Hypoellipticity for Fokker-Planck operators and Witten Laplacians[*]

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Foreword

This text is a completed version of notes initially written for a course given by the author in july 2006 at the Morningside Center for Mathematics in Beijing. It is intended to present some recent developments around hypoelliptic techniques applied to some drift-diffusion operators involved in several models, namely the Witten Laplacian acting on functions and the Fokker-Planck operator of kinetic theory. It is oriented towards the presentation of a conjecture stated by the author and B. Helffer which says that a Fokker-Planck operator has a compact resolvent if and only if the associated Witten Laplacian on 0-forms has a compact resolvent.

It is made essentially of three parts. In the first one, a simple (and probably old) way of proving the hypoellipticity of the operator $v.\partial_x - \Delta_v$ on $\mathbb{R}^{2d}_{x,v}$ is presented. After the introduction of the Witten Laplacian on 0-forms and the Fokker-Planck operator, an adaptation of the previous method with the help of some standard global pseudodifferential calculus validates the conjecture in some elliptic or weakly elliptic case. The last part is devoted to the analysis of the Witten Laplacians with an arbitrary rank condition, on the basis of Helffer-Nourrigat results concerned with the maximal hypoellipticity.

The text is written in a rather self-contained way in order to be accessible to ph-D students or researchers mainly familiar with analysis or PDEs. The first proof of hypoellipticity is written by avoiding any use of pseudodifferential calculus. But any critical reader will realize afterwards that the pseudodifferential calculus provides simplifications with a systematic way of thinking. With a similar pedestrian approach, a complete proof of the maximal hypoellipticity for Witten Laplacians with polynomial potentials is provided. Indeed this specific example allows to make an explicit presentation of Helffer-Nourrigat analysis without requiring the machinery of Kirillov theory nor sophisticated microlocal surgery. For this last point, the author hopes that this text will in a second step encourage people to investigate these topics rather than prevent them from learning them. A motivation for writing such an explicit example was twofold :

1. Present the core of Helffer-Nourrigat induction argument which is a beautiful and little known piece of analysis.
2. This basic presentation is certainly a good way to explore the Fokker-Planck operators with degenerate potentials for which nothing is known up to now.

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Contents

1 Introduction.

1.1 Hypoellipticity : a basic example.

We shall establish here with the simplest possible tools the hypoellipticity of the differential operator in $\mathbb{R}^{2d}_{x,v}$

$$P = v.\partial_x - \Delta_v = \sum_{j=1}^{d} v_j \partial_{x_j} - \partial_{v_j}^2 \,. \tag{1.1}$$

Notations : The Fourier transform of a function $u(x,v)$ is normalized according to

$$\hat{u}(\xi,\eta) = (Fu)(\xi,\eta) = \int_{\mathbb{R}^{2d}} e^{-i(\xi.x+\eta.v)} u(x,v)\; dxdv\,.$$

For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^{2d})$ is characterized by

$$\begin{aligned} \left(u \in H^s(\mathbb{R}^{2d})\right) \quad &\Leftrightarrow \quad \left(\int_{\mathbb{R}^{2d}} \langle \xi,\eta\rangle^{2s} \left|\hat{u}(\xi,\eta)\right|^2 \; d\xi d\eta < +\infty\right) \\ &\Leftrightarrow \quad \left((1-\Delta_{x,v})^{s/2} u \in L^2(\mathbb{R}^{2d}, dxdv)\right)\,, \\ \text{with} \qquad & \qquad \langle \xi,\eta\rangle^2 = 1 + |\xi|^2 + |\eta|^2\,. \end{aligned}$$

The space $H^s_{loc}(\mathbb{R}^{2d})$ is characterized by

$$\left(u \in H^s_{loc}(\mathbb{R}^{2d})\right) \Leftrightarrow \left(\forall \chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}), \quad \chi u \in H^s(\mathbb{R}^{2d})\right)\,.$$

Keep in mind that for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ the mapping $u \mapsto \chi u$ is continuous from $H^s(\mathbb{R}^{2d})$ (or $H^s_{loc}(\mathbb{R}^{2d})$) into $H^s(\mathbb{R}^{2d})$. Note also the consequence of Sobolev Lemma :

$$\mathop{\cap}_{s\in\mathbb{R}} H^s_{loc}(\mathbb{R}^{2d}) = \mathcal{C}^\infty(\mathbb{R}^{2d}) \qquad \mathop{\cup}_{s\in\mathbb{R}} H^s_{loc}(\mathbb{R}^{2d}) = \mathcal{D}'(\mathbb{R}^{2d})\,.$$

Definition 1.1. *A differential operator P in $\mathbb{R}^n$ is said to be* <u>*hypoelliptic*</u> *around x_0 when there is a neighborhood ω_{x_0} of x_0 such that the implication*

$$(Pu \in \mathcal{C}^\infty(\omega_{x_0})) \Rightarrow (u \in \mathcal{C}^\infty(\omega_{x_0}))$$

holds for any $u \in \mathcal{D}'(\mathbb{R}^n)$. It is said hypoelliptic in an open set Ω when it is hypoelliptic around any $x_0 \in \Omega$.

Proposition 1.2. *For the operator $P = v.\partial_x - \Delta_v$ introduced in* (1.1)*, $u \in H^s_{loc}(\mathbb{R}^{2d})$ and $Pu \in H^s_{loc}(\mathbb{R}^{2d})$ imply $u \in H^{s+1/4}_{loc}(\mathbb{R}^{2d})$.*

Corollary 1.3. *The operator $P = v.\partial_x - \Delta_v$ is hypoelliptic in $\mathbb{R}^{2d}$.*

Proof of the corollary : Assume that $Pu \in \mathcal{C}^\infty(\omega_{X_0})$ for some neighborhood ω_{X_0} of $X_0 = (x_0, v_0) \in \mathbb{R}^{2d}$. Let χ denote a cut-off function $\chi \in \mathcal{C}_0^\infty(\omega_{X_0})$, $\chi \equiv 1$ in a neighborhood of X_0. We will check that χu belongs to $H^{s_0+n/4}(\mathbb{R}^{2d})$ by induction $n \in \mathbb{N}$, while possibly adapting χ at every step.

- Since $u \in \mathcal{D}'(\mathbb{R}^{2d})$, there exists $s_0 \in \mathbb{R}$ such that $\chi u \in H^{s_0}(\mathbb{R}^{2d})$.
- Assume that $\chi_n u \in H^{s_0+n/4}(\mathbb{R}^{2d})$ for $n \in \mathbb{N}$ with $\chi_n \in \mathcal{C}_0^\infty(\omega_{X_0})$, $\chi_n \equiv 1$ in a neighborhood of X_0, and set $s = s_0 + n/4$. We want to prove $\chi u \in H^{s+1/4}(\mathbb{R}^{2d})$, when $\chi \in \mathcal{C}_0^\infty(\omega_{X_0})$ is chosen so that $\chi_n \equiv 1$ in a neighborhood of $\operatorname{supp} \chi$. We shall use of course the information on $\chi(Pu)$ but the point here is that we want to transform it into some information on $P(\chi u)$ in order to use the Proposition 1.2.

a) A uniform estimate with respect to $\delta \in (0; \delta_0]$ of $\left\|P(\chi w_\delta)\right\|_{H^s}$ when

$$w_\delta = (1 - \delta\Delta_{x,v})^{-1} w, \quad \delta \in (0, \delta_0], \quad w = \chi_n u, \tag{1.2}$$

will first be considered. We compute

$$\begin{aligned}
P(\chi w_\delta) &= \chi(Pw_\delta) + (v.\partial_x\chi)w_\delta - [\Delta_v, \chi]\, w_\delta \\
&= \chi(Pw_\delta) + (v.\partial_x\chi)w_\delta - 2\partial_v.(\partial_v\chi w_\delta) + (\Delta_v\chi)w_\delta \\
&= -2\partial_v.((\partial_v\chi)w_\delta) + R_\chi \\
\text{with} \quad & \left\|R_\chi\right\|_{H^s} \leq C_\chi \left(\left\|w_\delta\right\|_{H^s} + \left\|\chi P w_\delta\right\|_{H^s}\right) .
\end{aligned}$$

The problem is then to get an estimate on $-2\partial_v.((\partial_v\chi)w_\delta)$ in H^s or by changing the function χ on $\partial_v(\chi' w_\delta)$.

