Sensitivity Analysis of Credit Risk Measures in the Beta Binomial Framework

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Mixed binomial models (or Bernoulli mixture models) are common ways to model default risk in credit portfolios. To account for default dependency, these models assume that the common default probability is randomly distributed according to a *mixing* distribution. Credit portfolio managers usually find them very appealing because they are easy to simulate in Monte Carlo analysis and simple to calibrate on real data (Frey and McNeil [2003]). Actually, many standard industry models for managing credit portfolios are nothing else than specific Bernoulli mixture models.[1] The beta binomial approach plays a special role, nevertheless, as it often serves as a benchmark to assess the performance of others.

This article first reconsiders the beta binomial approach and introduces a new reparameterization of the beta mixing distribution. Both the expected default probability and the default correlation are favored as key input parameters.[2] Hereafter, this article will use the common default correlation for several reasons: 1) the expected default probability is often considered as fixed in homogenous credit portfolios; 2) the default correlation may vary for a given level of default risk (see Renault and Servigny [2004] for documented statistics);[3] 3) the literature does not make it clear how sensitive classical models and credit risk measures are to the level of default correlation.

Armed with this new parameterized mixing distribution, one can derive easy-to-implement analytical expressions that are very useful for analyzing the sensitivity of standard credit indicators to the default correlation.[4] Following standard practices, one mainly focuses on common credit risk measures, such as the credit at risk, the expected shortfall, and the tail function. Sensitivities and elasticities of these indicators are then studied with respect to the sole common default correlation (rather than the two statistical shape parameters of the distribution).

Numerical analysis shows that the correlation coefficient parameter plays an essential role. Interestingly, one finds that it impacts the considered credit risk measures quite differently. Sensitivities of the credit at risk and the tail function appear either positive or negative while that of the expected shortfall remains always positive. To highlight further this key role of the common default correlation, one examines the asymptotic tail functions associated to different tranches of CDOs. They show that the wealth of holders of the different tranches is differently influenced by the correlation parameter.

The next section presents the standard framework for analyzing a homogenous credit portfolio. The article then introduces the new reparameterization of the beta mixing distribution and analyzes homogenous credit portfolios. Further sections consider large portfolios and review standard credit risk indicators and then undertake the sensitivity analysis of these credit risk measures with respect to the default correlation.

THE STANDARD FRAMEWORK

We consider in this section a homogenous credit portfolio of N loans or bonds. In this article, homogeneity refers essentially to both the credit profile of borrowers and the design of credits. It is assumed that credit ratings are known and identical within the credit portfolio. The same is true for recovery rates or, equivalently, losses given default. By denoting by T the investment period and τ_i the default time of the ith borrower, the variable $X_i = 1_{\tau_i < T}$ plays the role of a default indicator. If face values are equal to 1, the value loss (suffered at the end of the investment period by the holder of the credit portfolio) is equal to the number of defaults. So this can be described by the sum of indicators $N_{\text{def}}(N) = \Sigma_{i=1}^{N} X_i$. Note that this latter assumption prevents tricky notations without modifying the salient feature of the credit risk modeling.

Mixed Binomial Models for Credit Risk Portfolios

Every loan or bond has the same rating, meaning that they share the same probability of default p. As a result, the above default indicators are identical Bernoulli distributed variables and the number of default is a binomial variable with parameters (N, p). More precisely, the random variable $N_{\text{def}}(N)$ takes values between 0 (no default) and N (all firms default) with a probability density described by

$$\Pr[N_{\text{def}}(N) = j] = \binom{N}{j} p^j (1-p)^{N-j}, \quad j \in \{0, 1, \ldots, N\} \tag{1}$$

where $\binom{N}{j} = C_j^N = \frac{N!}{j!(N-j)!}$ stands for the number of pairs of j defaults among the N borrowers. The cumulative density function is then given by $\Pr[N_{\text{def}}(N) \leq k] = \Sigma_{j=0}^{k} \Pr[N_{\text{def}}(N) = j]$, the mean loss is $E[N_{\text{def}}(N)] = N \times p$, and its variance $\sigma^2[N_{\text{def}}(N)] = N \times p \times (1-p)$. The average number of defaults is proportional to the number of borrowers, whereas its standard deviation is proportional to the square root of N.

The mixed binomial framework introduces dependence among default by letting the common default probability to be stochastic. If one assumes conditional independence of individual defaults (given the probability of default), then the probability of facing k defaults is given by:

$$\begin{aligned}\Pr[N_{\text{def}}(N) \leq k] &= E[\Pr[N_{\text{def}}(N) \leq k \mid p]] \\ &= \int_0^1 \sum_{j=0}^{k} \binom{N}{j} p^j (1-p)^{N-j} f(p)dp\end{aligned}$$

where f is the mixing distribution. Such a distribution is clearly central to modeling the default probability and the resulting dependence between defaults. It is also a good proxy for the (percentage) loss distribution of large homogenous credit portfolios. As mentioned previously, the beta mixing distribution is a classical way to randomize the default probability p. For the readers' convenience, it is useful to present a few results before introducing our own parameterization.

The Standard Beta Binomial Approach

The standard beta mixing distribution assumes that the probability density function of the default probability is well described by:

$$f(p;\alpha,\beta) := \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\int_0^1 p^{\alpha-1}(1-p)^{\beta-1}dp} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

where shape parameters α, β are positive real numbers and Γ is the gamma function. Properties of the beta distribution are well known. Its probability density function is humped, skewed, and leptokurtic. The α shape parameter controls the steepness of the hump, while the β parameter controls the fatness of the tail. The expected default probability and associated variance are respectively given by $E[\mathbf{p}] := \int_0^1 pf(p;\alpha,\beta)dp = \frac{\alpha}{\alpha+\beta}$ and $V[\mathbf{p}] = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$. The skewness is $Sk[\mathbf{p}] = \frac{2(\beta-\alpha)}{\sqrt{p}\sqrt{\alpha\beta}(\alpha+\beta+2)}$. The corresponding cumulative density function is known as the regularized incomplete beta function:

$$I_x(\alpha,\beta) := \frac{\int_0^x p^{\alpha-1}(1-p)^{\beta-1}dp}{\int_0^1 p^{\alpha-1}(1-p)^{\beta-1}dp} := \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)}$$

where $B(\alpha, \beta)$ and $B_x(\alpha, \beta)$ stand for the so-called beta function and incomplete beta function, respectively. Many useful identities and recurrence results exist on these functions and we refer to Abramowitz and Stegun [1972] for details. The incomplete beta distribution also admits useful relations with the generalized hypergeometric function since $B_x(\alpha,\beta) = \frac{1}{\alpha} x^{\alpha} {}_2F_1(\alpha, 1-\beta, \alpha+1; x)$.

