the considered American options. The exercise boundary and the exercise price can be denoted by S_G^* and K_G, respectively. Because the optimal exercise of perpetual American options does not depend on time-to-expiration, the exercise boundary S_G^* is constant through time. Prices of American call and put options depend on exogenous parameters so that they can be denoted by $C(S, K_C, r, \delta, \sigma)$ and $P(S, K_P, r, \delta, \sigma)$ respectively or, to simplify, $C(S, K_C)$ and $P(S,K_P)$. Standard conditions for call options with strike price K_C then yield to:

$$C(S_C^*,K_C)=S_C^*-K_C \qquad (4)$$

$$\lim_{S\to S_C^*}\frac{\partial C(S,K_C)}{\partial S}=1 \qquad (5)$$

Those associated with perpetual American put options give

$$P(S_C^*,K)=K_P-S_P^* \qquad (6)$$

$$\lim_{S\to S_P^*}\frac{\partial P(S,K_P)}{\partial S}=-1 \qquad (7)$$

Equations (4) and (6) are sometimes referred as value-matching conditions, whereas (5) and (7) are high-contact conditions. Each system of Equations (4, 5) and (6, 7) forms the smooth-pasting conditions. Solving (2, 4, 5) and (2, 6, 7) yields, respectively, to $C(S,K_C)=(S_C^*-K_C)(\frac{S_0}{S_C^*})^{\lambda_1}$ and $P(S,K_P)=(K_P-S_P^*)(\frac{S_0}{S_P^*})^{\lambda_2}$ where $S_C^*=\frac{\lambda_1}{\lambda_1-1}K_C$ and $S_P^*=\frac{\lambda_2}{\lambda_2-1}K_P$. Hence, exercise boundaries are deduced from r, σ, and δ. Standard perpetual American call options can be regarded as rebates of perpetual up-and-out barrier options whose barrier is S_C^* and whose pay off is $S_C^*-K_C$ in case of disappearance. Similarly, standard perpetual American put options can be regarded as rebates of perpetual down-and-out barrier options whose barrier is S_P^* and whose pay-off is $K_P-S_P^*$. Here, $(\frac{S_0}{S_C^*})^{\lambda_1}$ and $(\frac{S_0}{S_P^*})^{\lambda_2}$ are the discount functions of the respective payoffs. If T_C denotes the first time the stock price reaches S_C^*, the (call option) exercise boundary, then one has $(\frac{S_0}{S_C^*})^{\lambda_1}=E^{Q}[e^{-rT_C}]$. This expression means that $(\frac{S_0}{S_C^*})^{\lambda_1}$ is the Laplace transform of the probability density function of T_C.

Perpetual American strangles are strategies written on a single underlying asset S and equipped with a put-side strike and a call-side strike. These strike prices are denoted by K_P and K_C, respectively. American strangles involve a couple of exercise boundaries too. Because considered contracts are perpetual, these two boundaries are constant. One denotes by L and H the lower and upper exercise boundaries so that the set $[L, H]$ forms the continuation region (the corridor of inaction) whereas $[0, L]\cup[H, \infty]$ forms the so-called exercise regions. It is clear that the strike prices and exercise boundaries obey $L<K_P<S_0<K_C<H$. The full expression of the time 0 – price of the American strangle price may be written $A(S, K_P, K_C, r, \delta, \sigma)$, since exercise boundaries are functions of the exogenous parameters. But some parameters are omitted and the time 0 – price is simply denoted $A(S, K_P, K_C)$.

Viewed as a function of the underlying price process S, the price of perpetual American strangles solves Equation (2) with, in addition, four limit conditions:

$$A(H,K_P,K_C)=H-K_C \qquad (8)$$

$$A(L, K_P, K_C) = K_P - L \tag{9}$$

$$\lim_{S \to H} \frac{\partial A(S, K_P, K_C)}{\partial S} = 1 \tag{10}$$

$$\lim_{S \to L} \frac{\partial A(S, K_P, K_C)}{\partial S} = -1 \tag{11}$$

The price of American strangles hence solves simultaneously the smooth-pasting conditions related to the put option and those associated with the call option (recalled in 4, 5, 6, 7). Most of previous studies on American strangles (or straddles) solve some equivalent systems of equations with Laplace or Fourier transforms. One exception is Gerber and Shiu [1994a, b] who developed a pure probabilistic approach. The rest of our article instead exploits an analogy with double barrier options.

PRICING AND HEDGING PERPETUAL AMERICAN STRANGLES

This section presents and exploits the main idea of this article that a perpetual American Strangle can be viewed as an asymmetric rebate of a perpetual double knock-out barrier option (whose knock-out thresholds are constant). Barriers of the double knock-out barrier option play the role of early exercise boundaries L and H. One assumes, for a while, that these barriers are exogenous, i.e., known. The rebate is also supposed to pay $(H - K_C)$ if the upper barrier H is first reached and $(K_P - L)$ if the lower threshold L is first touched. This rebate is termed asymmetric because there is no reason that $(H - K_C)$ equals $(K_P - L)$. Denoting by T_L and T_H the first times that S hits L and H, respectively, one has: $T_L = \inf\{t \geq 0 \mid S_t = L\}$ and $T_H = \inf\{t \geq 0 \mid S_t = H\}$. The first time the underlying asset reaches a barrier is then given by $T_{L,H} = T_L \wedge T_H$. This is also the first time the underlying assets exits from the corridor $[L, H]$.

The price of the rebate of a perpetual double knock-out barrier option can be written:

$$\begin{aligned} &R_{DBO}(S, L, H, K_P, K_C) \\ &\quad = E^{\mathbb{Q}}[\exp(-rT_{L,H})[(H - K_C)1_{\{T_H < T_L\}} + (K_P - L)1_{\{T_L < T_H\}}]] \end{aligned}$$

Conditionally on $\{T_H < T_L\}$, one has $T_{L,H} = T_H$. As a result, the price becomes:

$$\begin{aligned} &R_{DBO}(S, L, H, K_P, K_C) \\ &\quad = (H - K_C)E^{\mathbb{Q}}[\exp(-rT_H) \mid T_H < T_L]\mathbb{Q}[T_H < T_L] \\ &\qquad + (K_P - L)E^{\mathbb{Q}}[\exp(-rT_L) \mid T_L < T_H]\mathbb{Q}[T_L < T_H] \end{aligned} \tag{12}$$

When both L and H are known (as assumed here), these expectations can be computed thanks to available analytical results on Brownian motions and exotic options (summarized in Appendix A). Determination of such exercise boundaries are, however, part of the pricing challenge of American strangles as revealed by the following proposition.

Proposition 1 *For $L < S_0 < H$, the price of perpetual American strangles is given by*:

$$\begin{aligned} A(S, K_P, K_C) &= (H - K_C) \times \frac{\left(\frac{S_0}{L}\right)^{\lambda_1} - \left(\frac{S_0}{L}\right)^{\lambda_2}}{\left(\frac{H}{L}\right)^{\lambda_1} - \left(\frac{H}{L}\right)^{\lambda_2}} \\ &\quad + (K_P - L) \times \frac{\left(\frac{S_0}{H}\right)^{\lambda_2} - \left(\frac{S_0}{H}\right)^{\lambda_1}}{\left(\frac{H}{L}\right)^{-\lambda_2} - \left(\frac{H}{L}\right)^{-\lambda_1}} \end{aligned} \tag{13}$$

where the exercise boundaries H and L solve the smooth-pasting conditions:

$$\begin{cases} \left.\dfrac{\partial A(S, K_P, K_C)}{\partial S}\right|_{S=L} = -1 \\ \left.\dfrac{\partial A(S, K_P, K_C)}{\partial S}\right|_{S=H} = 1 \end{cases} \tag{14}$$

All proofs are given in Appendix A, but some remarks deserve to be made here. First of all, the Expression (13) alone is the closed-form formula for pricing a rebate of a double knock-out barrier option that promises the above asymmetric payoffs. Second, associated with the system of Equations (14), this equation corroborates results of Gerber and Shiu [1994a], although they use another approach and other notation. Third, one can check that, when S is set to H (or L), the Expression (13) becomes Expression (8) (Eq. (9)). Finally, the system of Equations (14) does not involve the price of the underlying asset. This means that exercise boundaries are not only constant through time but also independent of S. The complete formula requires the simultaneous deter-

mination of L and H. This is done by solving the two Equations of (14) simultaneously. Once again, the parallel with a perpetual double knock-out barrier option turns out to be fruitful because these equations involve the delta of the rebate (i.e., $\frac{\partial R_{DBO}(S,L,H,K_P,K_C)}{\partial S}$). This delta is derived in the appendix and provided in Equation (15). Note that solving Equations (14) requires a numerical approach.

