Valuing Corporate Liabilities When the Default Threshold is not an Absorbing Barrier

Franck Moraux*

Université de Rennes 1
Crereg - Axe Finance

Institut de Gestion de Rennes
11, rue Jean Macé BP 1997
35019 Rennes Cedex, France.

franck.moraux@univ-rennes1.fr
Tel: 332 2323 3535
Fax: 332 2323 7800

*This paper has benefited from discussions with seminar participants at ESSEC, at the Université Paris 1 - Sorbonne, with participants of the EIR 2001 Annual Conference, the EFMA 2002 Annual Conference, the London Guildhall University Conference on "credit derivatives" and the "Credit 2002 Conference" organized by the GRETA in Venice in September. Special thanks are addressed for comments to Eric Bryis, Michael Dempster, Raphaël Douady, Bernard Dumas, Jan Ericsson, Hélyette Geman, Kay Gesiecke, Roland Gillet, Jean-Paul Laurent, Constantin Mellios, Patrick Navatte, Patrice Poncet, Roland Portait, Philippe Raimbourg, Alexander Reisz, Olivier Renault, Mickael Rockinger, Nicholas Sarantis and the CREFIB team. The Crereg is the UMR CNRS 6585.

Valuing Corporate Liabilities When the Default Threshold is not an Absorbing Barrier

Abstract

This paper studies the impacts of delays beyond the default event on the ex ante pricing of corporate liabilities and credit spreads. Such delays are often granted by courts within domestic bankruptcy codes. A Black-Scholes-Merton-Cox type framework is developed to account for both the subordination and the possible convertibility of debt to equity. In this structural approach, the firm assets value is allowed to cross the default barrier without causing an immediate liquidation. One shows that all the liquidation procedures based on the time spent by the firm in financial distress are bounded by a couple of ideal procedures. Interestingly, these latter lead to quasi analytical pricing formulae for the corporate liabilities. We adapt and extend the analytical formulae introduced by Chesney-Jeanblanc-Yor (1996) in the context of Parisian options, to derive these two bounds. We then conduct extensive numerical simulation. Among other results, experiments mainly conclude that the credit spreads increase or decrease monotonically w.r.t. the extra time granted beyond default, depending on the way the considered bond is secured or subordinated to others. Overall, this article complements François-Morellec (2002) who examine whether the Chapter 11 of the US Bankruptcy procedure impacts on capital structure choices and the strategic decision to default in lines of Leland (1994).

Keywords : Bankruptcy procedures, Debt pricing, Default, Liquidation.

JEL Classification Codes : G 130, A 910.

1 Introduction

Structural models for valuing corporate liabilities currently use the default barrier introduced by Black-Cox (1976) (Leland (1994), Longstaff-Schwartz (1995) and Leland-Toft (1996)). Exogenous, this threshold reflects the safety covenant allowing secured bondholders to bankrupt the firm. This boundary signals the early default and gives the 'liquidation' value of the firm assets too. Assuming an immediate modus operandi nonetheless overestimates effective creditors rights because the default and the liquidation cannot be considered as equivalent events. Empirical studies in USA have found that additional "survival" periods beyond the main default event last up to 3 years (Altman-Eberhart (1994), Betker (1995), Hotchkiss (1995)). Helwege (1999) reports that the longest default of the modern US junk bond market is seven years long. The legal environment is an important explanatory factor for this duration. In most countries, bankruptcy processes check first the reality of the financial distress before claiming definitive and complete liquidation of the firm assets[1]. Liquidation can be further postponed because codes often favor firm continuation against claims reimbursement (for political and social considerations)[2].

This paper aims at studying the impacts of delays on the ex ante pricing

[1]Paraphrasing the French code, the financial distress must be real and recognized whereas a default may result from a simple and brief mismatch between available liquid assets and the current obligations.

[2]Out-of-court negotiations between claimants and stockholders induce delays too but they are reputed to be briefer processes (Hotchkiss (1995)).

of corporate liabilities within a Black-Scholes-Merton-Cox style framework. The firm assets value can cross the default threshold without causing an immediate liquidation. This approach succeeds in pricing complex securities, it sheds light on the intricate relations between credit spreads, subordination and convertibility. This paper focuses on liquidation procedures that are explicitly based on the time spent by the firm in financial distress. It shows that these (close to reality) decision-making processes are bounded by a couple of idealized procedures. One then demonstrates that results of Black-Cox (1976) and Ingersoll (1977) on subordinated and convertible bonds (respectively) are robust to such procedures. Using results of Chesney-jeanblanc-Picqué-Yor (1995) and new materials on cumulative contingent claims, closed form formulae for pricing corporate liabilities can then be derived.

Numerical experiments conclude that both equity price and corporate credit spreads are sensitive to delays. Equityholders wealth strictly rises as the granted time below the thresholds increases. More interestingly, credit spreads are increasing or decreasing monotone functions (of the extra time) depending on the way the bond is secured or subordinated to others. Our results are contrasting with previous procedures designed by Black-Scholes-Merton and Black-Cox. Appealing features of the approach are to make a clear distinction between a default event and the financial distress and to allow successive default events. In the structural approach, strategic models in lines of Anderson-Sundaresan (1996) and Mella Barral-Perraudin (1997) are the only ones that manage to do so. These sequential models require however detailed assumptions on the behavior of investors. A notable exception is François-Morellec (2002) who examine whether the so-called exclusivity period defined by the Chapter 11 of the US Bankruptcy code impacts on capital structure choices and the strategic decision to default in lines of Leland (1994). These authors found that the leverage choices are ambiguously impacted whereas "unambiguously [...] credit spreads increase with the length of the exclusivity period". One will show that this second claim is not verified when the debt subordination is considered.

The rest of this paper is organized as follows. Section 2 develops a framework à la Black-Cox-Scholes-Merton. Section 3 considers the pricing of corporate liabilities. Section 4 presents an in-depth discussion of the liquidation procedures modelling. Section 5 turns to a numerical analysis of the liquidation procedures. Section 6 illustrates previous results on a complex capital structure.