Compute the H^s-scalar product according to

$$\begin{aligned}
\operatorname{Re} \left\langle \chi' w_\delta \,,\, P\chi' w_\delta \right\rangle_{H^s} &= \left\langle \chi' w_\delta \,,\, (1 - \Delta_{x,v})^s P(\chi' w_\delta) \right\rangle \\
&= -2 \operatorname{Re} \left\langle \chi' w_\delta \,,\, (1 - \Delta_{x,v})^s \partial_v.(\partial_v\chi')w_\delta \right\rangle \\
&\qquad\qquad + \left\langle \chi' w_\delta \,,\, (1 - \Delta_{x,v})^s R_{\chi'} \right\rangle \\
&\leq 2 \left\|\partial_v(\chi' w_\delta)\right\|_{H^s} \left\|(\partial_v\chi')w_\delta\right\|_{H^s} + \left\|\chi' w_\delta\right\|_{H^s} \left\|R_{\chi'}\right\|_{H^s}
\end{aligned}$$

Replace now P with $v.\partial_x - \Delta_v$:

$$\begin{aligned}
\operatorname{Re} \left\langle \chi' w_\delta \,,\, P\chi' w_\delta \right\rangle_{H^s} &= \operatorname{Re} \left\langle \partial_v(\chi' w_\delta) \,,\, (1 - \Delta_{x,v})^s \partial_v(\chi' w_\delta) \right\rangle \\
&\qquad + \frac{1}{2} \left\langle \chi' w_\delta \,,\, (1 - \Delta_{x,v})^s v.\partial_x(\chi' w_\delta) \right\rangle \\
&\qquad + \frac{1}{2} \left\langle v.\partial_x(\chi' w_\delta) \,,\, (1 - \Delta_{x,v})^s(\chi' w_\delta) \right\rangle \\
&= \left\|\partial_v(\chi' w_\delta)\right\|_{H^s}^2 \\
&\qquad - \frac{1}{2} \Big\langle \chi' w_\delta \,,\, \sum_{j=1}^d \left[v_j, (1 - \Delta_{x,v})^s\right] \partial_{x_j}(\chi' w_\delta) \Big\rangle \\
&\geq \left\|\partial_v(\chi' w_\delta)\right\|_{H^s}^2 - C_s \left\|\chi' w_\delta\right\|_{H^s}^2 ,
\end{aligned}$$

where the last inequality comes from the explicit computation of the commutator after a Fourier transform :

$$[v_j, (1-\Delta_{x,v})^s] \sim \left[\partial_{\eta_j}, (1+\xi^2+\eta^2)^s\right] = 2s\eta_j \left(1+\xi^2+\eta^2\right)^{s-1} .$$

We obtain

$$\begin{aligned}\left\|\partial_v(\chi' w_\delta)\right\|_{H^s}^2 \leq 2\left\|\partial_v(\chi' w_\delta)\right\|_{H^s}\left\|(\partial_v\chi')w_\delta\right\|_{H^s} + C_s\left\|\chi' w_\delta\right\|_{H^s}^2 \\ + C_{\chi'}\left\|R_{\chi'}\right\|_{H^s}\left\|\chi' w_\delta\right\|_{H^s}\end{aligned}$$

and finally

$$\left\|\partial_v(\chi' w_\delta)\right\|_{H^s}^2 \leq \varepsilon\left\|\partial_v(\chi' w_\delta)\right\|_{H^s}^2 + C_\varepsilon\left[\left\|w_\delta\right\|_{H^s}^2 + \left\|Pw_\delta\right\|_{H^s}^2\right] ,$$

With $\varepsilon < 1/2$ this leads to

$$\left\|\partial_v(\chi' w_\delta)\right\|_{H^s} \leq C\left[\left\|w_\delta\right\|_{H^s} + \left\|\chi' Pw_\delta\right\|_{H^s}\right] .$$

By going back to the first identity with χ, $\chi \equiv 1$ in a neighborhood of $\operatorname{supp}\chi$, we deduce $P(\chi w_\delta) \in H^s(\mathbb{R}^{2d})$ with a uniform estimate

$$\left\|P(\chi w_\delta)\right\|_{H^s(\mathbb{R}^{2d})} \leq C'\left[\left\|w_\delta\right\|_{H^s} + \left\|\chi' Pw_\delta\right\|_{H^s}\right] . \tag{1.3}$$

b) The definition (1.2) of w_δ implies $\left\|w_\delta\right\|_{H^s} \leq \left\|w\right\|_{H^s} = \left\|\chi_n u\right\|_{H^s}$ and a simple application of Lebesgue's theorem on $\int_{\mathbb{R}^{2d}}\langle\xi,\eta\rangle^{2s}\left|\hat{w_\delta} - \hat{w}\right|^2\, d\xi d\eta$ yields $\lim_{\delta\to 0}\left\|w_\delta - w\right\|_{H^s} = 0$.
The second term of the right-hand side of (1.3) is estimated by

$$\left\|\chi' Pw_\delta\right\|_{H^s} \leq \left\|\chi'\left(Pw_\delta - Pw\right)\right\|_{H^s} + \left\|\chi' Pw\right\|_{H^s} .$$

The relation $w = (1-\delta\Delta_{x,v})w_\delta$ implies

$$\begin{aligned} Pw &= (1-\delta\Delta_{x,v})Pw_\delta + \left[P, (1-\delta\Delta_{x,v})\right]w_\delta \\ Pw_\delta &= (1-\delta\Delta_{x,v})^{-1}Pw \\ &\qquad + (1-\delta\Delta_{x,v})^{-1}\delta[v.\partial_x, \partial_x^2+\partial_v^2]w_\delta \\ &= Pw + \left[(1-\delta\Delta_{x,v})^{-1}Pw - Pw\right] - (1-\delta\Delta_{x,v})^{-1}\delta\partial_v.\partial_x w_\delta \end{aligned}$$

Applying again Lebesgue's theorem on the integrals which define the H^s-norms, this leads to $\lim_{\delta\to 0}\left\|Pw_\delta - Pw\right\|_{H^s} = 0$. Hence the family $P(\chi w_\delta)$ is uniformly bounded in $H^s(\mathbb{R}^{2d})$ and its weak limit can be only $P(\chi w)$ for $\lim_{\delta\to 0} w_\delta = w$ in $H^s(\mathbb{R}^{2d})$. Finally after noticing that $\chi_n \equiv 1$ in a neighborhood of $\operatorname{supp}\chi' \supset \operatorname{supp}\chi$ implies $\chi'(Pw) = \chi'(Pu)$ and $P(\chi w) = P(\chi u)$, we have proved

$$\frac{\left\|P(\chi u)\right\|_{H^s}}{C'} \leq \liminf_{\delta\to 0}\left\|w_\delta\right\|_{H^s} + \left\|\chi'(Pw_\delta)\right\|_{H^s} = \left\|\chi_n u\right\|_{H^s} + \left\|\chi'(Pu)\right\|_{H^s} .$$