To manage American strangles, one needs to use hedging parameters. The following propositions highlight the Delta and the Gamma of perpetual American strangles. They are identical to the hedging parameters of the rebate of the perpetual double knock-out barrier option except that fixed barriers are endogenously chosen.

Proposition 2 *For $L < S_0 < H$, the delta of American strangles is given by*:

$$\Delta_{A(S,K_P,K_C)} = \frac{(H-K_C)}{S}\frac{\lambda_1(\frac{S}{L})^{\lambda_1} - \lambda_2(\frac{S}{L})^{\lambda_2}}{(\frac{H}{L})^{\lambda_1} - (\frac{H}{L})^{\lambda_2}} + \frac{(K_P-L)}{S}\frac{\lambda_2(\frac{S}{H})^{\lambda_2} - \lambda_1(\frac{S}{H})^{\lambda_1}}{(\frac{H}{L})^{-\lambda_2} - (\frac{H}{L})^{-\lambda_1}} \quad (15)$$

where H and L solve the system of Equations (14).

Proposition 3 *For $L < S_0 < H$, the gamma of American strangles is given by*:

$$\Gamma_{A(S,K_P,K_C)} = \frac{(H-K_C)}{S^2}\frac{\lambda_1(\lambda_1-1)(\frac{S}{L})^{\lambda_1} - \lambda_2(\lambda_2-1)(\frac{S}{L})^{\lambda_2}}{(\frac{H}{L})^{\lambda_1} - (\frac{H}{L})^{\lambda_2}} + \frac{(K_P-L)}{S^2}\frac{\lambda_2(\lambda_2-1)(\frac{S}{H})^{\lambda_2} - \lambda_1(\lambda_1-1)(\frac{S}{H})^{\lambda_1}}{(\frac{H}{L})^{-\lambda_2} - (\frac{H}{L})^{-\lambda_1}} \quad (16)$$

where H and L solve the system of Equations (14).

Finally, one can derive analytically 1) the value of S for which the price of a perpetual American strangle is minimum ($S_{\min}$) and 2) the associated contract price.

Proposition 4 *The price of perpetual American strangles, viewed as a function of S, is minimum at*:

$$S_{\min}(K_P,K_C) = \sqrt[\lambda_1-\lambda_2]{\frac{\lambda_2}{\lambda_1}\frac{\Phi(H-K_C)L^{-\lambda_2} - \Psi(K_P-L)\ H^{-\lambda_2}}{\Phi(H-K_C)L^{-\lambda_1} - \Psi(K_P-L)H^{-\lambda_1}}}$$

with $\Psi = ((\frac{H}{L})^{\lambda_1} - (\frac{H}{L})^{\lambda_2})$, $\Phi = (\frac{H}{L})^{-\lambda_2} - (\frac{H}{L})^{-\lambda_1}$ and where H and L solve the system of Equations (14). And the contract value is worth $A(S_{\min}(K_P, K_C), K_P, K_C)$.

Note that a couple of other expressions for $S_{\min}$ are given in the appendix. All of them reveal that $S_{\min}(K_P, K_C)$ is a not-so-simple function of the exogenous parameters. It also demonstrates that there is no reason that $S_{\min} = (K_P, K_C)$ necessarily equals $\frac{K_P+K_C}{2}$, as it may be thought. As is typical, the graph plotted in Chiarella and Ziogas [2005, p. 34] to present American strangles suggests a symmetric and centered behavior of the price around $\frac{K_P+K_C}{2}$. By contrast, simulations highlight that prices are essentially (and most of the time) asymmetric around $S_{\min}$ (K_P, K_C).

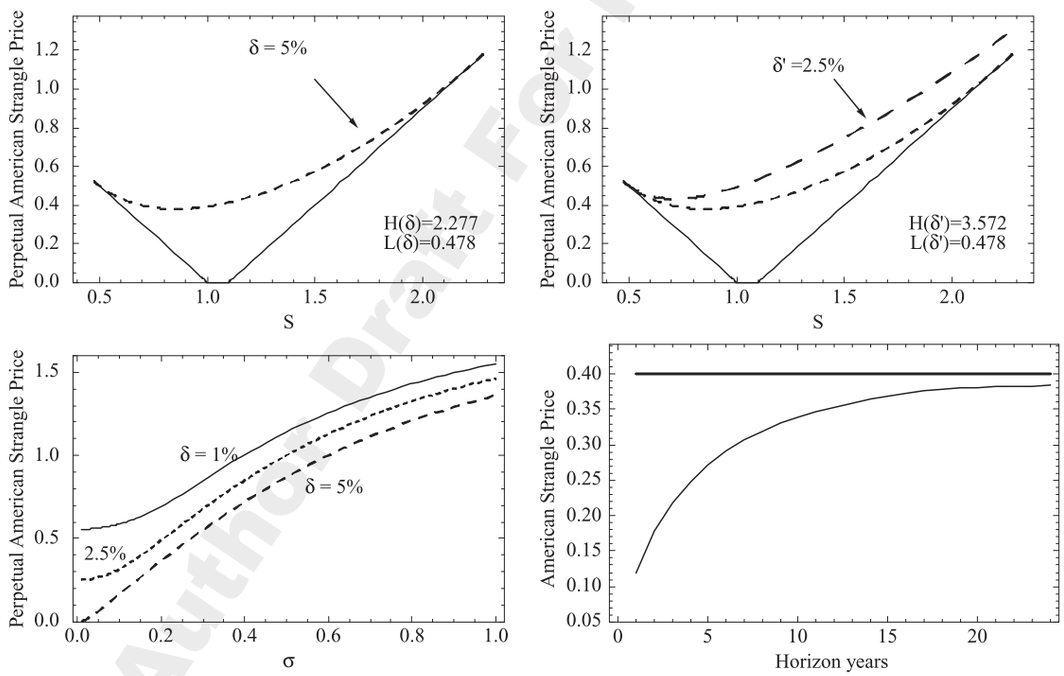
ANALYZING PERPETUAL AMERICAN STRANGLES

To analyze perpetual American options, one considers base-case parameters whose values are mostly inspired by the work of Chiarella and Ziogas [2005]. Strike prices of the American strangle are set to $K_C = 1$ and $K_P = 1.1$. The contract is written on an underlying asset whose volatility is set to $\sigma = 20\%$. Remaining parameters are the interest rate r and the dividend payout rate δ. Before setting values, the dividend deserves a few additional comments. The dividend notoriously plays a key role in both the pricing of American options and the optimal exercise policy. It is well-known that, for example, American calls on a non-dividend-paying stock should never be exercised early. If this American call is moreover perpetual, it won't be exercised—ever. Dividend payout may change this, but the holder still does not exercise if the promised dividend is too small (see Merton [1973, section 3] for a discussion of this point). In other words, the dividend has to be significant to justify early exercise of American calls. For American strangles, things are rather different because these contracts contain both a "put side" and a "call side." Holders of American strangles have another reason to exercise early the "call side" in addition to dividend capture. This is the threat that the optionality to exercise the "call side" is forfeited if the lower exercise boundary is first hit. Overall, one can expect the prices of perpetual American strangles and the optimal exercise policy to be rather sensitive to the dividend rate. For ordinary stocks, it is quite rare to have $\delta > r$ and dividend

yields of 10% and more are virtually nonexistent. These are the reasons why, in most simulations in this article, r is set to 5% and δ ranges from 0% to r. Note that I nevertheless conduct additional simulations with $\delta > r$ to investigate properties of the model. Of course, they are available upon request.