2 The structural framework

This section develops a Black-Scholes-Merton-Cox like framework capable of handling a given liquidation procedure. Most of the assumptions of the seminal setting are supposed to hold. Assets are traded continuously in a perfect and complete financial market. The interest rate level is assumed constant. One denotes r the continuously compounded risk free interest rate. The risk neutral price process of the underlying firm asset V is supposed to be well described by the following stochastic differential equation:

$$dV_t = rV_t dt + \sigma_V V_t dW_t. \tag{1}$$

where W is a standard Brownian motion and σ_V the constant volatility. Assumptions on the capital structure of the firm follow common practice too. Debts are zero coupon bonds that mature at time T. In case of liquidation, the absolute priority rule holds. Following Black-Cox (1976) and Longstaff-Schwartz (1995), the senior (the most secured) bond has a safety covenant with a net worth type criterion. The debtholder is theoretically allowed to force bankruptcy, when the firm assets value attains an exogenous threshold denoted V_B. In applications, the constant level V_B is equal either to the face value or to the discounted face value (with the risk free rate). The time-varying default barrier of Black-Cox (1976) is emploied too. Denoting F the face value, this default threshold at time t is given by $V_B(t) = \rho F e^{-r(T-t)}$ where $\rho \in [0,1]$ may be interpreted as a percentage of security. Assuming that the secured debt remains risky implies that $\rho < 1$.

Up to now, default and liquidation are undistinguishable events because the firm assets are immediately liquidated. In order to avoid confusion, some extra definitions are required. Let's define what a liquidation procedure is.

Definition 1 *A liquidation procedure is a (legal) decision-making process which aims at verifying whether the financial distress is real i.e. a recognized fact.*

It is implicitly assumes the ubiquity of the decision maker. In our setting, the net worth criterion of the safety covenant characterizes the default event. It also implies a natural definition of the financial distress.

Definition 2 *A firm is in financial distress when the value of its asset is lower than the default threshold.*

The rationale of the liquidation procedures must now be precised. One relates the procedure to the sojourn time in financial delays although any other criterion would induce delays.

Assumption *The rationale underlying the liquidation procedure is based on the time spent by the firm in financial distress.*

It must be stressed that this seemingly arbitrary assumption is in lines with observed practices within domestic codes and that no other requirement has been explicitly and reccurently found[3]. Bankruptcy codes may even insist on the duration of the financial distress. Hereafter, one refers to it as the "granted time". Due to the previous definition, the key point is the time spent by the underlying firm asset value beyond the standard default barrier. When the firm assets value attains the exogenous threshold (V_B), *a* default event is just signaled. Clearly, the liquidation event is strictly posterior to the default time. In addition, the default is no longer an absorbing state. The default threshold is no longer an absorbing barrier.

[3]Other arbitrary criteria for liquidation procedures could be investigated (e.g. a maximum number of default events). *Ad hoc* can one can construct rationale allowing quasi-analytical solution for the pricing of corporate liabilities. But this would lead to a choice unrelated to real practices and a model risk difficult to appreciate.

The way successive delays may be granted to the firm is a central question when modelling the considered liquidation procedure. First, delays can be given in response to successive and distinct distress periods. For example, the chapter 11 of the US Bankruptcy law may be used several times (cf. the TWA and its "Chapter 11, 22, 33"). Second, extra times can also be granted during the same financial distress. In France, e.g., a legal 3 months-length observation period is, to date, systematically granted by courts. This can be renewed once and exceptionally prolonged in the limit of six months on a discretionary basis. Anyway, a key difference between domestic environments stand in the way additional time is granted at the second (and subsequent) default events. Interestingly, it is demonstrated below that possible processes are bounded by a couple of idealized procedures. To this, one simply introduces them as follows.

Definition 3 *Under the procedure A, the liquidation is declared when the financial distress has last successively more than a d-length period.*

To compare with Black-Cox (1976), firm assets are liquidated as the time successively spent by the firm assets value V below the default is greater than d. This is the ideal procedure considered by François-Morellec (2002) to model the Chapter 11 of the US bankruptcy procedure. This is discussed in more depth in what follows.

Definition 4 *Under the procedure B, the liquidation is declared when the financial distress has last more than a d-length period during the debt's life .*

Firm assets are liquidated when the total time spent by the firm assets value V below the default is greater than d. Here d serves the role of maximum authorized duration in default. In the French code described above, it could be equal to one year. This couple of procedures leads to closed form solution for pricing corporate securities. After dealing with the corresponding pricing issues, one will emphasize the key difference between the procedures A and B and deduce some important implication. Note that d is supposed to be exogenously given.

3 The pricing of corporate liabilities

To refine the analysis, various random variables must be introduced. If necessary, the notations closely follow Chesney-Jeanblanc-Picqué-Yor (1995). Analytical formulae are then derived to price complex corporate liabilities. Their properties are finally considered.

3.1 Notations

The first random variable of importance is the first default event time. This is the first time the firm asset value reaches the default barrier. It is described by :

$$\tau_{V_B} = \inf\{0 \leq t \leq T : V_t = V_B\}$$

The second fundamental random variable is the last time before t the process $V = (V_t)_t$ has crossed the level V_B. This is denoted :

$$g^V_{V_B,t} = \sup\{s \leq t | V_s = V_B\}.$$

One can now focus on the duration or total duration of the financial distress. Under the procedure A, the liquidation time is the first time when the firm value process has spent consecutively more than the pre-specified value d below V_B. Here, the liquidation time is given by :

$$\tau^A_V(d) = \inf\{t \geq 0 : t - g^V_{V_B,t} \geq d, V_t \leq V_B\}.$$

Chesney-jeanblanc-Picqué-Yor (1997) precised this is a stopping time.
Under the procedure B, the liquidation is declared when the total time the firm assets value stands under the default barrier exceeds d. The cumulative time spent under the default threshold V_B also termed the occupation time is mathematically defined by:

$$A^-_t(V_B) = \int_0^t 1_{\{V_v \leq V_B\}}\, dv.$$

This random variable cumulates any excursion times between 0 and t. Here, the liquidation time is given by :

$$\tau^B_V(d) = \inf\{t \geq 0 : A^-_t(V_B) \geq d\}.$$

3.2 The equity

Following Black-Scholes (1973), let's assume that the firm has issued an unique non convertible zero-coupon bond maturing at T whose face value is K^4[4]. At maturity T, equityholders receive the firm assets value in excess of the promised face value if no early liquidation due to financial distress has been declared before. This is the case when the time spent (successively or overall) beyond the default threshold does not exceed d. Equityholders rights are well duplicated by an european contingent claim written on the firm assets. The price of the equity verifies :

$$Eq_i = E^Q\left[e^{-rT}\max\left(V_T - K, 0\right)1_{\{\tau_V^i(d)>T\}}\right],\ i = A, B \tag{2}$$

where $\tau_V^i(d)$, $i = A, B$ is defined above.