Proposition 1.2 implies $\chi u \in H^{s+1/4}(\mathbb{R}^{2d}) = H^{s_0+\frac{n+1}{4}}(\mathbb{R}^{2d})$. ∎

Proof of Proposition 1.2 : Assume $u \in H^s_{loc}(\mathbb{R}^{2d})$ and $Pu \in H^s_{loc}(\mathbb{R}^{2d})$. By taking $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, we have $\chi u \in H^s(\mathbb{R}^{2d})$ with a compact support and the same argument as in the previous induction step allows to assume

$$\partial_v(\chi u) \in H^s(\mathbb{R}^{2d}) \quad \text{and} \quad P(\chi u) \in H^s(\mathbb{R}^{2d})\,.$$

We will prove the hypoelliptic estimate

$$\|w\|_{H^{s+1/4}} \leq C\left[\|Pw\|_{H^s} + \|w\|_{H^s}\right] \tag{1.4}$$

for a compactly supported $w(=\chi u)$ by assuming $w \in H^{s+2}(\mathbb{R}^{2d})$. The same arguments as in the part **b)** of the previous proof with the the approximation $w_\delta = (1-\delta\Delta_{x,v})^{-1}w$ then yields the result for any compactly supported $w \in H^s(\mathbb{R}^{2d})$ such that $Pw \in H^s(\mathbb{R}^{2d})$.
Assume first $w \in H^{s+2}(\mathbb{R}^{2d})$. We already know

$$\|\partial_v w\|_{H^s} \leq C\left[\|Pw\|_{H^s} + \|w\|_{H^s}\right]\,.$$

We want to recover a similar estimate for the missing directions ∂_{x_k}, $k = 1,\ldots,d$.
Compute for $\varrho \geq 0$, the $H^{s+\varrho}$-norm in the following way (ϱ is fixed in the end) :

$$\begin{aligned}
\|w\|^2_{H^{s+\varrho}} &\leq \left\langle w\,,\,(1-\Delta_x-\Delta_v)^{s+\varrho-1}(1-\Delta_x-\Delta_v)w\right\rangle\\
&\leq -\sum_{k=1}^d \left\langle w\,,\,(1-\Delta_x-\Delta_v)^{s+\varrho-1}\,\partial_{x_k}\partial_{x_k}w\right\rangle + \|w\|^2_{H^{s+\varrho-1}} + \|\partial_v w\|^2_{H^{s+\varrho-1}}\,.
\end{aligned}$$

For the first term, use the fact that ∂_{x_k} equals the commutator

$$\partial_{x_k} = \left[\partial_{v_k}, v.\partial_x\right]\,, \tag{1.5}$$

namely write

$$\begin{aligned}
\left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial^2_{x_k}w\right\rangle &= \left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial_{x_k}\left[\partial_{v_k}, v.\partial_x\right]w\right\rangle\\
&= \left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial_{x_k}\left[\partial_{v_k}, P\right]w\right\rangle\\
&= \left\langle -v.\partial_x w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\,\partial_{x_k}\partial_{v_k}w\right\rangle\\
&\qquad -\left\langle w\,,\,\left[v.\partial_x, (1-\Delta_{x,v})^{s+\varrho-1}\right]\partial_{x_k}\partial_{v_k}w\right\rangle\\
&\qquad -\left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial_{x_k}\partial_{v_k}(v.\partial_x)w\right\rangle\\
&= -2\,\mathrm{Re}\,\left\langle \partial_{x_k}\partial_{v_k}w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}(v.\partial_x)w\right\rangle\\
&\qquad -\langle w\,,\,\sum_{j=1}^d\left[v_j\partial_{x_j}, (1-\Delta_{x,v})^{s+\varrho-1}\right]\partial_{x_k}\partial_{v_k}w\rangle
\end{aligned}$$

Finally, use the explicit form of the operator $P = v.\partial_x - \Delta_v$, in order to get

$$\begin{aligned}\left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial_{x_k}^2 w\right\rangle &= -2\,\mathrm{Re}\,\left\langle \partial_{x_k}\partial_{v_k} w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}Pw\right\rangle \\ &\quad +2\,\mathrm{Re}\sum_{j=1}^{d}\left\langle \partial_{x_k}\partial_{v_k}\partial_{v_j} w\,,\,(1-\Delta_{x,v})^{s+\varrho-1}\partial_{v_j} w\right\rangle \\ &\quad +c_{s,\varrho}\sum_{j=1}^{d}\left\langle w\,,\,(1-\Delta_{x,v})^{s+\varrho-2}\partial_{v_j}\partial_{x_j}\partial_{v_k} w\right\rangle \\ &= A_1 + A_2 + A_3\,.\end{aligned}$$

With $\varrho \leq 1$ the third term is estimated by

$$|A_3| \leq C\,\|w\|_{H^s}\,\|\partial_v w\|_{H^s}\ .$$

With $\varrho \leq 1/2$ the first term is estimated by

$$|A_1| \leq C\,\|\partial_v w\|_{H^s}\,\|Pw\|_{H^s}\ .$$

The second term requires some care

$$|A_2| \leq \left\|(1-\Delta_{x,v})^{\varrho-1}\partial_{v_k}\partial_{x_k}\partial_v w\right\|_{H^s}\,\|\partial_v w\|_{H^s}\ .$$

We write

$$\begin{aligned}&\left\|(1-\Delta_{x,v})^{\varrho-1}\partial_{v_k}\partial_{x_k}\partial_v w\right\|_{H^s}^2 \\ &\quad = \mathrm{Re}\,\left\langle (1-\Delta_{x,v})^{\frac{s}{2}+\varrho-1}\partial_{x_k}\partial_{v_k} w\,,\,P(1-\Delta_{x,v})^{\frac{s}{2}+\varrho-1}\partial_{x_k}\partial_{v_k} w\right\rangle \\ &\quad = -\,\mathrm{Re}\,\left\langle \partial_{v_k} w\,,\,\partial_{x_k}(1-\Delta_{x,v})^{s+2\varrho-2}\partial_{x_k}\partial_{v_k} Pw\right\rangle \\ &\qquad -\,\mathrm{Re}\,\left\langle \partial_{v_k} w\,,\,\partial_{x_k}[v.\partial_x, (1-\Delta_{x,v})^{s+2\varrho-2}\partial_{x_k}\partial_{v_k}]w\right\rangle \\ &\quad = A_{2,1} + A_{2,2}\,.\end{aligned}$$

The first term is bounded by

$$|A_{2,1}| \leq C\,\|\partial_v w\|_{H^s}\,\|Pw\|_{H^s}$$

provided that $\varrho \leq 1/4$. Finally an explicit computation of the commutator in $A_{2,2}$ implies

$$|A_{2,2}| \leq C\,\|\partial_v w\|_{H^{s+2\varrho-1/2}}\,\|w\|_{H^{s+2\varrho-1/2}}$$

which is estimated by H^s-norm when $\varrho \leq 1/4$. Gathering the estimates leads to (1.4) with $\varrho = 1/4$. ∎

This example is a basic example of type II Hörmander operator. Hörmander distinguished in [Hor1] two classes of hypoelliptic “sum of squares” :

Type I: $P_I = \sum_{j=1}^m X_j^2$

Type II: $P_{II} = X_0 + \sum_{j=1}^{m} X_j^2$

where the X_j's, $j = (0), 1, \ldots, m$ are $\mathcal{C}^\infty$ vector fields. Here with $-P = -v.\partial_x + \sum_{j=1}^{d} \partial_{v_j}^2$, take $X_0 = -v.\partial_x$ and $X_j = \partial_{v_j}$ for $j = 1, \ldots, d$. We recall for the differential operators $P_{I,II}$ acting on $\mathbb{R}^n$ the next classical result.