Exhibit 1 displays properties of perpetual American strangles in term of prices. The upper graphs plot, as functions of S, prices of perpetual American strangles (dashed or dotted lines) as well as the intrinsic value of the strategy (solid lines). The dividend payout rate δ is set to $\delta = 5\%$ or $\delta = 2.5\%$. The current value of the underlying asset S_0 ranges from the lower exercise threshold to the upper one (with $\delta = 5\%$). As explained in previous sections, they are obtained by solving the system of Equations (14). One finds $L(\delta) = L(5\%) \approx 0.478$ and $H(\delta) = H(5\%) = 2.277$, respectively. Corresponding exercised American strangles are worth $K_p - L(\delta) \approx 1 - 0.478 = 0.522$ and $H(\delta) - K_C \approx 2.277 - 1 = 1.277$, respectively. It is worth stressing that L and H are changing because the contextual environment is not the same. Mainly, both graphs show that the price of American strangles first decreases and then increases with the value of S. The upper-left graph also highlights how the price converges to the intrinsic value of the strategy as the price of the underlying assets tends to the exercise boundaries. The upper-right graph shows that the price of perpetual American strangles increases with a lower dividend payout rate although the lower exercise boundary remains approximately the same (see Exhibit 2 for additional details). This result is in line with intuition because, as the dividend rate narrows, the (risk-neutral) expected rate of growth of the underlying asset $(r - \delta)$ increases. The holder of an American strangle can therefore expect a larger (and more probable) payoff from a

EXHIBIT 1
Prices of Perpetual American Strangles



Notes: The upper graphs plot prices of perpetual American strangles (dotted lines) as well as the intrinsic value of the strategy (solid lines) as functions of S. Strike prices are given by $K_p = 1$ and $K_C = 1.1$, other fixed exogenous parameters are $r = 5\%$, $\sigma = 20\%$, and the dividend payout rate δ is set to either 5% or 2.5%. The underlying asset value ranges from $L(5\%) \approx 0.478$ to $H(5\%) \approx 2.277$. The lower-left graph plots prices of perpetual American strangles as functions of the volatility for three different values of the dividend payout rate. The solid line stands for $\delta = 1\%$, the other ones correspond to $\delta = 2.5\%$ and $\delta = 5\%$. The underlying asset value is $S = 1.05$. The lower-right graph plots prices of a couple of American strangles as functions of the time to expiration when $\delta = 5\%$. This graph illustrates how a numerical price estimate of the American strangle with finite expiration can converge to the price of the perpetual American strangle (the bold line).

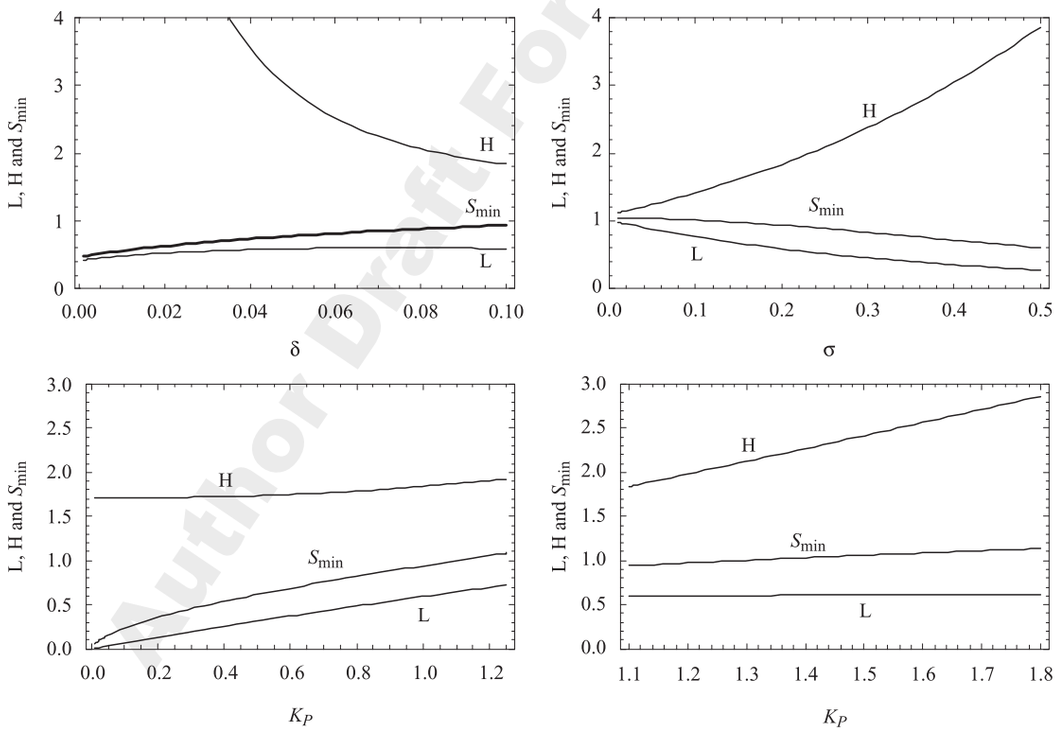
decision to exercise on the "call side." Finally, as predicted earlier, there is no special symmetric behavior of the price around the minimum $S_{\min}(K_P, K_C, L, H)$.

The lower graphs consider other interesting features of the prices of the American strangles. The lower-left graph plots prices of perpetual American strangles as functions of the volatility for three different values of the dividend payout rate. The solid line stands for $\delta = 1\%$, next ones corresponding to $\delta = 2\%$ and $\delta = 4\%$. The underlying asset value is $S_0 = 1.05$ (i.e., exactly between strikes). The underlying asset's volatility ranges from 0% to 100% which is clearly largely sufficient. For this range of values, the price of American strangles is a strictly increasing function of the underlying asset's volatility. There is nevertheless a slight change of convexity that is less pronounced for high levels of dividend. The lower-right graph compares the exact price of a perpetual American strangle (bold line) with a numerical price estimate of an American strangle with finite expiration. Both prices are plotted against time to expiration. This graph illustrates how American strangles with finite expiration converge to perpetual American strangles.

Exhibit 2 examines the way the exercise boundaries as well as the minimum behave as a function of δ, σ, K_P and K_C. We must recall that reaching boundaries means exercising and extinguishing the strategy. The upper-left graph mainly highlights that the dividend estimates are central for the determination of the optimal policy. Keeping in mind that the level of interest rate is set to 10% and is strictly larger than the dividend payout rate δ, the continuation region strictly narrows. This means that *ceteris paribus*, the expected time to exercise is shorter when the underlying asset has a significant dividend payment. Additional simulations show, however, that this region may enlarge when δ is greater than the spot rate. The present graph reveals that the minimum $S_{\min}(K_P, K_C, L, H)$ is an

EXHIBIT 2

Minima and Exercise Boundaries of Perpetual American Strangles



Notes: These graphs plot lower and upper boundaries (L and H) as well as $S_{\min}$ that are associated with perpetual American strangles as functions of 1) the dividend payout rate (upper-left graph), 2) the volatility (upper-right graph), 3) the put-side strike (lower-left graph), and 4) the call-side strike (lower-right graph). Parameters are set to: $r = 10\%$, $\delta = 10\%$, $\sigma = 20\%$, $K_P = 1$, and $K_C = 1.1$.

increasing function of the dividend rate. $S_{\min}$ is close to the lowest threshold when the dividend rate is moderate and gets closer to the largest boundary as δ increases. The upper exercise boundary appears quite large for small dividend rates. This is a direct consequence of the upside component of the contract. It decreases sharply, however, as δ increases and tends to the level of the spot interest rate. In few words, larger the dividend, better the stocks. And more interesting is the early exercise of the call-side. By comparison, the lower exercise boundary behaves moderately with respect to the dividend. This result confirms a comment made on Exhibit 1. The upper-right graph highlights that the upper exercise boundary is quite influenced by the level of the volatility. In line with intuition, the holder has a strong incentive to postpone the (call side) exercise when the volatility is high. The lower graphs show that the call-side (put-side) strike influenced almost linearly the upper (lower) exercise boundary.

To manage the optimal exercise policy of perpetual American strangles or to appreciate the overall effect of a change in exercise boundaries, it can be useful and informative to compute the expected first exercise time and the probability of exercising at the call side or the put side.