From a credit management viewpoint, the dependence between default events is the second dimension of interest in a credit portfolio (the first one being the expected default probability). It is useful to emphasize the following result.

Proposition 1 *In the mixed beta binomial framework, the common default correlation of an homogenous credit portfolio is*

$$\text{cor}[X_i, X_j] = \frac{1}{1+\alpha+\beta} := \rho$$

Since α and β are strictly positive, $\rho > 0$.

Proof. There are different ways to demonstrate this result. The following proof is among the simplest ones. It is well known that

$$\text{cov}[X_i, X_j] = \text{cov}[E[X_i|\mathbf{p}], E[X_j|\mathbf{p}]] + E[\text{cov}[X_i, X_j|\mathbf{p}]]$$

Because of the conditional independence, the second term is null. In addition, $\text{cov}[E[X_i|\mathbf{p}], E[X_i|\mathbf{p}]]$ is equal to $\text{cov}[\mathbf{p}, \mathbf{p}] = V[\mathbf{p}]$ for $i \neq j$. Correlation definition then yields to

$$\text{cor}[X_i, X_j] = \frac{V[\mathbf{p}]}{E[\mathbf{p}](1-E[\mathbf{p}])} = \frac{E[\mathbf{p}^2] - E[\mathbf{p}]^2}{E[\mathbf{p}] - E[\mathbf{p}]^2}, \quad i \neq j \tag{2}$$

□

The existence of analytical results makes the beta binomial framework suitable for modeling homogenous credit portfolios. For known shape parameters (α, β), $E[\mathbf{p}]$, $V[\mathbf{p}]$, and ρ are easy to compute (with previous expressions) and properties of the beta distribution are well known (see Appendix A).

Dealing with two shape parameters, however, is not so comfortable from a management viewpoint. Beyond the possible lack of understanding, key indicators for credit portfolio behave differently as we change α and β. Typically, sensitivities of the expected default probability ($\frac{\partial E[\mathbf{p}]}{\partial \alpha} = \frac{\beta}{(\alpha+\beta)^2}$ and $\frac{\partial E[\mathbf{p}]}{\partial \beta} = -\frac{\alpha}{(\alpha+\beta)^2}$) are, respectively, positive and negative.

Because the default probability is essentially fixed in an homogenous portfolio, one can rewrite $\beta(\alpha) = \alpha \frac{1-E[\mathbf{p}]}{E[\mathbf{p}]}$ to limit such a complexity. And, in that case, $V[\mathbf{p}] = \frac{E[\mathbf{p}]^2(1-E[\mathbf{p}])}{\alpha+E[\mathbf{p}]}$ and $\text{Cor}[X_i, X_j] = \frac{1}{1+\frac{\alpha}{E[\mathbf{p}]}}$. Hence (given $E[\mathbf{p}]$) both the variance of $\mathbf{p}$ and the default correlation are decreasing functions of α. This article suggests a new approach that makes the beta distribution a function of the common default correlation between issuers. As far as we know, such a parameterization has not been exploited anywhere else.

A CORRELATION-BASED BETA MIXING DISTRIBUTION FOR HOMOGENOUS CREDIT PORTFOLIOS

Mixed beta binomial models may be viewed as functions of the common default probability and the common default correlation ρ. To see this, it is sufficient to note that results of the previous section yields to

$$\begin{cases} \alpha = E[\mathbf{p}]\dfrac{1-\rho}{\rho} \\ \beta = (1-E[\mathbf{p}])\dfrac{1-\rho}{\rho} \end{cases} \tag{3}$$

with $\rho > 0$ (otherwise defaults are uncorrelated and the setting is a straight or pure binomial model). The beta distribution can now be reparameterized as $f(p; \alpha, \beta) = \varphi(p; E[\mathbf{p}], \rho)$, and one can even go a step further because the mean probability of default $E[\mathbf{p}]$ is essentially constant in homogenous credit portfolios. For an homogenous credit portfolio, let's finally define

$$\psi_{E[\mathbf{p}]}(p; \rho) \equiv f\left(p; E[\mathbf{p}]\frac{1-\rho}{\rho}, (1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)$$

Such a parameterization allows one to rephrase in financial terms most of well known properties. For instance, by virtue of the proof of Proposition 1, the variance of the default probability is now a simple increasing function of the default correlation given by $V[\mathbf{p}] = \rho E[\mathbf{p}](1 - E[\mathbf{p}])$. So the variance first increases with $E[\mathbf{p}]$ from 0 to $\frac{\rho}{4}$ (obtained for $E[\mathbf{p}] = \frac{1}{2}$) and then decreases to zero as $E[\mathbf{p}]$ gets to one.[5] The skewness can be rewritten $Sk = 4\frac{\frac{1}{2}-E[\mathbf{p}]}{\sqrt{E[\mathbf{p}](1-E[\mathbf{p}])}}\frac{\sqrt{\rho}}{1+\rho}$, and it highlights that the distribution is symmetric for $E[\mathbf{p}] = \frac{1}{2}$. All expressions of the previous section can also be rewritten in terms of the sole correlation parameter. The probability that j credit(s) defaults in the portfolio becomes

$$\Pr[N_{\text{def}}(N) = j] = \int_0^1 \binom{N}{j} p^j (1-p)^{N-j} \psi_{E[\mathbf{p}]}(p;\rho)dp \quad (4)$$

To illustrate the key role of the common default correlation, Exhibit 1 compares graphically the mixed beta binomial distribution parameterized by the correlation coefficient with the binomial density given in Equation (1). Exhibit 1 considers an homogenous credit portfolio of $N = 100$ loans or bonds. The shadow probability density function corresponds to the straight binomial model (with $E[\mathbf{p}] = 10\%$ and $\rho = 0\%$). Other ones correspond to the reparameterized beta binomial model. Here again, $E[\mathbf{p}]$ is set to 10%, but the correlation parameter ρ is now equal to either 2.5% or 10% (left-hand and right-hand graphs, respectively). Clearly, the default correlation impacts distributions. Probability density functions with non-zero correlation appear skewed; their (right) tails are heavier than that of the independent case.