Hörmander's Theorem : 1. *[Hor1] If at a point $x_0 \in \mathbb{R}^n$ the family of iterated commutators*

$$\left(X_I = [X_{i_1}, [X_{i_2}, \ldots [X_{i_{p-1}}, X_{i_p}] \ldots]\right)_{I=(i_1,i_2,\ldots,i_p), \quad \#I=p\leq r}$$

with some fixed $r \in \mathbb{N}$, span the tangent space $T_{x_0}\mathbb{R}^n$, then P_I (resp. P_{II}) is hypoelliptic around x_0.

In our example, this structure appears in (1.5) which allows to recover the missing directions of derivations. We have followed here Kohn's method, the simplest and the more flexible, but which does not provide the optimal Sobolev exponent ($\varrho = 1/4$ here while $\varrho = 2/3$ can be obtained with a more refined analysis). The key point of this methods are

- Recover the missing directions by an explicit handling of commutators with integration by parts.
- Use a little of pseudo-differential calculus in order to control the remainder terms. Here the commutators $[v_k, (1 - \Delta_{x,v})^s]$ were computed explicitely but this is possible only for this example.

A summary of Kohn's method for general type I and type II Hörmander operators may be found in [HelNi]. A variation of it has been recently proposed by C. Villani in [Vil] in order to handle nonlinear kinetic equations.
Other methods are :

- Fine microlocal and geometric methods in the derivation of subelliptic estimates. A presentation of this can be found in [Hor2]-XXVII. A slightly different treatment with analytic coefficients was developed by Maire in [Mai] after Treves in [Tr1][Tr2].
- Nilpotent Lie algebra techniques after Rothschild-Stein [RoSt], Helffer-Nourrigat [HeNo], Nourrigat [Nou]. This method which is very algebraic provides a lot of information. There is a local-global rigidity which can be bypassed with other methods and an interesting interplay between qualitative estimates (that is without any control of the constants) and quantitative asymptotic estimates with respect to some "frequency" parameters.

Both methods provide the optimal Sobolev exponent for our example.

1.2 From stochastic differential equations to Witten Laplacians and Fokker-Planck operators.

We simply review here how the Witten Laplacian and the kinetic Fokker-Planck operator (sometimes called Kramers operator) are related with stochastic processes. We refer the reader for example to [Ris][Ev] for more details. In this section, we simply make computations without taking care about regularity or decay assumptions.

1.2.1 A brief review on SDE.

Take the notations

$$\begin{array}{lll} X \in \mathbb{R}^n, & b : \mathbb{R}^n \to \mathbb{R}^n\,, & B \in M_n(\mathbb{R}) \\ & X \to b(X) & \\ \multicolumn{3}{l}{W \ n\text{-dimensional white noise} \quad dW_k dW_\ell = \delta_{k,\ell} dt\,,} \end{array}$$

and consider the stochastic differential equation

$$dX = b(X)dt + BdW\,. \tag{1.6}$$

Ito formulas : For a smooth function $(x,t) \to u(x,t)$, Ito's chain rule says[1] :

$$\begin{array}{c} du(X,t) = \dfrac{\partial u}{\partial t}\, dt + \displaystyle\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\, dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_j \partial x_j}\, dX_i dX_j \\ \text{with} \qquad dt^2 = 0, \quad dW dt = 0, \quad dW_k dW_\ell = \delta_{k,\ell} dt\,. \end{array}$$

When X solves the SDE (1.6), this leads to

$$\begin{aligned} d\left[u(X(t))\right] &= \frac{\partial u}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\left[b_i(X,t)\ dt + (B\ dW)_i\right] \\ &\qquad + \frac{1}{2}\sum_{i_1,j_1,i_2,j_2=1}^{n} \frac{\partial^2 u}{\partial x_{i_1}\partial x_{j_1}} B_{i_1 i_2} B_{j_1 j_2}\ dW_{i_2} dW_{j_2}\,. \\ &= \left[\partial_t u + b.\partial_x u + \frac{1}{2}\partial_x.(BB^t)\partial_x u\right]\ dt + \partial_x u.(B\ dW)\,. \end{aligned}$$

If v_0 is an observable independent of the t-variable, one gets

$$\begin{aligned} v_0(X(t)) = v_0(X(0)) + \int_0^t \left(b.\partial_x v_0 + \frac{1}{2}\partial_x.(BB^t)\partial_x v_0(X(s))\right)\ ds \\ + \int_0^t \partial_x v_0(X(s)).BdW\,. \end{aligned} \tag{1.7}$$

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[1] Formally compute the first order variation of $u(X,t)$ while assuming $dW = O(dt^{1/2})$.

The relationship between stochastic differential equations and drift-diffusion semigroups is obtained after computing the expectation value

$$v(x_0,t)=\mathbb{E}\left(v_0(X,t);X(0)=x_0\right)\,,$$

for some smooth and decaying observable v_0.
According to (1.7) we get

$$v(x_0,t)=v_0(x_0)+\int_0^t\left(b.\partial_x v+\frac{1}{2}\partial_x.(BB^t)\partial_x v\right)(x_0,s)\;ds+0\,,$$

because $\mathbb{E}\left(\int_0^t G(s)\;dW(s)\right)=0$, for any progressively measurable function G such that $\mathbb{E}\left(\int_0^T G(s)^2\;ds\right)<\infty$.

By setting

$$L=-b.\partial_x-\frac{1}{2}\partial_x.(BB^t)\partial_x\,. \tag{1.8}$$

we obtain

$$v(x_0,t)=v_0(x_0)+\int_0^t(-Lv)(s)\;ds\,,$$

which is the integral form of

$$v(t)=e^{-tL}v_0\,,\quad\text{or}\quad\left\{\begin{array}{l}\partial_t v=-Lv=b.\partial_x v+\frac{1}{2}\partial_x(BB^t)\partial_x v\\ v(t=0)=v_0\,.\end{array}\right. \tag{1.9}$$

1.2.2 Application 1 : Reversible diffusion processes.

Consider the case when the drift vector field is a gradient $b(x)=-\partial_x V(x)$ (in this case the diffusion process is said reversible) and $B=\sqrt{2}\mathrm{Id}$. The SDE (1.6) reads

$$dX=-\partial_x V(X)\;dt+\sqrt{2}dW$$

and the generator of the corresponding semigroup (1.8) equals

$$L=\partial_x V(x).\partial_x-\Delta_x=\left(\partial_x V(x)-\partial_x\right).\partial_x\,.$$

An invariant measure is then given by $\frac{d}{dt}\left\langle e^{-tL}v_0,\mu\right\rangle_{\mathcal{D}'}=0$, or equivalently

$$L'\mu=0\quad\text{with}\quad L'=-\partial_x.\partial_x V(x)-\Delta_x=-\partial_x.(\partial_x V(x)+\partial_x)\,.$$

Hence for a reversible diffusion process the invariant probability measure equals

$$\mu_V=\frac{e^{-V(x)}}{\int_{\mathbb{R}^n}e^{-V(x)}\;dx}\;dx$$

provided that $e^{-V}\in L^1(\mathbb{R}^n,dx)$.