3.2.1 Under the procedure A

Under such a liquidation, the equity is a Down and Out Parisian option written on the firm assets. The equity price may therefore be computed thanks to the results of Chesney-Jeanblanc-Picqué-Yor (1995). Denoting Eq_{BS} the standard Black-Scholes-Merton price of equity, one finds :

$$Eq_A = Eq_{BS} - e^{-(r+\frac{m^2}{2})T}\int_k^\infty e^{mu}(V_0e^{\sigma u} - K)h_{V_B}(T, u)\,du \tag{3}$$

where $k = \frac{1}{\sigma}\ln(\frac{K}{V_0})$, $m = \frac{1}{\sigma_V}(r - \delta_V - \frac{1}{2}\sigma_V^2)$. The second term in teh right hand side is the price of a Down and In parisian option. h_{V_B} is characterized by its Laplace transform given by :

$$\hat{h}_{V_B}(\lambda, y) = \frac{e^{b\sqrt{2\lambda}}}{d\sqrt{2\lambda}\varphi(\sqrt{2\lambda d})}\int_0^\infty ze^{-\frac{z^2}{2d}-|b-z-y|\sqrt{2\lambda}}dz$$

[4]More complex capital structure will be studied below.

with $b = \frac{1}{\sigma}\ln(\frac{V_B}{V_0})$ and $\varphi(u) = 1 + u\sqrt{2\pi}e^{\frac{1}{2}u^2}N(u)$. To invert this Laplace transform, a numerical procedure is required. Among many other choices, the gaussian quadrature algorithm of Piessens (1973) is emploied. This is the one used by Cathcart-El Jahel (1998) in a different context.

3.2.2 Under the procedure B

The second procedure exploits new materials on cumulative parisian options. These contingent claims depend on the total time spent by the underlying asset above or below a threshold level. The equity is priced as a Down and Out cumulative call option by the following equation :

$$Eq_B = V_0'\Psi^+_{m+\sigma}(T,k,b,d) - K'\Psi^+_m(T,k,b,d) \tag{4}$$

with $V_0' = V_0e^{-(r+\frac{m^2}{2})T}$, $K' = Ke^{-(r+\frac{m^2}{2})T}$, $b = \frac{1}{\sigma}\ln(V_B/V_0)$, $k = \frac{1}{\sigma}\ln(K/V_0)$. Assuming that $V_0 > V_B$, one has :

$$\begin{aligned}
\Psi^+_\mu(t,k,b,d) &= e^{\mu^2 t/2}\left(\Phi\left(d_{\Xi(\mu)}\left(t,V_0,V_B\vee K\right)\right) - (\frac{V_B}{V_0})^{2\mu/\sigma}\Phi(d_{\Xi(\mu)}(t,\frac{V_B^2}{V_0},V_B\vee K))\right) \\
&\quad + \int_d^T ds\int_{k\wedge b}^b e^{\mu x}\Upsilon\left(l-x,-b,s,t-s\right)dx \\
&\quad + \int_d^T ds\int_{k\vee b}^\infty e^{\mu x}\Upsilon\left(0,x-2b,s,t-s\right)dx
\end{aligned}$$

where

$$\Xi(\mu) = \begin{cases} 1 & \text{if} \quad \mu = m+\sigma \\ 2 & \text{if} \quad \mu = m \end{cases}$$

and

$$\Upsilon(a,b,u,v) = \int_0^\infty \frac{(x+a)(x+b)}{\pi(uv)^{\frac{3}{2}}}e^{-\frac{(x+a)^2}{2v}}e^{-\frac{(x+b)^2}{2u}}dx.$$

The derivation of this pricing formula follows the general work of Hugonnier (1999) on occupation time derivatives. It must be stressed however that it differs from the ones he proposed. Some details are given in appendix. Of course, numerical approaches could have been used. One refers to Avellaneda-Wu (1999), Haber-Schönbucher-Wilmott (1999) among many others.

3.3 Other liabilities

Here is the key interest of the chosen framework. One considers a general capital structure by assuming that the firm has issued n different debts (either bonds or loans), strictly ordered by their rank of priority in case of liquidation,which matures at the same date T. Denoted $(L_i)_{i=1,\ldots,n}$ the price of these ranked debts, L_1 stands for the most secured liabilities, L_n for the most subordinated debt (L_{n+1} could be assimilated to the equity price). Let $\mathcal{P}_j$ be the price of the pool of loans containing the j first liabilities $\mathcal{P}_j = \sum_{k=1}^{j} L_k$. Maturing at the same date T, the total amount of this pool is the sum of the j face values : $F(\mathcal{P}_j) = \sum_{k=1}^{j} F_k$. It is worthwhile to note that $F(\mathcal{P}_{j+1}) = F(\mathcal{P}_j) + F_{j+1}$.

3.3.1 Subordinated debts

In our setting, assuming that the Absolute Priority Rule holds is equivalent to suppose that the owners of the $j+1-th$ debt are equity holders until the $j-th$ debt is fully repaid. Hence,

Proposition 1 *Under the assumptions of section 2, any corporate liability* *(nor convertible, nor callable) can be valued as the difference of two equity prices. These equities correspond to identical firms but in the leverage. Denoting $Eq(K)$ the equity price of a K-leveraged firm, the $j-th$ liability is valued :*

$$L_j = Eq\left(F\left(\mathcal{P}_{j-1}\right)\right) - Eq\left(F\left(\mathcal{P}_j\right)\right)$$

where $F\left(\mathcal{P}_j\right)$ has been previously defined.

Some remarks merit to be done. First, this result is quite general. It is distribution free, independent of the interest rates behavior and even independent of the liquidation procedure (e.g. this is true within the Black-Cox (1976) setting (p 359)). Second, the valuation of a subordinated debt needs not the total value of the firm but only the knowledge of the current firm asset value. Third, an equivalent expression for subordinated debt that insists on the discount w.r.t. the risk free equivalent is easily found thanks to the parity relation. One has $L_j = F_j p_0(0,T) - (put(F(\mathcal{P}_j)) - put(F(\mathcal{P}_{j-1}))$ where $p_0(0,T)$ is the value of an equivalent risk free bond and put stands for the put opion value.

3.3.2 Convertible debts

Dealing with the convertible debt is an important point since they are very common in observed capital structures. Both the equity and the convertible debt are concerned with. For short, one will firts assume that a conversion is not optimal before maturity, within our context (there is no dividend payment).

Proposition 2 *Assuming that one of the debts is convertible and optimally convert only at maturity, the equity price is given by :*

$$Eq' = Eq(\bar{F} + D) - \xi Eq(\bar{F} + \frac{D}{\xi})$$

where ξ is a dilution factor, D the face value of the convertible debt and $\bar{F}$ the sum of any other face values to be repaid.