For completeness, Exhibit 2 provides the probability of k defaults within a portfolio of 10 assets ($\Pr[N_{\text{def}}(N) = k]$ with $N = 10$) for different values of default correlation given that the expected default probability is equal to 5% in all cases. The common default correlation ranges from about 0 to 10%. Such values are admissible in view of Table 5.2 of Renault and Servigny [2004]. These authors report, on the basis of "Standard & Poor's CreditPro" data, that the (one-year) default correlation within a given rating class is higher than 0% (AAA) and lower than 8.97% (CCC). These figures are estimated on observed defaults between 1981 and 2002. It must be noted furthermore that, as the default correlation increases, both the probability of no default in portfolio and the probability of the larger number of defaults increase. This is easily explained by the fact that, when the default correlation rises, underlying bonds or loans behave more and more similarly. Interestingly, one can observe that for $k = 2$, the probability of k defaults (as a function of ρ) is first increasing and then decreasing. This point is explored in the final section of this article.

We can also add results on the total number of defaults ($N_{\text{def}}(N)$) in the homogenous credit risk portfolio or (equivalently) on the loss rate in the portfolio $L(N)$, which is the proportion of default $\frac{N_{\text{def}}(N)}{N}$.

Proposition 2 *The total number of default among the N issuers verifies*

$$E[N_{\text{def}}(N)] = NE[\mathbf{p}] \quad (5)$$

$$V[N_{\text{def}}(N)] = [N + N(N-1)\rho]E[\mathbf{p}](1 - E[\mathbf{p}]) \quad (6)$$

EXHIBIT 1

Mixed Beta Binomial Distribution Parameterized by the Correlation Coefficient with the Binomial Density in Equation (1)

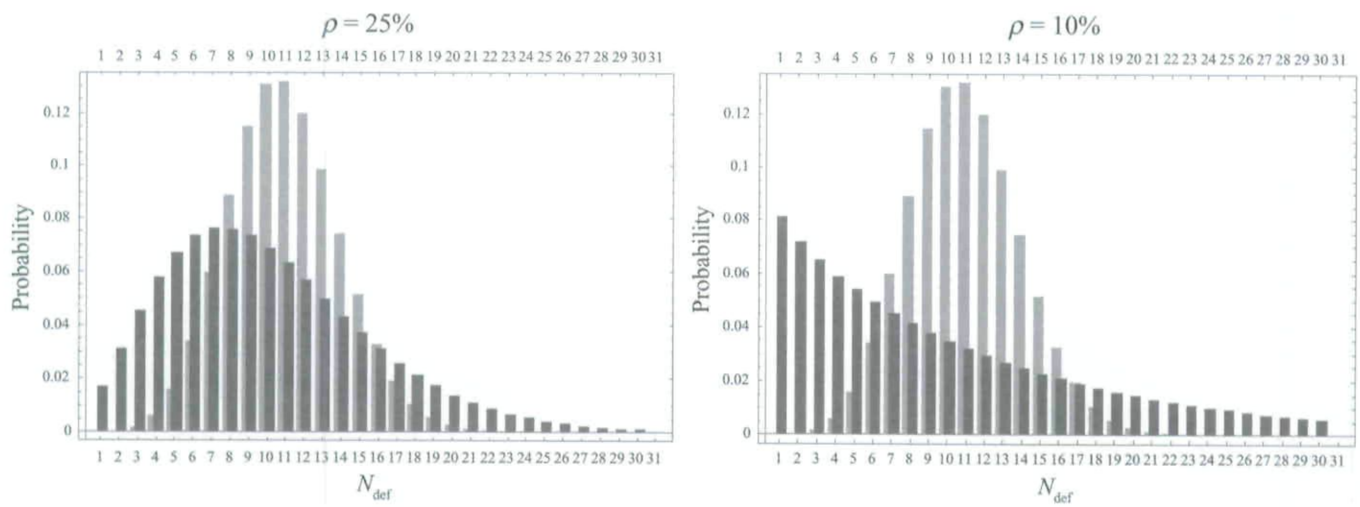


EXHIBIT 2

Probability of *k* Defaults within a Portfolio of 10 Assets

		Beta Binomial Model (ρ in %)				
$Pr[N_{Def} = k]$	**Binomial**	**0***	**1.25**	**2.5**	**5**	**10**
$k = 0$	59.87	59.87	61.56	63.08	65.75	70.02
$k = 1$	31.51	31.51	28.93	26.71	23.09	17.95
$k = 2$	7.46	7.46	7.59	7.87	7.77	7.08
$k = 3$	1.05	1.05	1.50	1.88	2.44	2.97
$k = 4$	0.10	0.10	0.23	0.39	0.70	1.23
$k = 5$	0.006	0.006	0.027	0.064	0.181	0.486
$k = 6$	0.000	0.000	0.003	0.009	0.041	0.176
$k = 7$	0.000	0.000	0.000	0.001	0.007	0.056
$k = 8$	0.000	0.000	0.000	0.000	0.001	0.015
$k = 9$	0.000	0.000	0.000	0.000	0.000	0.002
$k = 10$	0.000	0.000	0.000	0.000	0.000	0.000
$E[\mathbf{p}]$	5	5	5	5	5	5
$\Pr[N_{Def} > 5]$	0.000	0.000	0.003	0.010	0.050	0.250

Notes: The expected default probability is equal to 5% in all cases. All figures are expressed in %. "Binomial" stands for the pure binomial model described by Equation (1). 0 means negligible value.*

The mean and the variance of the loss rate are, respectively,

$$E[L(N)] = E[\mathbf{p}]$$

$$V[L(N)] = \frac{N + N(N-1)\rho}{N^2} E[\mathbf{p}](1 - E[\mathbf{p}])$$

The variance of the loss rate $L(N)$ is decreasing with the number of credits and tends to $\rho E[\mathbf{p}](1 - E[\mathbf{p}]) \equiv V[\mathbf{p}]$.