Property : For a reversible diffusion process with $e^{-V} \in L^1(\mathbb{R}^n; dx)$, the semigroup generator $L = (\partial_x V - \partial_x).\partial_x$ with domain $D(L) = \mathcal{C}_0^\infty(\mathbb{R}^n)$ is symmetric in the Hilbert space $L^2(\mathbb{R}^n, d\mu_V)$.

Another writing of L makes this property more obvious. Set $f = e^{\frac{V}{2}} u$:

$$\begin{aligned} & (f \in L^2(\mathbb{R}^n, d\mu_V)) \Leftrightarrow \left(u = e^{-\frac{V}{2}} f \in L^2(\mathbb{R}^n, dx)\right) \\ & \partial_x f = \partial_x (e^{\frac{V}{2}} u) = e^{\frac{V}{2}} (\partial_x + \frac{1}{2}\partial_x V(x)) u \\ e^{-\frac{V}{2}} L e^{\frac{V}{2}} &= -(\partial_x - \frac{1}{2}\partial_x V(x)).(\partial_x + \frac{1}{2}\partial_x V(x)) \\ &= -\Delta_x + \frac{1}{4}\left|\partial_x V(x)\right|^2 - \frac{1}{2}\Delta V(x)\,. \end{aligned}$$

Definition 1.4. *For a potential $V \in \mathcal{C}^\infty(\mathbb{R}^n)$, the Witten Laplacian (on 0-forms) associated with $V/2$ is the operator defined by*

$$\begin{aligned} \Delta_{V/2}^{(0)} &: D(\Delta_{V/2}^{(0)}) = \mathcal{C}_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dx) \\ & \qquad \Delta_{V/2}^{(0)} = -\Delta + \frac{1}{4}\left|\partial_x V(x)\right|^2 - \frac{1}{2}\Delta V(x)\,. \end{aligned}$$

Actually the name Witten Laplacian better refers to a deformation of Hodge theory, acting on p-forms[Wi][CFKS]. On a Riemannian manifold (M, g) it is defined as follows. Let d denote the differential operator $d : \mathcal{C}_0^\infty(M; \Lambda^p T^*M) \to \mathcal{C}_0^\infty(M; \Lambda^{p+1} T^*M)$ defined in local coordinates by

$$d\left(\omega_{i_1,\ldots,i_p}\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) = \sum_{i=1}^n \partial_{x_i}\omega_{i_1,\ldots,i_p}\, dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\,.$$

The codifferential $d^* : \mathcal{C}_0^\infty(M; \Lambda^p T^*M) \to \mathcal{C}_0^\infty(M; \Lambda^{p-1} T^*M)$ is then defined as the adjoint of d via the scalar product $\int_M \langle \omega\,,\, \eta \rangle_{g(x),p}(x)\ d\nu_g(x)$ where $\langle\ \ \rangle_{g(x),p}$ and $d\nu_g$ are the scalar product on p-forms at $x \in M$ and the measure on M associated with the metric g.

For a $\mathcal{C}^\infty$ function V on M and for $h > 0$, the deformed differential is defined as

$$d_{V/2,h} = e^{-V/2h}(hd)e^{V/2h} = hd + \frac{1}{2}dV\wedge$$

and its adjoint is

$$d_{V/2,h}^* = e^{V/2h}(hd^*)e^{-V/2h} = hd^* + \frac{1}{2}\mathbf{i}_{\nabla V}\,,$$

where $\wedge$ denotes the exterior product an $\mathbf{i}_X$ the interior product with the tangent vector X. These distorted differentials satisfy

$$d_{V/2,h} \circ d_{V/2,h} = 0 \quad \text{and} \quad d_{V/2,h}^* \circ d_{V/2,h}^* = 0\,.$$

The Witten Laplacian is then defined on $\mathcal{C}^\infty(M;\Lambda T^*M)$ as

$$\Delta_{V/2,h} = (d_{V/2,h}\circ d^*_{V/2,h} + d^*_{V/2,h}\circ d^h_{V/2}) = (d_{V/2,h}+d^*_{V/2,h})^2\,,$$

and its restriction on p-forms is usually denoted $\Delta^{(p)}_{V/2,h}$.
Here we focus on the case $p=0$ with $h=1$ and the writing $\Delta^{(0)}_{V/2} = d^*_{V/2}d_{V/2}$ is nothing but

$$\Delta^{(0)}_{V/2} = -\left(\partial_x - \frac{1}{2}\partial_x V(x)\right)\cdot\left(\partial_x + \frac{1}{2}\partial_x V(x)\right)\,.$$

Note that this writing ensures that $\Delta^{(0)}_{V/2}$ is a nonnegative Schrödinger operator with the $\mathcal{C}^\infty$ potential $\frac{1}{4}\left|\partial_x V(x)\right|^2 - \frac{1}{2}\Delta V(x)$. A result by Simader (see [Sima]) then says

Proposition 1.5. *When $V\in\mathcal{C}^\infty(\mathbb{R}^n)$, the Witten Laplacian $\Delta^{(0)}_{V/2}$ is essentially self-adjoint*[2] *on $\mathcal{C}^\infty_0(\mathbb{R}^n)$.*

Notation : In the sequel the same notation $\Delta^{(0)}_{V/2}$ will be used for the self-adjoint closure.

Exercise 1.6. *Check that $u\in$ Ker $(\Delta^{(0)}_{V/2})$ if and only if*

$$u\in L^2(\mathbb{R}^n)\quad \textit{and}\quad d(e^{\frac{V}{2}}u)=0 \textit{ in } \mathcal{D}'(\mathbb{R}^n;\mathbb{R}^n)\,.$$

(Approximate $u\in$ Ker $(\Delta^{(0)}_{V/2})$ by a $u_n\in\mathcal{C}^\infty_0(\mathbb{R}^n)$ in the graph norm $\|f\|_{V/2} = \|f\|_{L^2} + \left\|\Delta^{(0)}_{V/2}f\right\|$ and study the limit $d_{V/2}u_n$.)

From this result of which a variation will be studied later, we know that the unitary group $(e^{it\Delta^{(0)}_{V/2}})_{t\in\mathbb{R}}$ and the semigroup $(e^{-t\Delta^{(0)}_{V/2}})_{t>0}$ (therefore $(e^{-tL})_{t>0}$) are well defined. An important question in the study of stochastic processes and their application concerns the return to the equilibrium.
(QW0) Is there a constant $\alpha_W>0$ such that

$$\left\|e^{-t\Delta^{(0)}_{V/2}}u - c_u e^{-V/2}\right\|_{L^2} \leq Ce^{-\alpha_W t}\left\|u\right\|_{L^2}\ ?$$

Within the probabilistic framework it is often presented as the question of having a Poincaré inequality

$$\int_{\mathbb{R}^n}\left(f-\int_{\mathbb{R}^n} f\ d\mu_V\right)^2 d\mu_V \leq \frac{1}{\alpha_W}\int_{\mathbb{R}^n}\left|\nabla f\right|^2\ d\mu\,.$$

---

[2] Remember that this says that $\Delta^{(0)}_{V/2}$ admits a unique self-adjoint extension given by $D(\Delta^{(0)}_{V/2}) = \left\{u\in L^2(\mathbb{R}^n,dx)\,,\Delta^{(0)}_{V/2}u\in L^2(\mathbb{R}^n,dx)\right\}$.