To demonstrate this, note that the dilution factor ξ here is lower than one. One has $D/\xi > D$. Any ex ante earning G for stockholders is diluted and becomes $(1-\xi)G$ in case of conversion. If there is no conversion, the equity is worth at maturity the remaining value after repaiment of all debts. In the case of conversion, this value is simply diluated. Hence, one has formally :

$$Eq = E^Q[e^{-rT}(V_T - (\bar{F} + D))1_{\{V_T \in [\bar{F}+D; \bar{F}+\frac{D}{\xi}]\}}] + E^Q[e^{-rT}(1-\xi)(V_T - \bar{F})1_{\{V_T > \bar{F}+\frac{D}{\xi}\}}].$$

Decomposing the first member of the right hand side gives :

$$\begin{aligned} Eq &= E^Q[e^{-rT}(V_T - (\bar{F} + D))1_{\{V_T > \bar{F}+D\}}] \\ &- E^Q[e^{-rT}(V_T - (\bar{F} + D))1_{\{V_T > \bar{F}+\frac{D}{\xi}\}}] \\ &+ E^Q[e^{-rT}(1-\xi)(V_T - \bar{F})1_{\{V_T > \bar{F}+\frac{D}{\xi}\}}] \\ &= Eq(\bar{F} + D) - \xi Eq(\bar{F} + \frac{D}{\xi}). \end{aligned} \tag{5}$$

The last equation is found by noting that $V_T - (\bar{F} + D) - (1-\xi)(V_T - \bar{F}) = \xi(V_T - (\bar{F} + \frac{D}{\xi}))$.

Proposition 3 *In our Black-Scholes-Merton context with no dividend payment to stockholders, the optimal conversion policy is insensible to the liquidation procedure.*

As a result and paraphrasing Ingersoll (1977), *"a convertible security will never be converted prior to maturity"*. This has been demonstrated by Ingersoll (1977) in the context of Black-Scholes-Merton (one refers to Merton (1973) for a similar problem). This is also true whatever the liquidation is. First, in the case of liquidation, the position of debtholders is always safer than the one of equityholders (remind that the "absolute priority rule" is supposed to hold). Second, designing a liquidation procedure does not modify the potential upside behavior of the stock price (while the default event is a downside risk).

Armed with thess results, one can now turn to the case of subordinated convertible.

Proposition 4 *Under previous assumptions, a subordinated convertible debt is priced by:*

$$L_n = Eq(F(\mathcal{P}_{n-1})) - Eq(F(\mathcal{P}_n)) + \xi Eq(F(\mathcal{P}_n^{\xi}))$$

where $\mathcal{P}_n^{\xi} = F(\mathcal{P}_{n-1}) + F_n/\xi$ *with* ξ *the dilution factor.*

One of the simplest demonstration of this result is based on a differential rationale. It is sufficient to remark that this security is valued so that the market value of the corporate liabilities equal that of the firm. As a by-product, one can easily obtain an expression for the convertible rights at maturity for a subordinated convertible debt.

4 A closer look at the liquidation procedures

Under assumption 3, liquidation procedures (among which the A and B) appear quite similar at the first sights. They nonethless imply quite different modus operandi in the legal decision-making process, it is worthwhile to highlight and compare one another.

4.1 A comparative analysis of procedures A and B

The key difference between A and B stands in whether an equally d-length duration may be granted several times. To help the intuition, let's consider a "financial distress time counter" that adds the duration. In addition, recall that the liquidation procedure begins as the firm assets value gets lower than the default barrier.

Under the procedure A, each time the firm value process passes through and above V_B, the liquidation procedure is closed and the hypothetical distress counter is reset to zero. The next time a default event occurs, an identical procedure is run and an equal period of time d is granted. This mimics a decision-making process that does not keep in mind neither previous default events ($V_\tau = V_B$), nor past financial distress (any period when ($V_\tau < V_B$).

Under the procedure B, the distress counter is never reset to zero. Subsequent granted periods (and therefore tolerance) will be lower and lower as more default events and long financial distress will be observed. In fact, the granted time is lowered (each time) by the duration just used. Successive granted times are exactly equal to d minus the sum of any previous periods spent under the default barrier. This mimics a decision-making process that never forgets anything and depends on the whole history of the financial distress. By its very precise "count" process, this procedure is very idealized and thus appears unrealistic. Its main interest appears in the following paragraph[5].

4.2 Bounds for liquidation procedures

Here is the main result of the section.

Proposition 5 *Under the assumption 3, any possible liquidation processes are "bounded" by the procedures A and B in the sense that*

$$\tau^B \leq \tau \leq \tau^A$$

is verified for all liquidation dates τ

[5]It must also be stressed that considering only one procedure (e.g. procedure A) may lead to a spurious design. To illustrate this, let a leveraged firm have a debt in the form of a 10-years loan delivering a constant coupon rate whose most secured debtholders of the firm own a safety covenant. According to Black-Cox (1976), this latter implies the existence of a default threshold denoted V_B written on the firm assets value. One further assumes that the procedure A grants a period of $d = 2$ years before liquidation. Hence, the firm may regularly reach the default barrier without causing any assets liquidation. Let's now consider the exact time the firm assets value hits the default barrier for the first time. This is the exact time the firm enters in financial distress. The default threshold being reached, the debt service is suspended : there is no more coupon payment. If the state variable passes through the default threshold from below, the procedure is instantaneously stopped. When it comes back in the distress area, a period is once again granted with an identical delay d. Then, a couple of special cases are worth to be detailed. First, the state variable remains under the default threshold. Second, it spends consecutively 1 year and eleven months under V_B and then a month above it and so on until the end. In the former case, assets are liquidated and one may claim that the debt service has been "fairly" suspended. In the latter case, the procedure is triggered five times without causing the firm liquidation and the debt may be fully served only during 10 one-month periods. At the extreme, if renegociation costs capture the whole pay-out rate of the firm (not precised in equation (1)), bondholders may fear not to be (in any way) compensated for the incurred loss (whereas the defaulted firm is not liquidated). As this example shows, no need high bankruptcy costs to have the bankruptcy design fail.

If successive liquidation procedures are envisaged, either the maximum time d is granted each time or it is lowered. In this latter case, it should account for the previous distress duration. The extreme way to do this is the procedure B. Prices of corporate liabilities under effective liquidation procedures are therefore bounded.

Dealing with an interval instead of a single procedure is interesting. First, both limit procedures refer to a similar parameter d which yields to similar interpretation. Second, depending on the associated range, a detailed analysis and model of the real domestic liquidation procedure could be alleviated. Third, real life liquidation procedures are always hard to model in a way to obtain analytical results. Fourth, using a single proxy model surely leads to undesirable implications[6]. To exemplify this, let's assume, as François-Morellec (2002) did, that the idealized procedure A is a good proxy for the chapter 11 of the US Bankruptcy Code. Let's consider a general framework where there is a continuous debt service which depends on the firm financial wealth. Then, the coupon payment can clearly be suspended most of the time without the firm being ever liquidated because of the d-long duration condition. Finally, such an interval allows international comparison and arbitrage of corportae liabilities.