Proof.

$$\begin{aligned} V[N_{def}(N)] &= V[E[N_{def}(N) \mid \mathbf{p}]] + E[V[N_{def}(N) \mid \mathbf{p}]] \\ &= N^2 V[\mathbf{p}] + E[NV[X_1 \mid \mathbf{p}]] \\ &= N^2 V[\mathbf{p}] + NE[\mathbf{p}(1-\mathbf{p})] \\ &= N^2 V[\mathbf{p}] + NE[\mathbf{p}] - NE[\mathbf{p}^2] \end{aligned}$$

The second moment being $E[\mathbf{p}^2] = \rho E[\mathbf{p}] + (1-\rho)E[\mathbf{p}]^2$. Results on $L(N)$ are straightforward consequences. □

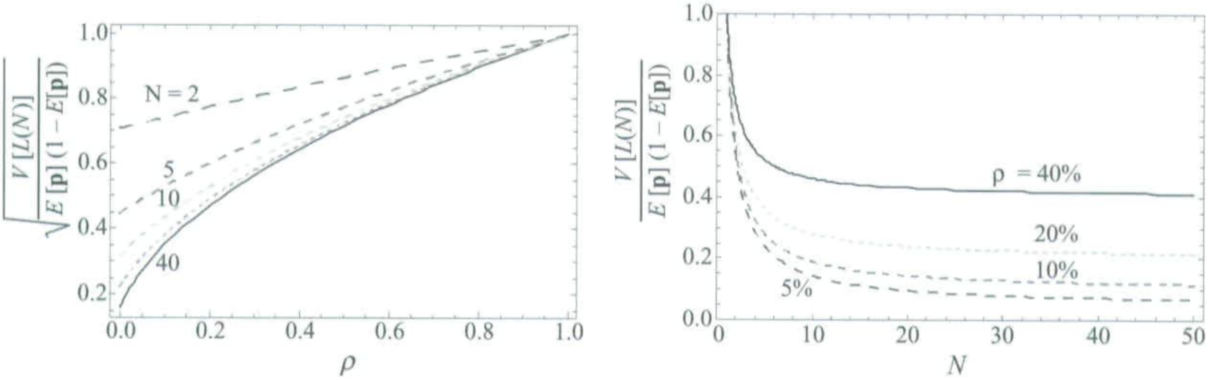
The above results imply that the variance of the loss rate $V[L(N)]$ is a strictly increasing function of ρ with a minimum and a maximum given by $\frac{1}{N}E[\mathbf{p}](1-E[\mathbf{p}])$ and $E[\mathbf{p}](1 - E[\mathbf{p}])$, respectively. The variance of the loss rate is a decreasing function of the number of credits in the portfolio with a maximum and a minimum given by $\frac{1+\rho}{2}E[\mathbf{p}](1-E[\mathbf{p}])$ and $\rho E[\mathbf{p}](1 - E[\mathbf{p}])$, respectively. Hence, the loss rate variable appears particularly suitable for analyzing credit risk portfolios (compared to the number of defaults $N_{def}(N)$). Exhibit 3 shows the properties of the loss rate with respect to the common default correlation and the number of credits. The left-hand graph draws the normalized standard deviation of the loss rate as a function of the common default correlation for different numbers of credits in the portfolio. This graph displays how the normalized standard deviation, computed by $\sqrt{\frac{V[L(N)]}{E[\mathbf{p}](1-E[\mathbf{p}])}}$, tends toward the minimum normalized standard deviation (given by $\sqrt{\rho}$), as the number of credits increases. The right-hand graph is inspired by the traditional portfolio theory. It plots the normalized variance of the loss rate as a function of the number of credits. This graph displays the diversification effect within a credit portfolio. Like more standard (stocks) portfolios, the variance decreases and tends to a non-zero value.

ANALYZING LARGE HOMOGENOUS PORTFOLIOS

Schönbucher [2003] has well explained that as the number of assets in the credit portfolio becomes large, the

EXHIBIT 3

Normalized Standard Deviation or the Normalized Variance of the Loss Rate for Different Values of Correlation and Different Numbers of Credits



loss rate statistic tends to be the relevant figure to consider. The proportion of defaults (whose conditional expectation is $\mathbf{p}$ for every N) tends to $\mathbf{p}$ as N gets large.[6] Models for large credit risk portfolios routinely exploit the fact that, for large N, $\Pr[L(N) \le l] \approx \Pr[\mathbf{p} \le l]$. In other words, tails of the true loss (rate) distribution of large homogeneous credit portfolios may be approximated by the tail of the mixture distribution. Our setting allows one to reconsider the loss rate distribution in the light of the common default correlation. The loss rate distribution in large homogenous credit portfolios is therefore described by

$$\Pr[L \le l] = \int_0^l \psi_{E[\mathbf{p}]}(p;\rho)dp := I_{E[\mathbf{p}]}(l;\rho)$$

Standard credit risk measures for analyzing credit portfolios are related to this probability. The tail function, defined by $TF_{E[\mathbf{p}]}(l;\rho) = \Pr[L > l] = 1 - I_{E[\mathbf{p}]}(l;\rho)$ is a first approach to highlight the extreme risk of credit portfolios. Denoting by c a confidence level (typically 99%, 99.9%), the credit at risk $CaR_c(L)$ is the value such that $TF_{E[\mathbf{p}]}(CaR_c(L);\rho) = \Pr[L > CaR_c(L)] = 1 - c$ or equivalently

$$\Pr[L \le CaR_c(L)] = I_{E[\mathbf{p}]}(CaR_c(L);\rho) = c \qquad (7)$$

This is the cth quantile of the reparameterized beta distribution. The cumulative density function $l \to I_{E[\mathbf{p}]}(l;\rho)$ being continuous and increasing, this may be rewritten $CaR_c(L) = I^{-1}_{E[\mathbf{p}]}(c;\rho)$. The expected shortfall is another important indicator to consider. Defined by $ES_c = E[L \mid L \ge CaR_c(L)]$ it has more desirable properties than the $CaR_c(L)$, as explained by Artzner et al. [1999]. This coherent measure of risk can be computed in the present framework by a couple of ways:

$$ES_c(\rho) = \frac{1}{1-c}\int_c^1 CaR_u(L)du = \frac{1}{1-c}\int_c^1 I^{-1}_{E[\mathbf{p}]}(u;\rho)du \qquad (8)$$