With the previous result it is a question about the spectrum of $\Delta^{(0)}_{V/2}$:

$$\inf[\sigma(\Delta^{(0)}_{V/2}) \setminus \{0\}] = \alpha_W > 0\,.$$

This leads to two questions :

(QW1) Is the resolvent $(1+\Delta^{(0)}_{V/2})^{-1}$ compact ?

(QW2) For the parameter dependent version $\Delta^{(0)}_{V/2,h}$, is it possible to have accurate expansions for $\alpha_W(h)$?

1.2.3 Application 2 : kinetic Fokker-Planck equation.

We consider now the stochastic dynamic in the phase-space $\mathbb{R}^{2d}_{x,v}$ (more realistic brownian motion also referred to as the Ornstein-Uhlenbeck process) given by :

$$\begin{aligned} dx &= vdt \\ dv &= -\frac{1}{m}\partial_x V(x)dt - \gamma v dt + \sqrt{\frac{2}{m\beta}}dW\,. \end{aligned}$$

The parameters are m, the particle mass, $\gamma > 0$ a friction coefficient, $\beta = \frac{1}{k_B T}$ the inverse temperature and W is a d-dimensional white noise. Following the rule of (1.8), we find that the evolution of observable is governed by the semigroup[3] generated by

$$K_1 = -v.\partial_x + \frac{1}{m}\partial_x V(x).\partial_v + \frac{\gamma}{m\beta}\left(-\partial_v + m\beta v\right).\partial_v\,.$$

Its formal adjoint equals

$$K_1' = v.\partial_x - \frac{1}{m}\partial_x V(x).\partial_v - \frac{\gamma}{m\beta}\partial_v(\partial_v + m\beta v)\,.$$

Hence an invariant probability measure (when it exists) is given by the normalized Maxwellian

$$M(x,v) = \frac{e^{-\beta(\frac{mv^2}{2}+V(x))}}{\int_{\mathbb{R}^{2d}} e^{-\beta(\frac{mv^2}{2}+V(x))}\,dxdv}\,dxdv\,.$$

Like in the derivation of the Witten Laplacian, the symmetry properties of the operators K_1 and K_1' are better understood after conjugating with

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[3] It will be checked that it is well defined.

$e^{-\beta\frac{mv^2+2V(x)}{4}}$. We are led to the analysis of

$$\begin{aligned}
e^{-\beta\frac{mv^2+V(x)}{4}}K_1 e^{\beta\frac{mv^2+V(x)}{4}} &= -v.\partial_x + \frac{1}{m}\partial_x V(x).\partial_v \\
&\qquad + \frac{\gamma}{m\beta}\left(-\partial_v + \frac{m\beta}{2}v\right).\left(\partial_v + \frac{m\beta}{2}v\right) =: K_- \\
e^{\beta\frac{mv^2+V(x)}{4}}K_1' e^{-\beta\frac{mv^2+V(x)}{4}} &= v.\partial_x - \frac{1}{m}\partial_x V(x).\partial_v \\
&\qquad + \frac{\gamma}{m\beta}\left(-\partial_v + \frac{m\beta}{2}v\right).\left(\partial_v + \frac{m\beta}{2}v\right) =: K_+ \\
\text{with} \qquad & K_\pm : \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{xv}) \subset L^2(\mathbb{R}^{2d}, dxdv) \to L^2(\mathbb{R}^{2d}_{xv})\,.
\end{aligned}$$

The real part of $K_\pm$,

$$\frac{1}{2}(K_+ + K_-) = \frac{\gamma}{m\beta}\left(-\partial_v + \frac{m\beta}{2}v\right).\left(\partial_v + \frac{m\beta}{2}v\right)$$

is a Witten Laplacian in the velocity variable (actually it is an harmonic oscillator hamiltonian).
The imaginary part

$$\frac{1}{2}(K_+ - K_-) = v.\partial_x - \frac{1}{m}\partial_x V(x).\partial_v$$

is the Hamiltonian vector field associated with the classical energy $p(x,v) = \frac{mv^2}{2} + V(x)$ and the symplectic form

$$\sigma(X_1, X_2) = mv_1.x_2 - mx_1.v_2, \quad X_i = (x_i, v_i)\,.$$

The question of the return to the equilibrium can be stated as
(QFP0) Is there a constant $\alpha_{FP} > 0$ such that

$$\left\|e^{-tK_+}u - c_u M^{1/2}\right\|_{L^2} \leq Ce^{-\alpha_{FP}t}\left\|u\right\|_{L^2}\,?$$

It can be decomposed into two steps

(QFP1) Is the resolvent $(1+K_\pm)^{-1}$ compact ?

(QFP2) Is it possible to compute or estimate $\alpha_{FP}(m,\gamma,\beta)$ in terms of the parameters (m,γ,β) ?

Here are two important differences with the case of the Witten Laplacian:

- $K_\pm$ is not elliptic. Only hypoellipticity can be expected.
- $K_\pm$ is not self-adjoint. This makes its spectral analysis and the introduction of a constant α_{FP} more subtle.

1.3 General remarks.

a) Once the operator $\Delta^{(0)}_{V/2}$ and $K_{\pm}$ have been derived from probabilistic arguments, it is no more necessary to restrict the analysis to the case when $e^{-V(x)} \in L^1(\mathbb{R}^d_x)$. Actually for some properties like the compactness of the resolvent, it is important to get rid of this restriction in order to have a better insight of what is really in the balance. It will be possible to consider a potential V such that $\lim_{x\to\infty} V(x) = \pm\infty$ and the discussion on the sign contains a lot of information.

b) The Witten Laplacian arises from a model with a complete diffusion (in all direction) and it is therefore elliptic. The reversibility of the stochastic process ensures it self-adjointness.

c) The two previous properties of the Witten Laplacian are lost for $K_{\pm}$. Nevertheless the strong relationships that can be exhibited between the two operators allow to get a quite accurate information on $K_{\pm}$. Actually the question **QFP1** and **QW1** about the compactness of the resolvent are intimately related. Similarly some accurate comparison exists between $\alpha_W(h)$ (see **QW2**) and $\alpha_{FP}(m,\gamma,\beta)$ (see **QFP2**), with $h \sim \beta^{-1}$. In some sense the Witten Laplacian will play the role of the Laplace operator Δ_x in our initial example (1.1).

d) The hypoelliptic Laplacian introduced recently by J.M. Bismut ([Bi] [BiLe] [Leb1] [Leb2]) is the geometric extension of kinetic Fokker-Planck operators on the contangent T^*M of a Riemannian manifold M. Written in a Hodge setting it appears as a microlocalization (or phase-space) version of Witten's deformation of Hodge theory. It raises a lot of interesting questions in analysis and provides a better information on some geometric structures.

Remark 1.7. *In these notes, we will focus on the qualitative problems namely* ***QW1*** *and* ***QFP1****. Nothing will be required about Hodge theory while it is a key point in the quantitative analysis of exponentially small eigenvalues done in [HelKlNi]. Actually the two questions* ***Q*1*** *and* ***Q*2*** *are not completely separated and this will appear especially when we will apply Helffer-Nourrigat results (techniques of adding or reducing variables).*

2 First properties and relations between $\Delta^{(0)}_{V/2}$ and $K_{\pm}$.

2.1 Maximal accretivity.

The references about maximal accretivity can be the books by Dautray-Lions [DaLi](Vol. 5, Chapter XVII), Reed-Simon [ReSi] or [Dav1]. Here is a summary. Let $\mathcal{H}$ be a complex (or real) Hilbert space.

Definition 2.1.