Proposition 5 *Let's consider L and H the exercise boundaries that solve the system of Equations (14). The expected optimal exercise time is then given by*:

$$E^{\mathsf{Q}}[T_{L,H}] = \frac{1}{r-\delta-\frac{\sigma^2}{2}}\left[\frac{1-(\frac{S}{L})^{\lambda_1+\lambda_2}}{1-(\frac{H}{L})^{\lambda_1+\lambda_2}}\ln\left(\frac{H}{L}\right)-\ln\left(\frac{S}{L}\right)\right] \tag{17}$$

$$= \frac{1}{r-\delta-\frac{\sigma^2}{2}}\left[\frac{S^{\lambda_1+\lambda_2}-L^{\lambda_1+\lambda_2}}{H^{\lambda_1+\lambda_2}-L^{\lambda_1+\lambda_2}}\ln\left(\frac{H}{L}\right)-\ln\left(\frac{S}{L}\right)\right] \tag{18}$$

whereas the risk-neutral probability of an exercise at the upper boundary H is:

$$\mathsf{Q}[T_H < T_L] := \frac{(\frac{S}{L})^{-(\lambda_1+\lambda_2)}-1}{(\frac{S}{L})^{-(\lambda_1+\lambda_2)}-(\frac{H}{L})^{-(\lambda_1+\lambda_2)}} = \frac{S^{-(\lambda_1+\lambda_2)}-L^{-(\lambda_1+\lambda_2)}}{S^{-(\lambda_1+\lambda_2)}-H^{-(\lambda_1+\lambda_2)}}$$

This probability is obviously equal to 1 minus the risk-neutral probability of an early exercise at the lower boundary.

A close examination of Expression (17) concludes that the expected exercise time is hump-shaped with respect to the underlying asset S. Interestingly, the maximum expected exercise time is not found at the minimum $S_{\min}(K_P, K_C, L, H)$. Simple calculus gives analytical expression for the maximum expected exercise time. Denoting by $f(S) := E^{\mathsf{Q}}[T^S_{H,L}]$, one finds $f'(S) \propto \frac{(\lambda_1+\lambda_2)S^{(\lambda_1+\lambda_2)}}{(H^{(\lambda_1+\lambda_2)}-L^{(\lambda_1+\lambda_2)})S}\ln(\frac{H}{L})-\frac{1}{S}$, which is zero at $S_{T^{\max}_{L,H}} = \sqrt[\lambda_1+\lambda_2]{\frac{1}{\lambda_1+\lambda_2}\frac{H^{\lambda_1+\lambda_2}-L^{\lambda_1+\lambda_2}}{\ln H-\ln L}}$. Clearly, $S_{T^{\max}_{L,H}}$ is not equal to $S_{\min}$.

Exhibit 3 displays numerical results related to previous simulations and obtained for different values for the underlying asset and different levels of volatility. The put-side strike is $K_P = 1$, while the call-side strike is $K_C = 1.5$. Chosen values for the underlying asset are 0.75, 1, 1.25, 1.50, 1.75, and $S_{\min}$. When $S = 1$ ($S = 1.5$), the American strangle is at-the-money on the put side (call side). Other parameters are $r = 5\%$ and $\delta = 5\%$. Very interestingly, one finds that increasing the volatility increases the range between exercise boundaries but decreases the expected time of early exercise. At $S = 1$, the expected time to early exercise is about 21% greater when the volatility is 20% than when the volatility is 30%. The effect of the volatility on the probability of an exercise at the upper boundary depends on the current value of the underlying asset. When S is fairly small (and close to the lower boundary), the probability of reaching the upper bounds is an increasing function of the volatility. When the value of S is larger and closer to the upper threshold, increasing the volatility lowers such a probability. Finally, one can check that the expected exercise time admits a maximum and that it is not observed at $S_{\min}$.

COMPARISON TO PORTFOLIOS OF AMERICAN INDIVIDUALS

As pointed out by Chiarella and Ziogas [2005], American strangles can be thought of as simple portfolios of American individuals. But this section highlights how this view may be misleading. The first concern is that such a portfolio is not extinguished once the first option is exercised. The second issue is on the pricing. A proxied value for an American strangle written on S is, for $S_0 \in [S_P^*; S_C^*]$:

$$\begin{aligned}\mathrm{Ptf}(S_0) &= C(S_0, K_C) + P(S_0, K_P)\\ &= (S_C^* - K_C)\left(\frac{S_0}{S_C^*}\right)^{\lambda_1} + (K_P - S_P^*)\left(\frac{S_0}{S_P^*}\right)^{\lambda_2}\end{aligned} \tag{19}$$

EXHIBIT 3

Values, Minima, and Exercise Boundaries of a Perpetual American Strangle

	$\sigma = 20\%$			$\sigma = 30\%$		
	$L = 0.492$			$L = 0.344$		
	$H = 2.966$			$H = 4.159$		
	$S_{\min} = 0.949$			$S_{\min} = 0.751$		
S	$A(S, K_P, K_C)$	$E[T_{L,H}]$	Pr (in %)	$A(S, K_P, K_C)$	$E[T_{L,H}]$	Pr (in %)
0.75	0.366	11.7	10.43	0.513	11.4	10.64
1.00	0.345	17.0	20.53	0.539	14.2	17.19
1.25	0.380	19.1	30.64	0.603	15.5	23.74
1.50	0.454	19.1	40.74	0.695	15.9	30.30
1.75	0.560	17.8	50.85	0.809	15.7	36.85
$S_{\min}$	0.343	16.2	18.46	0.513	11.4	10.65

Notes: Missing parameters are $K_P = 1$, $K_C = 1.5$, $r = 5\%$, $\delta = 5\%$. Pr stands for the probability that the American strangle is exercised upward (Pr = $Pr[T_H < T_L]$).

Exercise boundaries S_C^* and S_P^* solve Equations (5) and (7) which express high-contact conditions. The set $[S_P^*; \infty]$ forms the continuation region for the individual put option, whereas $[0; S_C^*]$ is that associated with the call option. Because $K_P < K_C$, it is clear that parameters obey $S_P^* < K_P < S_0 < K_C < S_C^*$. Examination of Equation (19) reveals that the price admits a minimum.

Both S_C^* and S_P^* can be compared to H and L respectively. The economic rationale behind the optimal exercise of American strangle positions suggests that the call-side boundary S_C^* is lower than H whereas the put-side boundary S_P^* is higher than L. Holders of individual American options hence exercise earlier than holders of American strangles, meaning that holders of American strangles can expect to receive a greater value at the exercise time. Because the early exercise of an individual American option does not extinguish the other (contrary to the American strangle position where the entire position is closed), holders of the portfolio of individuals still have an option to exercise later. Hence, to a certain extent, the value obtained from exercising American strangles also covers the opportunity cost of not exercising at the other boundary later.

For values of the underlying asset such that $S_P^* < S < S_C^*$, delta and gamma are:

$$\Delta_{\text{ptf}}(S) = \frac{(S_C^* - K_C)}{S}\lambda_1\left(\frac{S}{S_C^*}\right)^{\lambda_1} + \frac{(K_P - S_P^*)}{S}\lambda_2\left(\frac{S}{S_P^*}\right)^{\lambda_2} \quad (20)$$

$$\Gamma_{\text{ptf}}(S) = \frac{(S_C^* - K_C)}{S^2}\lambda_1(\lambda_1 - 1)\left(\frac{S}{S_C^*}\right)^{\lambda_1} + \frac{(K_P - S_P^*)}{S^2}\lambda_2(\lambda_2 - 1)\left(\frac{S}{S_P^*}\right)^{\lambda_2} \quad (21)$$

and similarities and differences with the delta (15) and the gamma (16) can be observed. Interestingly, the second component of Δ_{ptf} is negative (because λ_2 is), implying that there can exist a minimum value for the proxy portfolio. To demonstrate this, it is sufficient to set the Equation (20) to zero. The minimum of the portfolio is actually given by:

$$S_{\min}^{\text{ptf}} = \sqrt[\lambda_1 - \lambda_2]{\frac{\lambda_2}{\lambda_1}\frac{(S_P^* - K_P)(S_P^*)^{-\lambda_2}}{(S_C^* - K_C)(S_C^*)^{-\lambda_1}}}$$

Note that this square root is well defined because λ_2 and $(S_P^* - K_P)$ are both negative. Outside the interval $[S_P^*; S_C^*]$ the held position is not clear because it depends on the settlement chosen by the holder when he or she exercised his or her option. Observing $S > S_C^*$ means that the call has been exercised earlier and the portfolio now contains eventually the non-exercised American put option plus either some stocks (bought at the price K_C)

or some cash. To avoid such a dilemma, we will restrict our attention to the region delimited by $[S_P^*; S_C^*]$. Finally, the expected first exercise time is given by:

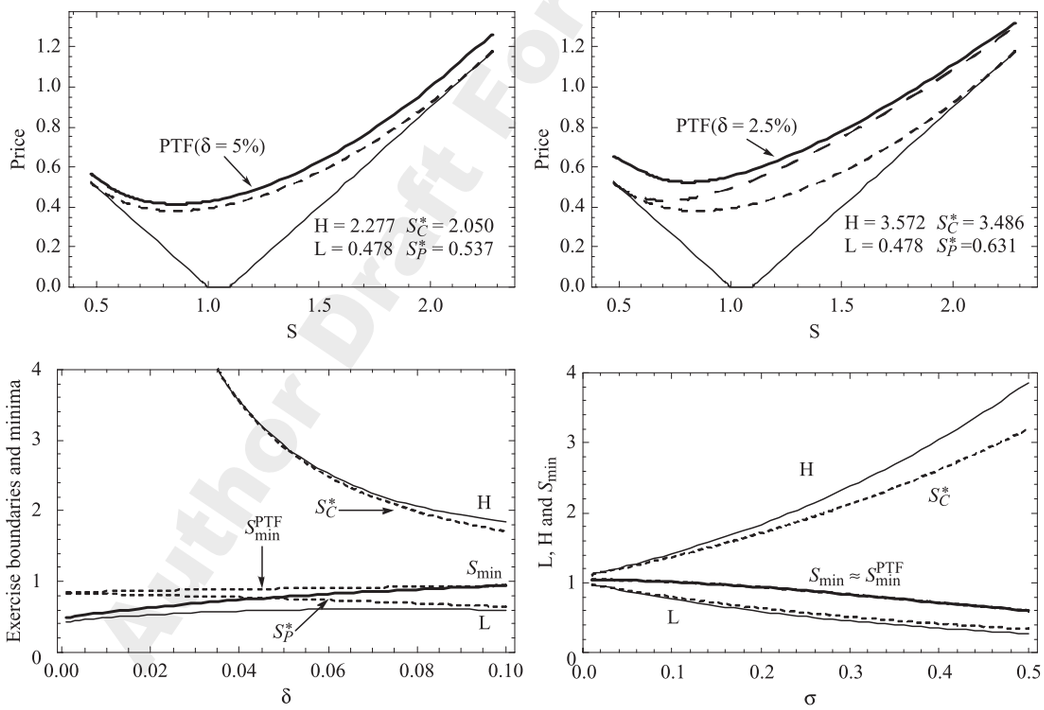
$$\min\{E^{\mathsf{Q}}[T_H], E^{\mathsf{Q}}[T_L]\} = \begin{cases} \dfrac{\ln(H/S)}{r-\delta-\frac{\sigma^2}{2}} & \text{if } r-\delta-\dfrac{\sigma^2}{2} > 0 \\ \dfrac{\ln(L/S)}{r-\delta-\frac{\sigma^2}{2}} & \text{if } r-\delta-\dfrac{\sigma^2}{2} < 0 \end{cases} \tag{22}$$

This equation must be contrasted to Equation (17).

The four graphs of Exhibit 4 reconsider the upper graphs of Exhibit 1 and those of Exhibit 2. Exogenous parameters are therefore the same as in Exhibits 1 and 2. In the upper graphs, we have added the price of the portfolios described by (19). As expected, the price of the portfolio is larger than that of the American strangle. Actually, the difference may be substantial. A couple of factors can explain this difference. The first one is obviously the difference of pricing formulas. The second factor is indirect and concerns the values of exercise boundaries considered in each case. Each graph displays its respective upper and lower bounds. One can observe that H is systematically greater than S_C^* while L is systematically lower than S_P^*. This result suggests that proxying the strangles exercise policy by the exercise policy of the portfolio of individuals is clearly suboptimal as it can precipitate the decision to exercise. In some context, however, the difference remains moderate inducing only small opportunity costs. This is the point explored in the two lower graphs. These graphs add to the upper graphs of Exhibit 2, the exercise boundaries of the portfolio of individuals S_C^* and S_P^* as well as the minimum $S_{\min}^{\text{Ptf}}$. The lower-left graph first focuses on the dividend effect. For the chosen parameters, it documents only small differences between exercise

EXHIBIT 4

Perpetual American Strangles versus Prices of Portfolios of American Individuals



Notes: These graphs compare portfolios of individual American options to perpetual American strangles in term of prices (upper graphs) and in term of exercise boundaries and minima (lower graphs). Except for the bold lines, the upper graphs are identical to those of Exhibit 1. Except dotted lines, lower graphs are identical to the upper graphs in Exhibit 2. The parameters are those of Exhibits 1 and 2. Note, in the lower-left graph, that $S_{\min}$ and S_P^* cross one another.

boundaries. The main point here is that $S_P^*(\delta)$ (viewed as a function of the dividend) cross $S_{\min}(\delta)$; this means that a trader who manages his American strangle as if it was a portfolio of individuals may exercise his strategy at the minimum value of the strategy! Such circumstances are actually rare and special, but they are useful pedagogically to illustrate the significant bias that exists when people infer American strangle properties from those of a portfolio of individuals. Further simulations have been done in different contexts for robustness checks. In agreement with Chiarella and Ziogas [2005], one finds that the divergence depends on the relative value of δ with respect to r. When $\delta >> r$, the call-side boundaries (H and S_C^*) are divergent, while when $\delta << r$, only the put-side boundaries are. When $\delta \approx r$ (as is the case here), there exist divergences in both sides but of smaller magnitude. The lower-right graph explores a potential volatility effect. And it shows how early exercise boundaries of American strangles diverge from those of a proxy portfolio as the volatility varies. It can be observed that the only important change concerns the call-side boundaries. Assuming that $\sigma = 40\%$, H is 16% larger than S_C^* and the difference represents 38% of the strike price! For comparison, Chiarella and Ziogas [2005] find for American strangles with finite and moderate maturity that the difference never exceeds 10% of the strike price. Overall, this graph shows that the gap between boundaries may be rather substantial in the perpetual case.

Exhibit 5 explores numerically some results evidenced in Exhibit 4. This exhibit must be viewed in comparison with Exhibit 3. It can be observed how far the portfolio of American options is from the American strangles in terms of early exercise boundaries, minimum, prices, expected time of first exercise, and probability of exercising the call-side first. The expected time of exercise estimated for a portfolio of individual options is higher than its counterpart. But it decreases with the volatility too. The expected times are longer and longer as the underlying increases. There is no maximum. Detailed inspection of numerical figures shows that, as the volatility increases, this difference in prices increases. When S = 1.25 and σ = 20%, the value of the portfolio is about 5.25% higher than that of the corresponding strangle. When σ = 30%, the relative difference is more than 6.5%, meaning that the pricing bias is at least 23.8% higher in a context of high volatility.

Finally, the graphs in Exhibit 6 plot values of parameters commonly used for hedging purposes. Note that we restrict our attention to the domain of definition of these parameters, that is, $[S_P^*; S_C^*]$ and $[L; H]$. The upper graphs of Exhibit 6 mainly show that the risk perceived by the portfolio of American options is slightly exacerbated compared to that of the American strangles. The lower graphs plot deltas and gammas as functions of the dividend rate, and they suggest that the alternative values of delta and gamma can be rather different. Note, finally, that simulations on a potential volatility effect, not reported here, show that they behave quite similarly with respect to this dimension.