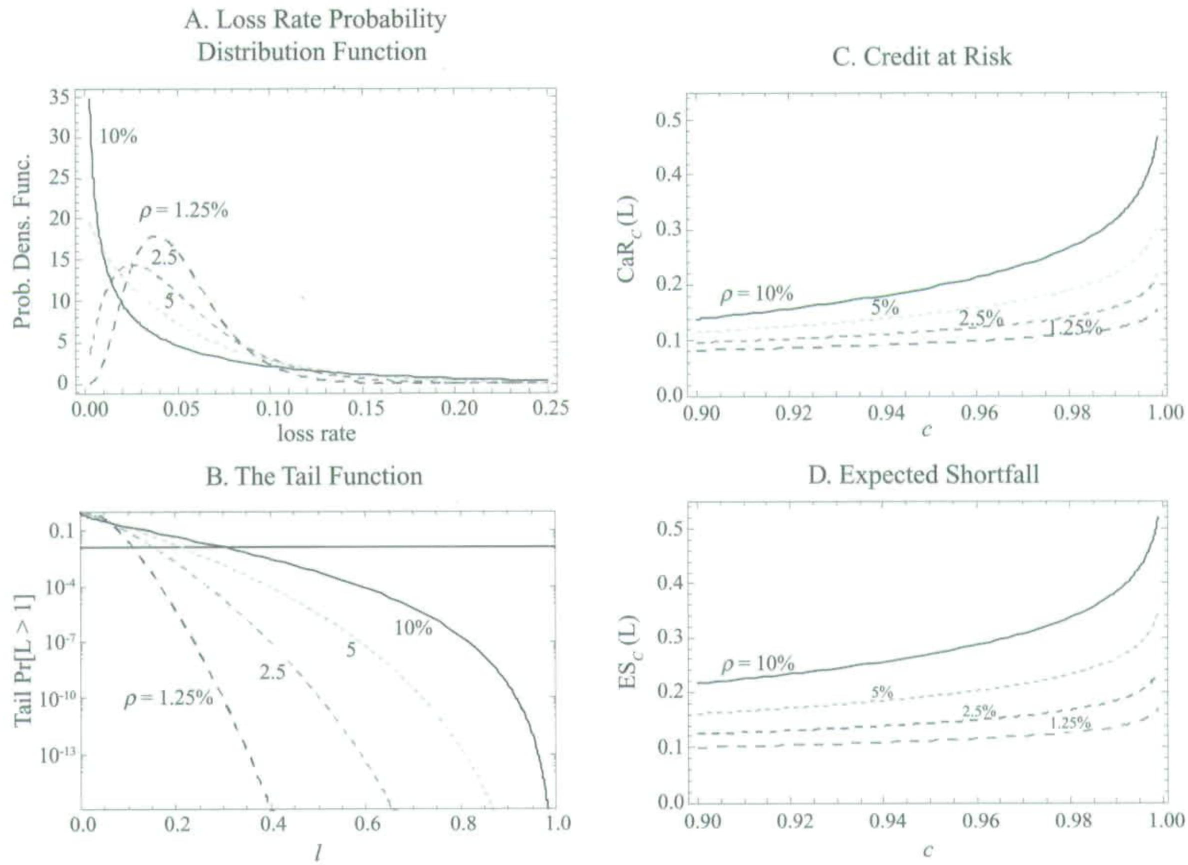
$$ES_c(\rho) = CaR_c(L) + \frac{1}{1-c}\int_{CaR_c(L)}^1 [1 - I_{E[\mathbf{p}]}(l;\rho)]dl \qquad (9)$$

One can remark that both require numerical integration techniques.[7]

Exhibit 4 displays various credit risk indicators: the loss rate probability distribution function, the tail function, the credit at risk, and the expected shortfall of a large homogenous credit portfolio with identical average default probability. Following usual practices, the tail function is drawn on a logarithmic y-scale where a horizontal line represents the 99th percentile case and can serve as a benchmark. One can verify that the expected shortfall is larger than the credit at risk as predicted by Equation (9). All these graphs illustrate that the default correlation impacts significantly on the perceived risk of the large homogeneous portfolio (as measured by the different credit indicators).

Beyond this graphical approach, it is worthwhile to assess quantitatively the sensitivity of credit indicators with respect to the default correlation. This is the aim of the following section. Before introducing analytical expressions, Exhibit 5 provides direct percentage differences of credit at risk and expected shortfall for a reference default correlation of 1.25%. As becomes clear in this exhibit, the credit risk assessment is dramatically affected by any misestimation

EXHIBIT 4
Various Credit Risk Indicators



Notes: The expected default probability of the credit portfolio is equal to 5%. In the Panel B, the horizontal line stands for the 99th percentile level.

in default correlation. For example, in a credit portfolio with a 5% expected default probability, the credit at risk for a 10% default correlation is 2 times larger than that computed for a 1.25% correlation. The expected shortfall is even about 2.5 times larger. As suggested by Exhibit 4, percentage errors are worse as the confidence level c increases. We notice however that errors for *CaR* and *ES* become of same order for huge confidence level. Interestingly, for the expected shortfall, the largest correlation case displays a maximum percentage errors at the $c = 99.99\%$ level. Additional simulations reveal that the same is true for the credit at risk but at an even larger confidence level.

SENSITIVITY ANALYSIS OF CREDIT RISK INDICATORS

This section provides closed-form formulae for credit indicators to analyze their sensitivity to default correlation. Our reparameterization of the beta distribution suggests to write the cumulative density function and its inverse function as $I_{E[\mathbf{p}]}(l;\rho) = I_x(\alpha,\beta)$ and $I^{-1}_{E[\mathbf{p}]}(c;\rho) = I^{-1}_x(\alpha,\beta)$, respectively. Due to expressions (7), (8), and (9), derivatives formulae are available to the extent we can compute $\frac{\partial I_x(\alpha,\beta)}{\partial\alpha}$, $\frac{\partial I_x(\alpha,\beta)}{\partial\beta}$, $\frac{\partial I^{-1}_x(\alpha,\beta)}{\partial\alpha}$ and $\frac{\partial I^{-1}_x(\alpha,\beta)}{\partial\beta}$. Some expressions are exposed in Appendix A. One then finds

$$\begin{aligned}\frac{\partial I_{E[\mathbf{p}]}(l;\rho)}{\partial\rho} &= \frac{\partial I_x}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} + \frac{\partial I_x}{\partial\beta}\frac{\partial\beta}{\partial\rho} \\ &= -\frac{1}{\rho^2}\left[E[\mathbf{p}]\frac{\partial I_x}{\partial\alpha}\left(E[\mathbf{p}]\frac{1-\rho}{\rho}, (1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right. \\ &\quad \left. + (1-E[\mathbf{p}])\frac{\partial I_x}{\partial\beta}\left(E[\mathbf{p}]\frac{1-\rho}{\rho}, (1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right]\end{aligned}$$

Other analytical expressions are derived along similar lines. The sensitivity of the tail function with respect to default correlation is simply given by $\frac{\partial TF_{E[\mathbf{p}]}(l;\rho)}{\partial\rho} = -\frac{\partial I_{E[\mathbf{p}]}(l;\rho)}{\partial\rho}$.