- *Let A be an unbounded operator in $\mathcal{H}$ with domain $D(A)$. We say that A is accretive if*

$$\operatorname{Re}\langle Ax \mid x\rangle_{\mathcal{H}} \geq 0 \;,\; \forall x \in D(A) \;. \tag{2.1}$$

- *An accretive operator A is maximally accretive if there there is no accretive extension $\tilde{A}$ with strict inclusion of $D(A)$ in $D(\tilde{A})$.*

Actually the notion of accretivity corresponds to a semi-symmetry and the notion of maximal accretivity corresponds to a semi-self-adjointness, according to the following table.

$A = iB$	
A accretive : $\operatorname{Re}\langle Ax\,,\,x\rangle \geq 0$	B symmetric : $\langle Bx\,,\,x\rangle \in \mathbb{R}$ $\operatorname{Re}\langle Ax\,,\,x\rangle \leq 0$ and ≥ 0
$(A \text{ accretive}) \Rightarrow \big(\overline{A} \text{ accretive}\big)$	$(B \text{ symm.}) \Rightarrow \big(\overline{B} \text{ symm.}\big)$
$(A \text{ accretive}) \Rightarrow (A^* \text{ accretive})$	$(B \text{ symm.}) \Rightarrow (B^* \text{ symm.})$
A maximally accretive	B self-adjoint
Hille-Yosida theorem for contraction semigroup : $((e^{-tA})_{t\geq 0}) \Leftrightarrow (A \text{ max. acc.})$	Stone theorem for unitary group : $((e^{-itB})_{t\in\mathbb{R}}) \Leftrightarrow (B \text{ self-adj.})$
$(A$ max. acc.$)$ implies $(\sigma(A) \subset \{\operatorname{Re} z \geq 0\})$	$(B$ self-adj.$)$ implies $(\sigma(B) \subset \mathbb{R})$

By following this correspondence it is possible to introduce the notion of essential maximal accretivity when the domain of the closed operator cannot be made explicit.

Definition 2.2. *An accretive operator A in $\mathcal{H}$ with domain $D(A)$, is said essentially maximally accretive if it admits a unique maximally accretive extension.*

The equivalence of the next statements can easily be checked :

1. A is essentially maximally accretive.
2. $\overline{A}$ is maximally accretive.
3. There exists $\lambda_0 > 0$ such that $A^* + \lambda_0 I$ is injective.
4. There exists $\lambda_1 > 0$ such that the range of $A + \lambda_1 I$ is dense in $\mathcal{H}$.

Remark 2.3. *A particular case is when A is a differential operator with $\mathcal{C}^\infty$ coefficients initially defined with $D(A) = \mathcal{C}_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. The domain of its closure equals*

$$D(\overline{A}) = \left\{ f \in L^2(\mathbb{R}^n),\ Af \in L^2(\mathbb{R}^n) \right\} .$$

According to the point 4, the essential maximal accretivity of A is true if for some $\lambda_1 > 0$

$$\left((\lambda_1 + A')f = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n), f \in L^2(\mathbb{R}^n) \right) \Rightarrow (f = 0)$$

where A' is the formal adjoint of A.

2.2 Essential maximal accretivity of the Fokker-Planck operator.

The next result is a variation of Simader's theorem [Sima] which says that a semi-bounded Schrödinger operator with a $\mathcal{C}^\infty$ potential is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n)$. For the sake of simplicity, we take $m = \beta = \gamma = 1$ and we write $K = K_+$ (its formal adjoint K_- shares the same properties).

Proposition 2.4.
Let V be a C^∞ potential on $\mathbb{R}^d$, then Fokker-Planck operator defined on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ defined by

$$K := -\Delta_v + \frac{1}{4}|v|^2 - \frac{d}{2} + X_0 \ , \tag{2.2}$$

where

$$X_0 := v \cdot \partial_x - \nabla V(x) \cdot \partial_v \tag{2.3}$$

is essentially maximally accretive.

Remark 2.5. *Actually this result is somehow surprising. Nothing is required about the sign or the behaviour of the potential $V(x)$ at infinity. The following discussion can be made after introducing the friction coefficient $\gamma \geq 0$ (with $m = \beta = 1$). When $\gamma = 0$ the Fokker-Planck operator is reduced to the Hamiltonian vector field $v.\partial_x - \partial_x V(x).\partial_v$ and it is known that the dynamics is* *not well defined (globally in time) when $V(x)$ goes to $-\infty$ faster than $-Cx^2$. The previous result simply says that the dynamics (that is the semigroup $(e^{-tK_\pm})_{t\geq 0}$) is well defined as soon as $\gamma > 0$, whatever the potential $V(x)$ does at infinity.*

Proof: We apply the abstract criterion taking $\mathcal{H} = L^2(\mathbb{R}^{2d})$ and $A = K$. The operators being real, we can consider everywhere real functions. The accretivity on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ is clear. Changing K in $T := K + (\frac{d}{2}+1)I$, we would like to show that its range is dense.
Let $f \in L^2(\mathbb{R}^2 d)$ be such that

$$< f \mid Tu >_{\mathcal{H}} = 0 \ , \ \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \ . \tag{2.4}$$

We have to show that $f = 0$.
Because K is real, one can assume that f is real.
We first observe that (2.4) implies that :

$$(-\Delta_v + v^2/4 + 1 - X_0)f = 0 \ , \quad \text{in } \mathcal{D}'(\mathbb{R}^{2d}) \ .$$

The standard hypoellipticity theorem for the Hörmander operators of type 2 implies that $f \in C^\infty(\mathbb{R}^{2d})$. Actually this can be recovered from Proposition 1.2 and Corollary 1.3 by writing

$$-v.\partial_x f - \Delta_v f = -\partial_x V(x)\partial_v f - v^2/4 f - f$$

and checking first the regularity of $\partial_v f$ from the ellipticity in the v variable.
We now introduce a family of cut-off functions $\zeta_k := \zeta_{k_1,k_2}$ by

$$\zeta_{k_1,k_2}(x,v) := \zeta(x/k_1)\zeta(v/k_2) \ , \ \forall k \in \mathbb{N}^2 \ , \tag{2.5}$$

where ζ is a C^∞ function satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(0,1)$ and $\operatorname{supp} \zeta \subset B(0,2)$.
For a derivation ∂ and a cut-off ζ, we start with the formula

$$\partial(\zeta u) = (\partial\zeta)u + (\zeta\partial u)$$

and its adjoint relation

$$-\zeta(\partial.X) = -\partial.(\zeta X) + (\partial\zeta).X \ .$$

We obtain

$$\begin{array}{rcl} -\zeta\partial.(\partial(\zeta u)) & = & -\zeta\partial.[(\partial\zeta)u] - \zeta\partial.[\zeta(\partial u)] \\ & = & -\partial.\left[\zeta(\partial\zeta)u\right] + |\partial\zeta|^2 u - \partial.\left[\zeta^2\partial u\right] + \zeta(\partial\zeta).(\partial u) \\ & = & +\partial.[\zeta(\partial\zeta)u] + |\partial\zeta|^2 u - \partial.\partial[\zeta^2 u] + \zeta(\partial\zeta).(\partial u) \ . \end{array}$$

Apply this with $\partial = \nabla_v$, $\zeta = \zeta_k$, $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and take the scalar product with $f \in L^2$:

$$\begin{aligned}\int \nabla_v(\zeta_k f)\cdot\nabla_v(\zeta_k u)\; dxdv \;&=\; \int f(-\Delta_v(\zeta_k^2 u)) + \left|\nabla_v\zeta_k\right|^2 fu\; dxdv\\ &\quad + \sum_{i=1}^d \int \left(f\partial_{v_i}u - \partial_{v_i}fu\right)(\partial_{v_i}\zeta_k)\zeta_k\; dxdv\end{aligned}$$