[6] E.g. do the corresponding prices represent a maximum or a minilmum ? In any case, the model risk must be simultaneously appreciated.

4.3 Other bounds

Whatever the real liquidation procedure is (assumption 3 is nonetheless supposed verified), special cases arise as the granted period gets larger than the maturity of the debt or drops to zero. To sum up, one states the following result.

Proposition 6 *For any liquidation procedure described in assumption 3, the price of any corporate liabilities L verifies*

$$\begin{aligned} L(d) &\underset{d\to T}{\longrightarrow} L_{BSM} \\ L(d) &\underset{d\to 0}{\longrightarrow} L_{BC} \end{aligned} \tag{6}$$

where L_{BSM} (resp. L_{BC}) is the price of the security under a Black-Scholes (resp. Black-Cox) liquidation procedure.

Let's first consider the procedures A and B. When the $d-$length duration gets larger than the maturity, no early liquidation before maturity can be signaled. This is equivalent to the process designed by Black-Scholes-Merton, one has:

$$Eq_i(d) \underset{d\to T}{\longrightarrow} Eq_{BSM}, i = A, B.$$

When the $d-$length duration gets to zero, no extra time is allowed beyong the default event. The liquidation time collapses to the first hitting time i.e. the first default time. This is the procedure considered in Black-Cox (1976). One has :

$$Eq_i(d) \underset{d\to 0}{\longrightarrow} Eq_{BC}, i = A, B.$$

Thanks to the proposition 8, this is true for any process within the interval formed by the procedures A and B. As this is true for equity price, by virtue of the previous section, this is true for any other corporate liablities.

5 Numerical analysis

This section aims at analyzing the delay effects emphasized in the previous section, Figure 1 plots equity prices for granted periods lying in $[0, T]$ within the procedure A. To compare, the default models of Black-Scholes-Merton and Black-Cox are represented too. Recalling that the unique debt is totally secured in a Black-Cox setting, the corresponding equity price is intuitively the cheapest. Values for d are chosen from 0 up to 25 weeks.

FIGURE 1

It appears that time delays substantically appreciate equity values. It has a non linear effect on prices. The safety covenant held by creditors in the Black-Cox model is significantly lowered by the granted period. Above 12 weeks, the effect of the default barrier is divided by two. Ten weeks suffice to fill the gap between extremely secured creditors and poorly ones. Additional simulations could also show that the designed liquidation model continuously changes from one to the other as the length time d increases. Above two years, the default thresholds is as unactive. To conclude, the time spent below the default threshold appears an important parameter.

6 A detailed example

Using a Black-Scholes-Merton style framework allows one to price various corporate securities, among which subordinated and convertible debts. Under the "absolute priority rule" assumption, corporate liabilities have been shown equivalent to portfolios of options written on the firm assets value.

6.1 A complex capital structure for the firm

Let it be a firm whose capital structure is composed by N_p stocks and three different debts maturing at the same date $T = 5$ years. The debt prices are denoted L_i, $i = 1, 2, 3$. One denotes F_1, F_2, F_3 the different face values. These liabilities are strictly ordered by priority in default. L_1 is the price of the most secured debt, L_3 is that of the most subordinated one. A safety covenant allows the senior debtholders to bankrupt the firm. Thanks to Proposition 6, the senior debt is valued as the difference of two equity prices. The first corresponds to an unlevered firm and the second to a firm facing a repaiment equal to F_1. One has $L_1 = Eq(0) - Eq(F_1) = v(V_0) - Eq(F_1)$ where $v(V_0)$ is the market firm value[7]. Along similar lines, the first and second subordinated debts (L_2 and L_3) are respectively price by $L_2 = Eq(F_1) - Eq(F_1 + F_2)$, $L_3 = Eq(F_1 + F_2) - Eq\left(\sum_{i=1}^{3} F_i\right)$. In some case, the most subordinated liability is supposed convertible at maturity. Convertible in N_n stocks, the

[7]Note that no closed form expression for the market value of the firm has yet been found. Today's firm value is therefore approximated by today's value of firm assets : $L_1 \approx V_0 - Eq(F_1)$.

dilution factor is $\xi = \frac{N_n}{N_p+N_n}$. By Proposition 8, its price is given by:

$$Eq\left(F_1+F_2\right) - Eq\left(F_1+F_2+F_3\right) + \xi Eq\left(F_1+F_2+\frac{F_3}{\xi}\right).$$

For numerical analysis, face values F_1, F_2, F_3 are 90, 25 and 10 respectively. If necessary, the dilution factor is equal to 1/3. The firm assets value is 100, its volatility 40 %. This base case depicts firms with substancial default risk to highlight the effects of deviations from safety covenants.

6.2 The prices of corporate securities

Table 1 presents the prices of corporate securities for different structural models of default and liquidation. The Black-Scholes-Merton model (BSM) assumes that there is neither default nor liquidation (nor early signal of financial distress) before the common maturity. At the other extreme, the Black-Cox model (BC) introduces an absorbing default barrier, $V_B(t) = V_B e^{-\gamma(T-t)}$ which implies an immediate liquidation of assets. Here, the default barrier increases at the rate of γ. Following Black-Cox' alternatives, this boundary is chosen either constant, BC($\gamma = 0$), or exponentially time-dependent, BC($\gamma \neq 0$). By choosing $\gamma = r$, the default barrier mimics a secured claim. The ratio of this barrier on a discounted secured value remains constant through time. For computation, one considers respectively 1% and 5 % below the face value of the most secured debt. In-between, (Cum) denotes the proposed liquidation model based on the occupation time. It distinguishes the default and liquidation dates. Recall that the liquidation is declared only if the cumulated period spent beyond the threshold exceeds the total granted time d (equal to one month or one year).

TABLE 1

Two main remarks merit to be made. First, the less risky debt (whose safety covenant justifies, according to Black-Cox (1976), the existence of a default threshold) is worth more as its reimbursment becomes more and more secured (from left to right). The senior debt is worth 53.570 if one neglects any safety covenant, 69.532 if there exists a "99 %"-safety barrier[8]. Symmetrically, this early default mechanism deteriorates both equity and subordinated loans prices. Second, the subordinated debt (2) is the riskiest liability for every liquidation procedure. Nonetheless, this loan seems to decrease in value when safety covenants are strengthened than the subordinated debt (1).