EXHIBIT 5

Relative Differences of Credit at Risk and Expected Shortfall for a Large Homogenous Credit Portfolio

	Default Correlation (in %)		
$\frac{CaR_c(\rho)}{CaR_c(1.25\%)}-1$	2.5	5	10
$c = 95.00\%$	22.32	55.45	103.85
$c = 97.50\%$	26.64	67.84	131.80
$c = 99.00\%$	31.08	80.52	159.76
$c = 99.90\%$	38.45	100.83	199.98
$c = 99.99\%$	42.82	111.54	215.05
$c = 99.999\%$	45.52	116.90	216.99
$\frac{ES_c(\rho)}{ES_c(1.25\%)}-1$	2.5	5	10
$c = 95.00\%$	28.21	72.24	140.90
$c = 97.50\%$	31.37	81.21	160.45
$c = 99.00\%$	34.73	90.64	180.07
$c = 99.90\%$	40.56	106.05	207.59
$c = 99.99\%$	44.10	114.13	216.26
$c = 99.999\%$	46.30	117.92	214.87

Notes: All figures are expressed in %. The expected default probability is 5%.

The $CaR_c(\rho)$ sensitivity to the default correlation is assessed by

$$\frac{\partial CaR_c(\rho)}{\partial\rho} = \frac{\partial I^{-1}_{E[\mathbf{p}]}(c;\rho)}{\partial\rho}$$
$$= -\frac{1}{\rho^2}\left[E[\mathbf{p}]\frac{\partial I_c^{-1}}{\partial\alpha}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right) + (1-E[\mathbf{p}])\frac{\partial I_c^{-1}}{\partial\beta}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right]$$

The sensitivity of the expected shortfall admits a couple of expressions depending on the considered definition. One finds either

$$\frac{\partial ES_c(\rho)}{\partial\rho} = \frac{1}{1-c}\int_c^1 \frac{\partial I^{-1}_{E[\mathbf{p}]}(u;\rho)}{\partial\rho}du$$

or

$$\frac{\partial ES_c(\rho)}{\partial\rho} = -\frac{1}{1-c}\int_{CaR_c(L)}^1 \frac{\partial I_{E[\mathbf{p}]}(l;\rho)}{\partial\rho}dl$$

This latter (perhaps surprisingly simple) expression comes from the differentiation of Equation (9) with respect to ρ and simplification

$$\frac{\partial ES_c(\rho)}{\partial\rho} = \frac{\partial CaR_c(\rho)}{\partial\rho} - \frac{1}{1-c}\int_{CaR_c(\rho)}^{\infty} \frac{\partial I_{E[\mathbf{p}]}(l;\rho)}{\partial\rho}dx - \frac{1}{1-c}\frac{\partial CaR_c(\rho)}{\partial\rho}[1-I_{E[\mathbf{p}]}(CaR_c(\rho);\rho)]dx$$

Once again, this latter expression is expected to be less time-consuming because no inversion is involved. However, the following analysis favors the former one because it involves the same underlying quantile function as the credit at risk. Note that every formula has been checked with approximate numerical derivatives. Armed with these expressions, one can comfortably undertake a sensitivity analysis of the credit at risk and expected shortfall, as shown in Exhibit 6.

Exhibit 6 displays interesting results concerning sensitivities of credit measures to the default correlation. The four graphs show that the two measures are affected quite differently. Mainly, the sensitivity and elasticity of the credit at risk may be either positive or negative while those of the expected shortfall are strictly positive for the considered values.

APPLICATIONS TO CDOS

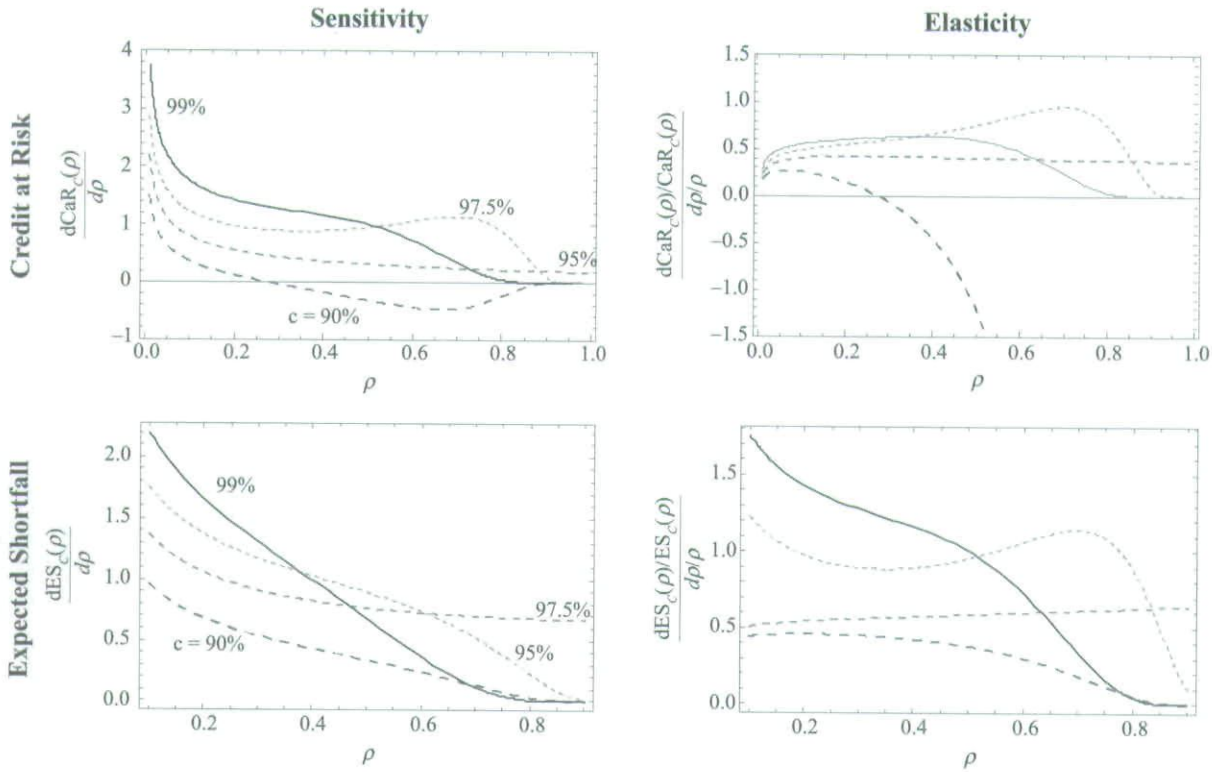
Portfolios with a limited number of credits are obviously common in asset management. Typically, a credit portfolio's underlying CDOs contain 125 different names. The above analysis can be easily extended to account for this. Technically, the exercise is straightforward, and for instance, the probability that no more than k credits default within the portfolio of size N is given by