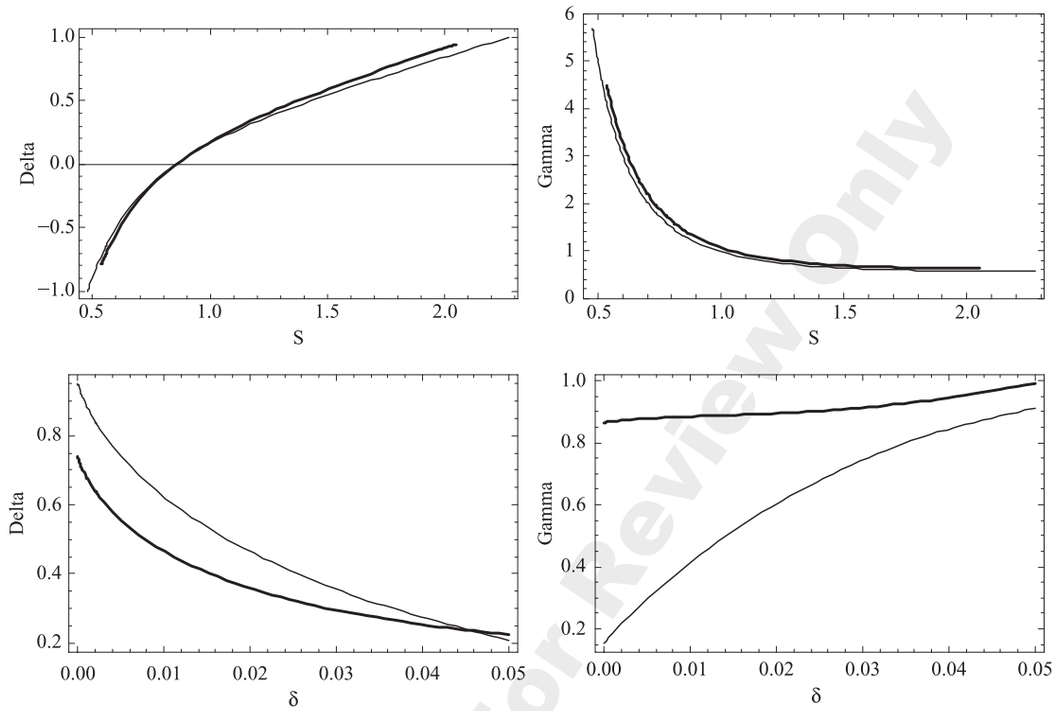
EXHIBIT 5

Values, Minima, and Exercise Boundaries of a Portfolio of Perpetual American (Individual) Options

	$\sigma = 20\%$			$\sigma = 30\%$		
	$S_P^* = 0.537$			$S_P^* = 0.400$		
	$S_C^* = 2.795$			$S_C^* = 3.750$		
	$S_{\min}^{\text{ptf}} = 0.955$			$S_{\min}^{\text{ptf}} = 0.758$		
S	Ptf	$E[\tau]$	Pr (in %)	Ptf	$E[\tau]$	Pr (in %)
0.75	0.390	16.7	9.45	0.548	13.9	10.45
1.00	0.366	31.1	20.51	0.574	20.4	17.91
1.25	0.402	42.3	31.59	0.641	25.3	25.37
1.50	0.479	51.4	42.66	0.737	29.4	32.84
1.75	0.589	59.1	53.73	0.856	32.8	40.30
$S_{\min}$	0.365	28.8	18.52	0.548	14.2	10.69

Notes: The missing parameters are $K_P = 1$, $K_C = 1.5$, $r = 5\%$, $\delta = 5\%$. Pr stands for the probability that the first option to be exercised is the call ($Pr = Pr[T_C < T_P]$).

EXHIBIT 6
Hedging Parameters



Notes: These graphs compare the delta and the gamma of perpetual American strangles (thin lines) to those associated with portfolios of individual American options (bold lines). The upper graphs plot hedging parameters as functions of S, lower graphs consider the dividend rate. Base case parameters are used, and the value of the underlying asset either ranges from L to H or it is set to 1.05.

APPLICATIONS TO PERPETUAL AMERICAN STRADDLES

A standard straddle strategy may be viewed as a European strangle whose call-side exercise price and put-side exercise price have the same value ($K_C = K_P$). Long straddles are classical strategies that essentially bet on increasing volatility levels. Short straddles are short volatility trades. The following corollary adapts some previous analytical results to analyze American straddles by simply setting $K_C = K_P = K$. This somewhat straightforward result contrasts with the numerical approach of Chiarella and Ziogas [2005] where it is necessary to set $K_C = K_P + \varepsilon$ for a small ε.

Corollary 6 For $L < S_0 < H$, the prices of perpetual American straddles are given by:

$$A(S, K) = A(S, K, K) \tag{23}$$

$$= H \times \Pi_1 - L \times \Pi_2 + K \times (\Pi_2 - \Pi_1) \tag{24}$$

where $\Pi_1 = \frac{\left(\frac{S_0}{L}\right)^{\lambda_1} - \left(\frac{S_0}{L}\right)^{\lambda_2}}{\left(\frac{H}{L}\right)^{\lambda_1} - \left(\frac{H}{L}\right)^{\lambda_2}}$ *and* $\Pi_2 = \frac{\left(\frac{S_0}{H}\right)^{\lambda_2} - \left(\frac{S_0}{H}\right)^{\lambda_1}}{\left(\frac{H}{L}\right)^{-\lambda_2} - \left(\frac{H}{L}\right)^{-\lambda_1}}$. *Delta and gamma are given by* $\Delta_{A(S,K)} = \Delta_{A(S,K,K)}$ *and* $\Gamma_{A(S,K)} = \Gamma_{A(S,K,K)}$, respectively. *The minimum is given by* $S_{\min}(K, H, L) = S_{\min}(K, K, H, L)$. *Note that all these expressions that involve L and H depend on the unique strike price K.*

CONCLUSION

This article continues the work of Chiarella and Ziogas [2005] in investigating American strangles, the main difference being that I considered perpetual options rather than finite expiry options. American strangles can only be early exercised once, so that the holder must choose to receive the call or the put payoff but not both. The main idea is that the rebate of a double knock-out barrier option (with well-designed payoffs) succeeds in duplicating the standard American position. The key subtlety is that there

are *two* unknown early exercise boundaries that need to be calculated. This requires the numerical solution of two nonlinear simultaneous equations. In this article, I priced perpetual American strangles by modeling the early exercise policy, and I analyzed 1) the relative location of these exercise boundaries to each other and to the underlying asset price, 2) the lowest option value; 3) the expected times to expiry, and 4) the probability of taking the call option or put option payoffs. I also investigated the effect of changing the dividend yield and the asset volatility. The analysis is not straightforward because of the nonlinearity. Finally, I pointed out drawbacks of modeling American strangles with a portfolio of an American put option and an American call option. In particular, this way to proceed can lead the holders to exercise their self-closing position at the lowest possible value.

APPENDIX A

This appendix presents key materials to prove propositions in the core text. The most important tools are related to the first exit time of a Brownian motion from the corridor $[L, H]$. Readers interested in double barrier options may consult Douady [1998] or any advanced textbooks dealing with options, such as Wilmott, Dewynne, and Howison [1993] or Wilmott [2006]. It must be remembered, however, that the pricing of American strangles also requires solving the optimal exercise issue.

Let $Z = (Z_t)_t$ be the new process defined by $Z_t = \frac{1}{\sigma}\ln(S_t/S_0)$. One has $Z_t = \nu t + B_t$ with $\nu = \frac{1}{\sigma}(r - \delta - \frac{\sigma^2}{2})$, and the risky asset price may be rewritten $S_t = S_0 e^{\sigma Z_t}$. The process Z is notoriously key in option-pricing theory because the Girsanov theorem can yield to a new probability for which Z is a standard Brownian motion. One denotes by $l = \frac{1}{\sigma}\ln(L/S_0) < 0$ and $h = \frac{1}{\sigma}\ln(H/S_0) > 0$ the lower bound and the upper bound associated with Z (of course, $h - l = \frac{1}{\sigma}\ln(H/L) > 0$). T_k stands for the first time Z attains the threshold k, whereas $T_{l,h}$ correspond to the first time time Z reaches one of the two barriers. It is well-known (see e.g., Douady [1998]) that the expected first hitting time is

$$E^{\mathbf{Q}}[T_{l,h}] = \frac{[\exp(-2\nu l) - 1]h + [1 - \exp(-2\nu h)]l}{\nu[\exp(-2\nu l) - \exp(-2\nu h)]} \tag{25}$$

and that

$$E^{\mathbf{Q}}[\exp(-rT_{l,h}) \mid T_h < T_l]\,\mathbf{Q}[T_h < T_l] = \frac{\exp(\nu h)\sinh\left[\sqrt{\nu^2 + 2r}\,|l|\right]}{\sinh\left[\sqrt{\nu^2 + 2r}\,(h - l)\right]} \tag{26}$$

$$E^{\mathbf{Q}}[\exp(-rT_{l,h}) \mid T_l < T_h]\,\mathbf{Q}[T_l < T_h] = \frac{\exp(\nu l)\sinh\left[\sqrt{\nu^2 + 2r}\,h\right]}{\sinh\left[\sqrt{\nu^2 + 2r}\,(h - l)\right]}$$

where the probability of reaching the upper bound first is given by

$$\mathbf{Q}[T_h < T_l] = 1 - \mathbf{Q}[T_h > T_l] = \frac{\exp(-2\nu l) - 1}{\exp(-2\nu l) - \exp(-2\nu h)} \tag{27}$$