To support this claim, credit spreads are computed and displayed in Table 2. This is a simple way to compare the riskyness of different loans. Figures in Table 2 confirm that the credit spread of the most subordinated debt is (of course) the highest whatever the liquidation procedure is. It also appears that the difference between the two subordinated debts decreases as the liquidation procedure gets stronger. More precisely, when the most secured debt becomes almost riskless, the subordinated liabilities become more and more

[8] The price of a comparable riskless debt is $90e^{-0,05.5} \approx 70,10$.

similar with regards to their credit risk (and as expected, the credit spread of the secured debt tends to zero).

TABLE 2

By signaling successive default events and an (non equivalent) early liquidation and limiting the undesirable risk transfert implied by the classical default model, the total time spent under the default threshold is a useful way to model debt covenants and observed deviations. It represents an adjustable alternative between the Black-Scholes-Merton default procedure and the Black-Cox one. As a result, while the former underestimates creditors rights, the latter may not be appropriate for very long term debt.

7 Conclusion

This paper has developed a structural methodology capable of pricing complex corporate securities when the default threshold is not an absorbing barrier. Real-life liquidation procedures whose rationale is based on the time spent in financial distress are shown to be bounded by a couple of idealized procedures. One of these is similar to the one used by François-Morellec (2002) to model the Chapter 11 of the US Bankruptcy law. It is also demonstrated that the results of Black-Cox (1976) and Ingersoll (1977) on respectively subordinated and convertible bonds are robust when a liquidation procedure is taken in account. Numerical experiments illustrate that delays impacts on the ex ante price of corporate liabilties. Equity price strictly rises as the granted time below the thresholds increases. Credit spreads increase or decrease depending on the way the considered bond is secured or subordinated to others. As a final word, it appears that credit spreads of the secured bonds are less sensible to the way the Black-Cox (1976) default barrier is designed than to the time the firm can spend beyond the default event.

References

[1] Altman E., Eberhart A. (1994) : "Do priority provisions protect a bondholder's investment ?", *Journal of Portfolio Management*, 20, 67-75.

[2] Avellaneda M., L. Wu (1999) : "Pricing Parisian-Style Options with a Lattice Method", *International Journal of Theoretical and Applied Finance*, 2 (1), 1-16.

[3] Anderson R., Sundaresan S. (1996) : "Design and Valuation of Debt Contracts", *Review of Financial Studies*, 9(1), 37-68.

[4] Betker B. (1995) : "An empirical examination of prepackaged bankruptcy", *Financial Management*, 24, 3-18.

[5] Black F., Scholes M. (1973) : "The pricing of options and corporate liabilities", *Journal of Political Economy*, 81, 637-654.

[6] Black F., Cox J. (1976) : "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", *Journal of Finance*, 31(2), 351-367.

[7] Cathcart L., El-Jahel L. (1998) : "Valuation of Defaultable Bonds", *Journal of Fixed Income*, 8(1), 65-78.

[8] Chesney M., Jeanblanc-Picqué M., Yor M. (1995) : "Brownian Excursions and Parisian Barrier Options", *Advances in Applied Probability*, 29, 165-184.

[9] Fan H., Sundaresan S. [2000] : "Debt Valuation, Renegociation and Optimal Dividend Policy", *Review of Financial Studies*, 13, 1057-1099.

[10] François P., Morellec E. (2002) : "Capital Structure and Asset Prices : Some effect of Bankruptcy Procedures", *Journal of Business*, to appear.

[11] Haber R., Schönbucher P., Wilmott P. (1999) : "Pricing Parisan Options", 6(3), 71-79.

[12] Helwege J. (1999) : "How Long Do Junk Bonds Spend in Default ?", *Journal of Finance*, 54(1), 341-357.

[13] Hotchkiss E. (1995) : "Postbankruptcy performance and management turnover", *Journal of Finance*, 50, 3-21.

[14] Hugonnier J. (1999) : "The Feynman-Kac Formula and Pricing Occupation Times Derivatives", *International Journal of Theoretical and Applied Finance*, 2(2), 153-178.

[15] Ingersoll, J. (1977) : "A Contingent-Claims Valuation of Convertible Securities", *Journal of Financial Economics*, 4, 289-321.

[16] Leland H. (1994) : "Corporate Debt Value, Bond Covenants and Optimal Capital Structure", *Journal of Finance*, 49(4), 1213-1252.

[17] Leland H., Toft K. (1996) : "Optimal Capital Structure, Endogenous Bankruptcy and the Term Structure of Credit Spreads, *Journal of Finance*, 51(3), 987-1019.

[18] Longstaff F., Schwartz E. (1995) : "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt", *Journal of Finance*, 5, 789-820.

[19] Mella-Barral P., Perraudin P. (1997) : "Strategic debt service", *Journal of Finance*, 52(2), 531-556.

[20] Merton R. (1974) : "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance*, 29, 449-470.

[21] Moraux F. (2002) : "On Cumulative Parisian Options", *Finance*, 23 (Special Issue : Financial Mathematics), 127-132.

[22] Piessens R. (1971) : "Gaussian Quadrature Formulas for the Numerical integration of Bormwich's Integral and the Inversion of the Laplace Transform", *Journal of Engineering Mathematics*, 5(1), 1-9.

[23] Schönbucher P. (1998) : "Term Structure Modelling of Defaultable Bonds", *Review of Derivatives Research*, 2(2/3), 161-193.

Figure 1: The granted period and equity prices : from the Black-Cox model to the Black-Scholes-Merton one.

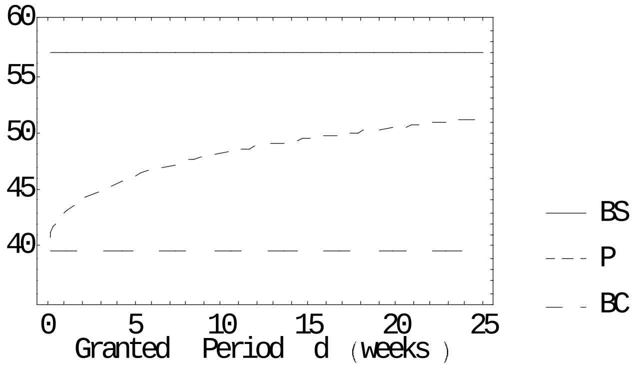

Figure 2: Firm parameters are $V_0 = 100, \delta_V = 0, \sigma_V = 30\%$. The single corporate debt matures in $T = 10$ years, its face value is 80. and the default threshold is 70. $r = 5\%$.

Table 1: Corporate securities, the default signaling and different liquidation procedures.