$$\Pr[N_{\text{def}}(N) \le k] = \sum_{j=0}^{k}\int_0^1 \binom{N}{j} p^j(1-p)^{N-j}\psi_{E[\mathbf{p}]}(p;\rho)dp$$

The expression clearly highlights that the dependence on the common default correlation comes from $\psi_{E[\mathbf{p}]}(p;\rho)$

EXHIBIT 6

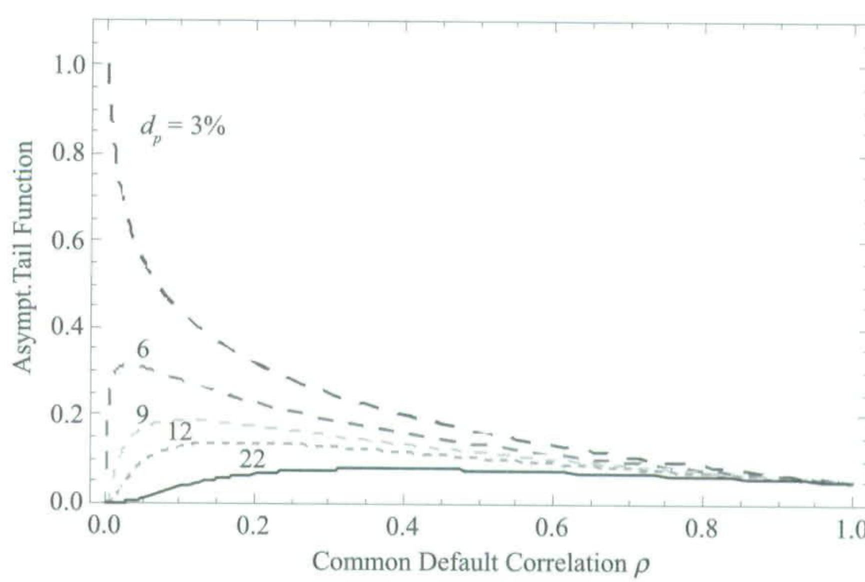
Sensitivities and Elasticities of Credit at Risk and the Expected Shortfall with Respect to the Common Default Correlation



only. Additional simulations could have nevertheless shown that distributions for a 125-name portfolio are very close to their asymptotic counterpart. So, we favor asymptotic distribution to investigate how holders of the different tranches of a CDO are impacted by a change in the default correlation. Holders of the so-called equity tranche are exposed to the first defaults in the portfolios while holders of the last tranche are impacted only if the number of defaults is significant. To fix this idea, let's consider the tranching of the Itraxx contracts for which attachment points are 0%, 3%, 6%, 9%, 12%, 22%. The associated detachment points correspond to the upper limit of the losses covered by the tranche. Exhibit 7 plots, for each considered tranche, the probability that losses will exceed the detachment point versus the common default correlation ρ.

EXHIBIT 7

Probability That Losses Will Exceed the Detachment Point versus the Common Default Correlation



Notes: d_p stands for detachment point and corresponds to the upper limit of the losses covered by the tranche. The plotted tail function is the probability that the loss exceeds the considered detachment point.

Exhibit 7 exposes how holders of the different tranches are differently impacted by the common default correlation. It can be observed first that holders of the equity tranche (whose detachment point d_p is 3%) benefit from any increase of the common default correlation—a complete loss being less probable. A reason for this is that the underlying references behave more identically as correlation increases; meaning that the common survival correlation increases too. Holders of the other tranches are clearly differently affected. Among them, investors in the second tranche remain rather exposed to the correlation risk.

CONCLUSION

This article reconsiders the beta binomial approach for modeling homogenous credit portfolios. It favors both the expected default probability and the common default correlation to parameterize the mixing distribution. This article makes standard credit risk indicators functions of the correlation only and it sheds lights on the model risk associated with that parameter. Analytical expressions have been reported to allow sensitivity analysis. Simulations conclude that default correlation is a key parameter to account for. Finally, it must be stressed that the idea exposed in the article is applicable to every mixing distribution to the extent there exists a suitable function transforming structural parameters into the expected default probability and the common default correlation.

Appendix A

PROPERTIES OF THE BETA DISTRIBUTION

This appendix displays some well-known properties of the beta distribution with respect to its shape parameters. If $\alpha = \beta = \gamma$, then the beta distribution is symmetric with respect to $E[\mathbf{p}] = \frac{1}{2}$. Straightforward computations give $V[\mathbf{p}] = \frac{1}{4(2\gamma+1)}$ and $\text{cor}[X_i, X_j] = \frac{1}{1+2\gamma} = 4V[\mathbf{p}]$. The standard deviation of the (random) default probability is bounded by $\frac{1}{2}$. This limit corresponds to $\gamma \approx 0$ for which $\text{cor}[X_i, X_j] = 1$. In such a case, either all issuers survive or all default. The corresponding distribution weights only 0 and 1. If instead, $\gamma = 1$, the symmetric beta distribution is the uniform one with $V[\mathbf{p}] = \frac{1}{12}$ and $\text{cor}[X_i, X_j] = \frac{1}{3}$.