Of course $E^{\mathbf{Q}}[\exp(-rT_{l,h}) \mid T_h < T_l] = E^{\mathbf{Q}}[\exp(-rT_h) \mid T_h < T_l]$ because, conditionally on $\{T_h < T_l\}$, $T_{l,h} = T_h$. It is interesting to have a closer look at the probability of reaching the upper bound first. To this end, let's denote by $Y = (Y_t)_t$ the process defined by $Y_t = \exp((-2\nu)B_t - \frac{1}{2}(-2\nu)^2 t) := \exp(-2\nu Z_t)$. By construction, the process Y is a martingale ($E^{\mathbf{Q}}[Y_t \mid Y_s] = Y_s$), such that $E^{\mathbf{Q}}[Y_t] = 1$, $\forall t$. The expected value of Y at time $T_{l,h}$ also obeys

$$\begin{aligned} E^{\mathbf{Q}}[Y_{T_{l,h}}] &= E^{\mathbf{Q}}\left[\exp(-2\nu h)1_{\{T_h < T_l\}} + \exp(-2\nu l)1_{\{T_l < T_h\}}\right] \\ &= \exp(-2\nu l) + [\exp(-2\nu h) - \exp(-2\nu l)]\,\mathbf{Q}[T_h < T_l] \end{aligned}$$

The probability of reaching h first ($\mathbf{Q}[T_h < T_l]$) is therefore given by

$$\mathbf{Q}[T_h < T_l] = \frac{\exp(-2\nu l) - 1}{\exp(-2\nu l) - \exp(-2\nu h)}$$

and the probability of reaching l first is of course $\mathbf{Q}[T_l < T_h] = \frac{1 - \exp(-2\nu h)}{\exp(-2\nu l) - \exp(-2\nu h)}$. In order to evaluate $E^{\mathbf{Q}}[T_{l,h}]$, one can use a similar trick. The expected value for Z at time $T_{l,h}$ is, on the one hand,

$$E^{\mathbf{Q}}[Z_{T_{l,h}}] = E^{\mathbf{Q}}[B_{T_{l,h}} + \nu T_{l,h}] = \nu E^{\mathbf{Q}}[T_{l,h}]$$

and it obeys, on the other hand,

$$\begin{aligned} E^{\mathbf{Q}}[Z_{T_{l,h}}] &= E^{\mathbf{Q}}[h1_{\{T_h < T_l\}} + l1_{\{T_l < T_h\}}] \\ &= h\mathbf{Q}[T_h < T_l] + l\mathbf{Q}[T_l < T_h] \\ &= l + (h - l)\,\mathbf{Q}[T_h < T_l] \end{aligned}$$

Hence, one has

$$E^{\mathbf{Q}}[T_{l,h}] = \frac{1}{\nu}(l + (h - l)\,\mathbf{Q}[T_h < T_l])$$

Proposition 1 is deduced from the analogy between a perpetual American strangle and the rebate of a perpetual double

barrier option (with constant barriers) designed as in the core text. The price of this rebate is actually given by:

$$\begin{aligned} R_{DBO}(S,L,H,K_P,K_C) &= (H-K_C)E^{\mathbf{Q}}[\exp(-rT_H)\,|\,T_H<T_L] \\ &\quad \times \mathbf{Q}[T_H<T_L]+(K_P-L) \\ &\quad \times E^{\mathbf{Q}}[\exp(-rT_L)\,|\,T_L<T_H] \\ &\quad \times \mathbf{Q}[T_L<T_H] \end{aligned} \tag{28}$$

Results of this appendix imply

$$\begin{aligned} & E^{\mathbf{Q}}[\exp(-rT_H)\,|\,T_H<T_L]\,\mathbf{Q}[T_H<T_L] \\ &\quad = E^{\mathbf{Q}}[\exp(-rT_{l,h}\,|\,T_h<T_l)]\,\mathbf{Q}[T_h<T_l] \\ &\quad = \left(\frac{S_0}{H}\right)^{-\frac{r-\delta-\frac{\sigma^2}{2}}{\sigma^2}} \frac{\left(\frac{S_0}{L}\right)^{\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}} - \left(\frac{S_0}{L}\right)^{-\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}}}{\left(\frac{H}{L}\right)^{\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}} - \left(\frac{H}{L}\right)^{-\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}}} \end{aligned}$$

$$\begin{aligned} & E^{\mathbf{Q}}[\exp(-rT_L)\,|\,T_L<T_H]\,\mathbf{Q}[T_L<T_H] \\ &\quad = E^{\mathbf{Q}}[\exp(-rT_{l,h})\,|\,T_l<T_h]\,\mathbf{Q}[T_l<T_h] \\ &\quad = \left(\frac{S_0}{L}\right)^{-\frac{r-\delta-\frac{\sigma^2}{2}}{\sigma^2}} \frac{\left(\frac{H}{S_0}\right)^{\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}} - \left(\frac{H}{S_0}\right)^{-\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}}}{\left(\frac{H}{L}\right)^{\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}} - \left(\frac{H}{L}\right)^{-\frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}}} \end{aligned}$$

Setting $a=\frac{\nu}{\sigma}=\frac{r-\delta-\frac{\sigma^2}{2}}{\sigma^2}$ and $b=\frac{\sqrt{(r-\delta-\frac{\sigma^2}{2})^2+2r\sigma^2}}{\sigma^2}$ and plugging these two expressions into Equation (28) yield to:

$$\begin{aligned} R_{DBO}(S,L,H,K_P,K_C) &= (H-K_C)\left(\frac{S_0}{H}\right)^{-a}\frac{(\frac{S_0}{L})^b-(\frac{S_0}{L})^{-b}}{(\frac{H}{L})^b-(\frac{H}{L})^{-b}} \\ &\quad + (K_P-L)\left(\frac{S_0}{L}\right)^{-a}\frac{(\frac{S_0}{H})^{-b}-(\frac{S_0}{H})^{b}}{(\frac{H}{L})^b-(\frac{H}{L})^{-b}} \end{aligned}$$

This equation can now be rewritten as

$$\begin{aligned} R_{DBO}(S,L,H,K_P,K_C) &= (H-K_C)\frac{(\frac{S_0}{L})^{-a}}{(\frac{H}{L})^{-a}}\frac{(\frac{S_0}{L})^b-(\frac{S_0}{L})^{-b}}{(\frac{H}{L})^b-(\frac{H}{L})^{-b}} \\ &\quad + (K_P-L)\frac{(\frac{S_0}{H})^{-a}}{(\frac{L}{H})^{-a}}\frac{(\frac{S_0}{H})^{-b}-(\frac{S_0}{H})^{b}}{(\frac{H}{L})^b-(\frac{H}{L})^{-b}} \\ &= (H-K_C)\frac{(\frac{S_0}{L})^{b-a}-(\frac{S_0}{L})^{-b-a}}{(\frac{H}{L})^{b-a}-(\frac{H}{L})^{-b-a}} \\ &\quad + (K_P-L)\frac{(\frac{S_0}{H})^{-b-a}-(\frac{S_0}{H})^{b-a}}{(\frac{H}{L})^{b+a}-(\frac{H}{L})^{-b+a}} \end{aligned}$$

As shown in the case of standard American options, it is a common practice in option theory to consider $\lambda_i=-a-(-1)^i b$ (see Equation (3)). One then obtains:

$$\begin{aligned} R_{DBO}(S,L,H,K_P,K_C) &= (H-K_C)\frac{(\frac{S_0}{L})^{\lambda_1}-(\frac{S_0}{L})^{\lambda_2}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}} \\ &\quad + (K_P-L)\frac{(\frac{S_0}{H})^{\lambda_2}-(\frac{S_0}{H})^{\lambda_1}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}} \end{aligned}$$

This is the Equation (13) we are looking for.