	BSM	Cum		BC	BC	
Threshold		constant		constant	exponential	
					95 %	99 %
Granted period		1 year	1 month	0	0	0
Equity	35.409	34.270	29.991	26.879	26.746	25.173
if convertible	25.205	24.330	21.188	18.939	18.811	17.679
Sub. Debt (2)	2.767	2.526	1.989	1.682	1.594	1.449
if convertible	12.973	12.465	10.790	9.621	9.529	8.942
Sub. Debt (1)	8.253	7.313	5.573	4.637	4.276	3.845
Senior Debt	53.570	55.89*	62.445*	66.801	67.383	69.532
Firm value	100	100*	100*	100	100	100

Every corporate liabilities mature in $T = 5$ years. Other parameters are $V_0 = 100, \delta_V = 0, \sigma_V = 40\%, r = 5\%$. The different face values are 90, 25, 10 by recovery priority. Default thresholds are worth $V_B = 0.99 F_1 e^{-rT}$ when constant and respectively egal to $V_B(\tau) = 0.99 F_1 e^{-r(T-\tau)}$.and $V_B(\tau) = 0.95 F_1 e^{-r(T-\tau)}$ when exponential as in Black-Cox (1976). The dilution factor for bonds convertible at maturity is 1/3. * recalls that the price has been valued by $V_0 - Eq(F1)$.

Table 2: Corporate credit Spread for different liquidation procedures (in percentage).

	BSM	Cum		BC	BC	
Threshold		constant		constant	exponential	
					95 %	99 %
Granted period		1 year	1 month	0	0	0
Sub. Debt (2)	20.70	22.52	27.30	30.65	31.73	33.63
Sub. Debt (1)	17.17	19.58	25.02	28.70	30.32	32.44
Senior Debt	5.38	4.53	2.31	0.96	0.79	0.16

Chosen parameters are those of Table 1. No credit spread has been computed for convertible debts.

8 Appendix

This appendix breifly considers the analytical pricing of cumulative parisian options[9]. Analytical valuation of occupation time derivatives have early been studied by Chesney-Jeanblanc-Yor (1997) and Hugonnier (1999). Here, one follows Hugonnier (1999)'s approach and derives new pricing formulae (among which the one used in the text). The short demonstration in this appendix uses the framework and arguments of Chesney-Jeanblanc-Yor (1997) and Hugonnier (1999). A Cumulative Parisian Call Option (CPC hereafter) is *an option whose pay-off is that of a standard call provided that the underlying asset has spent more than a prespecified time* [d] *beyond the barrier level* [L] (Hugonnier (1999)). A superscript ($^+$or $^-$) precises whether the occupation time is considered above or below the threshold level. Now, let it be an option, maturing at T, that counts the time spent above the threshold L. Let it be an underlying price process correctly described under the risk neutral measure by :

$$dS_t = (r - \delta)\, S_t dt + \sigma S_t dW_t. \tag{7}$$

where δ is the dividend rate. One denotes $\nu = \frac{1}{\sigma}(r - \delta - \frac{1}{2}\sigma^2)$, K the exercise price, T the maturity, $d(t, a, b, c) = \frac{\ln(a/b)}{\sigma\sqrt{T-t}} + c\sigma\sqrt{T-t}$, and Ψ the normal probability distribution function, it's well known that the vanilla call option

[9] These options are knocked in/knocked out when the total time spent by the underlying asset beyond a known barrier exceeds a prescribed value.

at time 0 is given by

$$C_{BSM}(T) = S_0 e^{-\delta T}\Psi(d_1(0, S_0, K)) - Ke^{-rT}\Psi(d_2(0, S_0, K))$$

where $d_1(t, a, b) = d(t, a, b, \nu + \sigma)$, $d_2(t, a, b) = d(t, a, b, \nu)$. The price of cumulative Parisian call option at time $t = 0$ can be written

$$CPC^{+}(d) = e^{-(r+\frac{\nu^2}{2})T}\mathcal{C}^{+}_{CPC} \tag{8}$$

where

$$\mathcal{C}^{+}_{CPC} = S_0\Psi^{+}_{\nu+\sigma}(T, k, l, d) - K\Psi^{+}_{\nu}(T, k, l, d) \tag{9}$$

with $l\sigma = \ln(L/S_0)$, $k\sigma = \ln(K/S_0)$and Ψ to be precised. Note that Chesney-Jeanblanc-Yor (1997) have termed $\mathcal{C}_{CPC}$ the (r, ν)-discounted value of the cumulative parisian call option. All other call options are then given by :

$$CPC^{\mp}(d) + CPC^{\pm}(T - d) = C(T).$$

If the occupation time under a barrier level is limited to d(until maturity T), the stock price is equivalentely expected to spend more than $T - d$time above it. A similar relation holds for put options. Parity relations between cumulative calls and puts exist too. Summing up Chesney-Jeanblanc-Yor (1997), one has :

$$CPP^{\mp}(T, S_0, K, L; r, \delta) = S_0 KCPC^{\pm}(T, \frac{1}{S_0}, \frac{1}{K}, \frac{1}{L}; \delta, r).$$

The derivation of Ψ^{+}appears thus critical for pricing cumulative parisian options. Its expression is as follows.

Proposition 7 *If $S_0 < L$,*

$$\Psi^+_\mu(t,k,l,d) = \int_d^T ds\Big[\int_{k\wedge l}^{l} e^{\mu x}\Upsilon(2l-x,0,s,t-s)\,dx + \int_{k\vee l}^{\infty} e^{\mu x}\Upsilon(l,x-l,s,t-s)\,dx\Big].$$

If $S_0 > L$,

$$\begin{aligned}\Psi^+_\mu(t,k,l,d) &= e^{\mu^2 t/2}\Big(\Phi(d_{\Xi(\mu)}(t,S_0,L\vee K)) - (\frac{L}{S_0})^{2\mu/\sigma}\Phi(d_{\Xi(\mu)}(t,\frac{L^2}{S_0},L\vee K))\Big)\\ &\quad + \int_d^T ds\Big[\int_{k\wedge l}^{l} e^{\mu x}\Upsilon(l-x,-l,s,t-s)\,dx + \int_{k\vee l}^{\infty} e^{\mu x}\Upsilon(0,x-2l,s,t-s)\,dx\Big]\end{aligned}$$

where

$$\Upsilon(a,b,u,v) = \int_0^\infty \frac{(z+a)(z+b)}{\pi(uv)^{1.5}}\exp(-\frac{(z+a)^2}{2v})\exp(-\frac{(z+b)^2}{2u})\,dz$$

and

$$\Xi(\mu) = \begin{cases} 1 & if \quad \mu = m+\sigma \\ 2 & if \quad \mu = m \end{cases} \tag{10}$$

This proposition is partly similar to Hugonnier's result (proposition 14 p. 106). It differs however for $S_0 > L$. Here, Ψ^+_μ is shown to be a (much more) complex fonction of μ. First, $\Xi(\mu)$ must be introduced. Second, the power of $\frac{L}{S_0}$ depends on it.