Analytical Expressions for Sensitivities

Due to the non-uniqueness of their representations, various expressions could be derived and reported for the derivatives of the cumulative density function of the beta distribution and its associated inverse function. Expressions below are very appealing for programming on Mathematica—the package I use throughout the article:

$$\frac{\partial I_x(a,b)}{\partial a} = [\ln(x) - \psi(a) + \psi(a+b)]I_x(a,b) - \frac{\Gamma(a)\Gamma(a+b)}{\Gamma(b)} x^a \,{}_3F_2(a,a,1-b;a+1,a+1;x)$$

$$\frac{\partial I_x(a,b)}{\partial b} = -[\ln(1-x) - \psi(b) + \psi(a+b)]I_{1-x}(b,a) + \frac{\Gamma(b)\Gamma(a+b)}{\Gamma(a)} (1-x)^b \,{}_3F_2(b,b,1-a;b+1,b+1;1-x)$$

and

$$\begin{aligned}\frac{\partial I_x^{-1}(a,b)}{\partial a} &= (1 - I_x^{-1}(a,b))^{1-b}[I_x^{-1}(a,b)]^{1-a} \\ &\times [-B_{I_x^{-1}(a,b)}(a,b)[\ln(I_x^{-1}(a,b)) - \psi(a) + \psi(a+b)] \\ &+ [I_x^{-1}(a,b)]^a \Gamma(a)^2 \,{}_3F_2(a,a,1-b;a+1,a+1;I_x^{-1}(a,b))]\end{aligned}$$

$$\begin{aligned}\frac{\partial I_x^{-1}(a,b)}{\partial b} &= (1 - I_x^{-1}(a,b))^{-b}(I_x^{-1}(a,b) - 1)[I_x^{-1}(a,b)]^{1-a} \\ &\times [-B_{1-I_x^{-1}(a,b)}(b,a)[\ln(1 - I_x^{-1}(a,b)) - \psi(b) + \psi(a+b)] \\ &+ [1 - I_x^{-1}(a,b)]^b \Gamma(b)^2 \,{}_3F_2(b,b,1-a;b+1,b+1; \\ &\times 1 - I_x^{-1}(a,b))]\end{aligned}$$

where $B_x(a, b)$ is the incomplete beta function defined by $B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$. ${}_3F_2(a_1, a_2, a_3; b_1, b_2, b_3; z)$ is the regularized hypergeometric function and ψ is the digamma function. The digamma function is the logarithm derivative of the Euler gamma function:

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{where } \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

The function Γ is a generalization to complex numbers of the factorial function since, for any integer n, $\Gamma(n) = (n-1)!$ ${}_3F_2(a_1, a_2, a_3; b_1, b_2, b_3; z)$ is defined by

$$ {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!} $$

with $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer's symbol. See Abramowitz and Stegun [1972] for more details on these functions.

ENDNOTES

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[1]A mixed logit-normal distribution is explicitly used in CreditPortfolioView (see Wilson [1997a,b]); a probit-normal one is used in Creditmetrics. Frey and McNeil [2002] have demonstrated that the CreditRisk+ solution implicitly uses a beta mixing distribution for the default probability. As a result, the present article admits connections with the CreditRisk+ framework, but this point is left for future research.

[2]The key point here is to develop an easy-to-understand way to manage credit portfolios within the beta binomial framework with no reference to the traditional (and rather obscure) shape parameters of the mixing distribution. This feature is desirable because everybody involved in the credit industry is not necessary "fluent" in statistics. Moreover, it is not so evident that people involved in the credit business interpret beta's shape parameters in the same way. It is well known (see Frey and McNeil [2001]) that, when one of the two first moments of the random default probability is fixed, the second moment or the default correlation determines the shape parameters of the beta mixing distribution. However, to our knowledge, no research has developed this way of reasoning further.

[3]Analytical expressions exposed in this article provide formulae to assess the impact of a correlation *shift*. This article therefore admits some closed connections with the recent stream of research dedicated to the introduction of non-constant default correlation in credit portfolios (see Burtschell, Gregory, and Laurent [2007] for references). But this issue is left for future research.

[4]Avoiding simulations, the approach is worthwhile for credit analysts for at least a couple of reasons. First of all, it can speed up computations and subsequent decisions making. Second, it prevents drawbacks of rival simulation-based methods. It is well known that the estimates they provide can be fairly unstable, as they depend on the number of simulation runs and on the way the random figures are generated. These methods can even fail to provide safe results. Credit risk measures are indeed intimately related to the tail of the distribution, which is challenging to capture by simulation.

[5]Note that, except speculators, very few investors would invest in credit portfolios with a such a significant expected default probability.

[6]To see this, it is sufficient to note that the conditional variance of the proportion $\sigma^2[\frac{N_{def}(v)}{N} \mid \mathbf{p}] = \frac{\mathbf{p}(1-\mathbf{p})}{N}$ tends to 0.

[7]The latter appears less time-consuming than the former because no inversion is required. This remark may be helpful, for who wants to make intensive computations.

REFERENCES

Abramowitz, M., and I. Stegun. *Handbook of Mathematical Functions.* Dover, 1972.

Artzner, Ph., F. Delbaen, J.-M. Eber, and D. Heath. "Coherent Measures of Risk." *Mathematical Finance*, Vol. 9, No. 3 (1999), pp. 203–228.

Burtschell, X., J. Gregory, and J.P. Laurent. "Beyond the Gaussian Copula: Stochastic and Local Correlation." *Journal of Credit Risk*, Vol. 3, No. 1 (2007), pp. 31–62.

Frey, R., and A. McNeil. "Modelling Dependent Defaults." Working Paper, ETH Zürich, 2001.

——. "VaR and Expected Shortfall in Portfolios of Dependent Credit Risks: Conceptual and Practical Insights." *Journal of Banking & Finance*, Vol. 26, No. 7 (2002), pp. 1317–1334.

——. "Dependent Defaults in Models of Portfolio Credit Risk." *Journal of Risk*, Vol. 6, No. 1 (2003), pp. 59–92.

Renault, O., and A. de Servigny. *Measuring and Managing Credit Risk* (The Standard and Poor's Guide). McGraw-Hill, 2004.

Schönbucher, P. "Taken to the Limit: Simple and Not-So-Simple Loan-Loss Distributions." *WILMOTT Magazine*, 2 (2003), pp. 63–73.

Wilson, T. "Portfolio Credit Risk I." *Risk*, 10 (September 1997a), pp. 111–117.

——. "Portfolio Credit Risk II." *Risk*, 10 (October 1997b), pp. 56–61.

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