Proposition 2 is essentially derived by 1) computing the first derivative of the price of the rebate of the perpetual double barrier option with respect to the price of the underlying asset S_0 and by 2) plugging into it the optimal exercise boundaries. Proposition 3 follows strictly the same line for the second derivative. An argument among others is that exercise boundaries do not depend on the price of the underlying asset. Actually, four expressions of the form $(\frac{S}{L})^{\lambda}$ have to be derived once and then twice with respect to S. For the delta presented in the core text, use repeatedly $\frac{\partial}{\partial S}[(\frac{S}{L})]^{\lambda}=\lambda(\frac{1}{L})^{\lambda}S^{\lambda-1}=\frac{\lambda}{S}(\frac{S}{L})^{\lambda}$. For the gamma, use $\frac{\partial^2}{\partial S^2}[(\frac{S}{L})^{\lambda}]=\frac{\partial}{\partial S}[\lambda(\frac{1}{L})^{\lambda}S^{\lambda-1}]=\lambda(\lambda-1)(\frac{1}{L})^{\lambda}S^{\lambda-2}=\lambda(\lambda-1)\frac{1}{S^2}(\frac{S}{L})^{\lambda}$ (instead of differentiating the final expression for delta). Finally, the delta of the rebate is

$$\begin{aligned} \Delta_R(S) = \frac{\partial R_{DBO}(S)}{\partial S} &= \frac{(H-K_C)}{S}\frac{\lambda_1(\frac{S}{L})^{\lambda_1}-\lambda_2(\frac{S}{L})^{\lambda_2}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}} \\ &\quad + \frac{(K_P-L)}{S}\frac{\lambda_2(\frac{S}{H})^{\lambda_2}-\lambda_1(\frac{S}{H})^{\lambda_1}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}} \end{aligned} \tag{29}$$

To derive Proposition 4, one searches for $S_{\min}$ such that $\Delta_R(S_{\min})=0$. Equating Equation (29) to zero yields

$$\begin{aligned} & \frac{(H-K_C)}{S_{\min}}\frac{\lambda_1(\frac{S_{\min}}{L})^{\lambda_1}-\lambda_2(\frac{S_{\min}}{L})^{\lambda_2}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}} \\ &\quad + \frac{(K_P-L)}{S_{\min}}\frac{\lambda_2(\frac{S_{\min}}{H})^{\lambda_2}-\lambda_1(\frac{S_{\min}}{H})^{\lambda_1}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}}=0 \end{aligned} \tag{30}$$

Different expressions may be derived from this equation, depending on chosen simplifications. One mainly focuses here on that exposed in the core text. Dividing the numerator by $S_{\min}^{\lambda_2}$ and the denominator by $S_{\min}$ yield to:

$$\begin{aligned} & (H-K_C)\frac{\lambda_1 S_{\min}^{\lambda_1-\lambda_2}L^{-\lambda_1}-\lambda_2 L^{-\lambda_2}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}} \\ &\quad + (K_P-L)\frac{\lambda_2 H^{-\lambda_2}-\lambda_1 S_{\min}^{\lambda_1-\lambda_2}H^{-\lambda_1}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}}=0 \end{aligned}$$

Inspection of the left-hand side of the equation reveals a linear function of the form $\text{slope} \times S_{\min}^{\lambda_1-\lambda_2} + \text{const}$ where

$$\text{slope} = (H-K_C)\frac{\lambda L^{-\lambda_1}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}} - (K_P-L)\frac{\lambda_1 H^{-\lambda_1}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}}$$

$$\text{const} = (K_P-L)\frac{\lambda_2 H^{-\lambda_2}}{(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}} - (H-K_C)\frac{\lambda_2 L^{-\lambda_2}}{(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}}$$

The solution is therefore $S_{\min}^{\lambda_1-\lambda_2} = \frac{-\text{const}}{\text{slope}}$, and simplifications give

$$S_{\min}^{\lambda_1-\lambda_2} = \frac{\lambda_2}{\lambda_1}\frac{[(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}](H-K_C)L^{-\lambda_2}-[(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}](K_P-L)H^{-\lambda_2}}{[(\frac{H}{L})^{-\lambda_2}-(\frac{H}{L})^{-\lambda_1}](H-K_C)L^{-\lambda_1}-[(\frac{H}{L})^{\lambda_1}-(\frac{H}{L})^{\lambda_2}](K_P-L)H^{-\lambda_1}}$$

This is the one exposed in the core text. The initial Equation (30) provides many other expressions too. Factoring $S_{\min}^{\lambda_2}$ at the numerator and either $(\frac{H}{L})^{\lambda_2}$ or $(\frac{H}{L})^{-\lambda_1}$ at the denominator yield

$$\frac{(H-K_C)}{S_{\min}}\frac{S_{\min}^{\lambda_2}[\lambda_1 S^{\lambda_1-\lambda_2}L^{-\lambda_1}-\lambda_2 L^{-\lambda_2}]}{(\frac{H}{L})^{\lambda_2}[(\frac{H}{L})^{\lambda_1-\lambda_2}-1]} + \frac{(K_P-L)}{S_{\min}}\frac{S_{\min}^{\lambda_2}[\lambda_2 H^{-\lambda_2}-\lambda_1 S^{\lambda_1-\lambda_2}H^{-\lambda_1}]}{(\frac{H}{L})^{-\lambda_1}[(\frac{H}{L})^{-\lambda_2-\lambda_1}-1]} = 0$$

This in turn implies

$$\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}}[\lambda_1 S^{\lambda_1-\lambda_2}L^{-\lambda_1}-\lambda_2 L^{-\lambda_2}] + \frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}[\lambda_2 H^{-\lambda_2}-\lambda_1 S^{\lambda_1-\lambda_2}H^{-\lambda_1}] = 0$$

or

$$\lambda_1 S^{\lambda_1-\lambda_2}\left[L^{-\lambda_1}\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}} - H^{-\lambda_1}\frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}\right] - \lambda_2\left[L^{-\lambda_2}\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}} - H^{-\lambda_2}\frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}\right] = 0 \quad (31)$$

This equation can provide straighforwardly:

$$S_{\min} = {}^{\lambda_1-\lambda_2}\sqrt{\frac{\lambda_2}{\lambda_1}\frac{\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}}L^{-\lambda_2}-\frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}H^{-\lambda_2}}{\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}}L^{-\lambda_1}-\frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}H^{-\lambda_1}}}$$

Otherwise, after factoring $L^{-\lambda_1}$ in the first term and $H^{-\lambda_2}$ in the second term, one finds:

$$\lambda_1 L^{-\lambda_1}S^{\lambda_1-\lambda_2}\left[\frac{(H-K_C)}{(\frac{H}{L})^{\lambda_2}}-(K_P-L)\right] - \lambda_2 H^{-\lambda_2}\left[(H-K_C)-\frac{(K_P-L)}{(\frac{H}{L})^{-\lambda_1}}\right] = 0 \quad (32)$$

and

$$\lambda_1 L^{\lambda_2-\lambda_1}S^{\lambda_1-\lambda_2}[H^{-\lambda_2}(H-K_C)-(K_P-L)L^{-\lambda_2}] - \lambda_2 H^{\lambda_1-\lambda_2}[(H-K_C)H^{-\lambda_1}-(K_P-L)L^{-\lambda_1}] = 0$$

whose solution is

$$S_{\min} = LH^{\lambda_1-\lambda_2}\sqrt{\frac{\lambda_2}{\lambda_1}\frac{(H-K_C)H^{-\lambda_1}-(K_P-L)L^{-\lambda_1}}{(H-K_C)H^{-\lambda_2}-(K_P-L)L^{-\lambda_2}}}$$

After noting that $\frac{2v}{\sigma} = 2a = \lambda_1 + \lambda_2$, Proposition 5 is a direct consequence of Expressions (25) and (27):

$$\mathbf{Q}[T_H < T_L] = 1 - \mathbf{Q}[T_H > T_L] = \frac{(\frac{S_0}{L})^{-(\lambda_1+\lambda_2)}-1}{(\frac{S_0}{L})^{-(\lambda_1+\lambda_2)}-(\frac{S_0}{H})^{-(\lambda_1+\lambda_2)}}$$

$$E^{\mathbf{Q}}[T_{H,L}^S] = \frac{1}{r-\delta-\frac{\sigma^2}{2}}\left[\frac{1-(\frac{S}{L})^{\lambda_1+\lambda_2}}{1-(\frac{H}{L})^{\lambda_1+\lambda_2}}\ln\left(\frac{H}{L}\right)-\ln\left(\frac{S}{L}\right)\right]$$

ENDNOTE

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