In order to demonstrate this proposition, more materials are needed. First, let's recall a result :

Lemma 1 (Hugonnier (1999)) *Let f be the pay-off function of an occupation time derivative. Let $\widehat{f} : \mathbb{R}^+ \times [0,T] \longrightarrow \mathbb{R}^+$ be the function defined by $\widehat{f}(z,s) = e^{\nu z} f(S_0 e^{\sigma z}, s)$. The $(r,\nu)-$discounted price at time $t = 0$ of the* *underlying contract is given by :*

- for $l > 0$: $\int_{-\infty}^{l} \widehat{f}(z,0)\Lambda_l(T,z)\,dz \quad +$

$$\int_0^T ds\bigg[\int_{-\infty}^{l} \widehat{f}(z,s)\Upsilon(2l-z,0,s,T-s)\,dz + \int_l^{\infty} \widehat{f}(z,s)\Upsilon(l,z-l,s,T-s)\,dz\bigg]$$

- for $l < 0$: $\int_l^{\infty} \widehat{f}(z,T)\Lambda_{-l}(T,-z)\,dz \quad +$

$$\int_0^T ds\bigg[\int_{-\infty}^{l} \widehat{f}(z,s)\Upsilon(l-z,-l,s,T-s)\,dz + \int_l^{\infty} \widehat{f}(z,s)\Upsilon(0,z-2l,s,T-s)\,dz\bigg]$$

To succeed in computing terms (6) and (7) with lemma 2, two hypothetical pay-off functions (f) must be identified. To the expectation in (6), that provides $\Psi_{\nu+\sigma}$ in the pricing formula (3), one associates the pay-off $f(u,v) = 1_{u\geq K}1_{v\geq d}\frac{u}{S_0}$. To the expectation in (7), providing Ψ_ν in (3), $f(u,v) = 1_{u\geq K}1_{v\geq d}$ is chosen. Let's denote respectively $\widehat{f}_{\nu+\sigma}, \widehat{f}_\nu$, the associated $\widehat{f}$ functions. Lemma 2 then tells us that must be considered :

$$\int_{-\infty}^{l} \widehat{f}_\mu(z,0)\Lambda_l(T,z)\,dz, \qquad l < 0 \tag{11}$$

$$\int_l^{\infty} \widehat{f}_\mu(z,T)\Lambda_{-l}(T,-z)\,dz, \qquad l > 0 \tag{12}$$

where either $\mu = \nu + \sigma$ or $\mu = \nu$. Recalling that by definition d is strictly positive and that, whatever μ is, $\widehat{f}_\mu(.,t)$ involves $1_{t\geq d}$, one concludes that $\widehat{f}_\mu(z,0) = 0$ in Equation (8). As a result, when $S_0 < L$, the only term in Ψ_μ is the double integral. This has already been stated by Hugonnier (1999).

Let's now turn to Equation (9) *i.e.* when $S_0 > L$. Since Λ_l verifies $\sqrt{2\pi t}\Lambda_l(t,x) = e^{-\frac{x^2}{2t}} - e^{-\frac{(2l-x)^2}{2t}}$, the equation may also be written :

$$\int_{k\vee l}^{\infty} e^{\mu z} e^{-\frac{z^2}{2t}} \frac{dz}{\sqrt{2\pi t}} - \int_{k\vee l}^{\infty} e^{\mu z} e^{-\frac{(2l-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} = \Gamma_1(\mu) - \Gamma_2(\mu) = \Gamma_\mu.$$

Each term can be integrated thanks to well known quadrature relations. Hence, denoting by n and Φ respectively the gaussian probability and gaussian cumulative density functions, one obtains :

$$\begin{aligned}
\Gamma_1(\mu) &= e^{\mu^2 t/2} \int_{k\vee l}^{\infty} e^{-\frac{(z-\mu t)^2}{2t}} d(\frac{z}{\sqrt{2\pi t}}) = e^{\mu^2 t/2} \int_{\frac{k\vee l-\mu t}{\sqrt{t}}}^{\infty} n(z)\, dz = e^{\mu^2 t/2} \Phi\left[-\frac{k\vee l-\mu t}{\sqrt{t}}\right] \\
&= e^{\mu^2 t/2} \Phi\left[\frac{\ln(S/(K\vee L)) + \mu\sigma t}{\sigma\sqrt{t}}\right], \\
\Gamma_2(\mu) &= e^{\mu^2 t/2} e^{2l\mu} \int_{k\vee l}^{\infty} e^{-\frac{(z-2l-\mu t)^2}{2t}} d(\frac{z}{\sqrt{2\pi t}}) = e^{\mu^2 t/2} (\frac{L}{S})^{2\mu/\sigma} \int_{\frac{k\vee l-2l-\mu t}{\sqrt{t}}}^{\infty} n(z)\, dz \\
&= e^{\mu^2 t/2} (\frac{L}{S})^{2\mu/\sigma} \Phi\left[-\frac{k\vee l-2l-\mu t}{\sqrt{t}}\right] \\
&= e^{\mu^2 t/2} (\frac{L}{S})^{2\mu/\sigma} \Phi\left[\frac{\ln(L^2/S(K\vee L)) + \mu\sigma t}{\sigma\sqrt{t}}\right].
\end{aligned}$$

As a result, one has :

$$\begin{aligned}
\Gamma_{\nu+\sigma} &= e^{(\nu+\sigma)^2 t/2} \Big(\Phi[d_1(0,S,K\vee L)] - (\frac{L}{S})^{2(\nu+\sigma)/\sigma} \Phi[d_1(0,L^2/S,K\vee L)] \Big) \\
\Gamma_\nu &= e^{\nu^2 t/2} \Big(\Phi[d_2(0,S,K\vee L)] - (\frac{L}{S})^{2\nu/\sigma} \Phi[d_2(0,L^2/S,K\vee L)] \Big).
\end{aligned}$$

Or, for short, $\Gamma_\mu = e^{\mu^2 t/2} \Big(\Phi[d_{\Xi(\mu)}(.)] - (\frac{L}{S})^{2\mu/\sigma} \Phi[d_{\Xi(\mu)}(.)] \Big)$ with

$$\Xi(\mu) = \begin{cases} 1 & \text{if} \quad \mu = m + \sigma \\ 2 & \text{if} \quad \mu = m \end{cases}$$

This ends the proof.