By adding the terms of K other than $-\Delta_v$ one gets :

$$\begin{aligned}&\textstyle\int \nabla_v(\zeta_k f)\cdot\nabla_v(\zeta_k u)\; dxdv + \int \zeta_k(x,v)^2(v^2/4+1)u(x,v)\,f(x,v)\; dx\,dv\\ &\quad + \textstyle\int f(x,v)(X_0(\zeta_k^2 u))(x,v)\; dx\,dv\\ &\qquad = \textstyle\int |(\nabla_v\zeta_k)(x,v)|^2 u(x,v)f(x,v)\; dx\,dv\\ &\qquad\quad + \textstyle\sum_{i=1}^d \int \left(f(\partial_{v_i}u) - u(\partial_{v_i}f)\right)(x,v)\zeta_k(x,v)(\partial_{v_i}\zeta_k)(x,v)\; dx\,dv\\ &\qquad\quad + \langle f(x,v)\mid T\zeta_k^2 u\rangle\;.\end{aligned}\tag{2.6}$$

When f satisfies (2.4), we get :

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^d} \nabla_v(\zeta_k f)\cdot\nabla_v(\zeta_k u)\; dxdv + \int \zeta_k^2(v^2/4+1)u(x,v)\,f(x,v)\; dx\,dv\\ &\quad + \textstyle\int f(x,v)(X_0(\zeta_k^2 u))(x,v)dx\,dv\\ &\qquad = \textstyle\int |(\nabla_y\zeta_k)(x)|^2 u(x)f(x,v)\; dx\,dv\\ &\qquad\quad + \textstyle\sum_{i=1}^d \int \left(f(\partial_{v_i}u) - u(\partial_{v_i}f)\right)(x,v)\zeta_k(x,v)(\partial_{v_i}\zeta_k)(x,v)\; dx\,dv\;,\end{aligned}\tag{2.7}$$

for all $u \in C^\infty(\mathbb{R}^{2d})$. In particular, we can take $u = f$.
We obtain

$$\begin{aligned}&< \nabla_v(\zeta_k f)\mid\nabla_v(\zeta_k f) > + \textstyle\int \zeta_k^2(v^2/4+1)|f(x,v)|^2\; dx\,dv\\ &+ \textstyle\int f(x,v)(X_0(\zeta_k^2 f))(x,v)\; dx\,dv\\ &= \textstyle\int |\nabla_v\zeta_k|^2|f(x,v)|^2\; dx\,dv\;.\end{aligned}\tag{2.8}$$

With an additional integration by part, we get

$$\begin{aligned}&< \nabla_v(\zeta_k f)\mid\nabla_v(\zeta_k f) > + \textstyle\int \zeta_k^2(v^2/4+1)|f(x,v)|^2\; dx\,dv\\ &+ \textstyle\int \zeta_k f(x,v)^2(X_0\zeta_k)(x,v)\; dx\,dv\\ &= \textstyle\int |\nabla_v\zeta_k|^2|f(x,v)|^2\; dx\,dv\;.\end{aligned}\tag{2.9}$$

This leads to the existence of a constant C such that, for all k,

$$\begin{aligned}&||\zeta_k f||^2 + \tfrac{1}{4}||\zeta_k v f||^2\\ &\leq C\tfrac{1}{k_2^2}||f||^2 + C\tfrac{1}{k_1}||v\zeta_k f||\;||f|| + C\tfrac{1}{k_2}||\nabla V(x)\zeta_k f||\;||f||\;.\end{aligned}\tag{2.10}$$

(The constant C will possibly be changed from line to line). This leads to

$$||\zeta_k\, f||^2 + \frac{1}{8}||\zeta_k\, v\, f||^2 \leq C(\frac{1}{k_2^2} + \frac{1}{k_1^2})||f||^2 + C(k_1)\frac{1}{k_2}||\zeta_k f||\;||f||\;, \tag{2.11}$$

where

$$C(k_1) = \sup_{|x|\leq 2k_1} |\nabla_x V(x)|$$

This implies

$$||\zeta_k f||^2 \leq C(\frac{\tilde{C}(k_1)}{k_2^2} + \frac{1}{k_1^2})||f||^2 \,. \tag{2.12}$$

This finally leads to $f = 0$. For example, one can take first the limit $k_2 \to +\infty$, which leads to

$$||\zeta(\frac{x}{k_1})f||^2 \leq \frac{C}{k_1^2}||f||^2 \,,$$

and then the limit $k_1 \to +\infty$. ∎

Notation : Like for Witten Laplacians, the same notation will be used for the Fokker-Planck operator $K_\pm$ with $D(K_\pm) = \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and its closure with

$$D(K_\pm) = \left\{f \in L^2(\mathbb{R}^{2d}),\, K_\pm f \in L^2(\mathbb{R}^{2d})\right\} \,.$$

2.3 An elementary link between QW1 and QFP1.

We recall that K cannot have a compact resolvent as soon as there exists an orthonormal sequence $(U_k)_{k\in\mathbb{N}}$ in $L^2(\mathbb{R}^{2n})$, with $U_k \in D(K)$, such that $\|KU_k\|$ remains bounded.

Theorem 2.6.
Assume that V is C^∞ function. If the operator $K_\pm$ has a compact resolvent then $\Delta^{(0)}_{V/2}$ has a compact resolvent.

Proof: By contradiction, assume that $(1 + \Delta^{(0)}_{V/2})^{-1}$ is not compact. Since $\Delta^{(0)}_{V/2}$ is self-adjoint, there exists an orthonormal sequence $(u_k)_{k\in\mathbb{N}}$ such that

$$\langle u_k \mid \Delta^{(0)}_{V/2} u_k\rangle = \left\|d_{V/2}u_k\right\|^2$$

is bounded. The sequence given by

$$U_k(x,v) = u_k(x)(2\pi)^{-d/4}e^{-v^2/4}$$

is an orthogonal sequence and satisfies

$$\forall k \in \mathbb{N},\; KU_k = (2\pi)^{-d/4}(d_{V/2}u_k(x)).ve^{-v^2/4} \quad \text{in } \mathcal{D}'(\mathbb{R}^{2d}) \,.$$

Hence every U_k belongs to $D(K)$ and we have found an orthonormal (after normalization) sequence $(U_k)_{k\in\mathbb{N}}$ such that $\|KU_k\|_{L^2}$ is uniformly bounded. ∎

Exercise 2.7. *Prove*

$$\left(0 \in \sigma_{ess}(\Delta^{(0)}_{V/2}\right) \Rightarrow (0 \in \sigma_{ess}(K_\pm))\,.$$

We recall that $\lambda \in \sigma_{ess}(A)$ as soon as there exists an orthonormal sequence $(U_k)_{k\in\mathbb{N}}$ in $D(A)$ such that $\lim_{k\to\infty} \|(A-\lambda)\,U_k\| = 0$ and that this condition is necessary and sufficient when A is self-adjoint.

2.4 A conjecture.

In [HelNi] the next conjecture was stated and studied.

Conjecture 2.8. *[HelNi] For $V \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, the Fokker-Planck operator $K_\pm$ has a compact resolvent if and only if the Witten Laplacian $\Delta^{(0)}_{V/2}$ has a compact resolvent.*

We will show in the next sections that it can be partly proved... and that it is far from being completely solved.

3 Global Kohn method for Fokker-Planck operators.

We follow the same lines as in Proposition 1.2 and Corollary 1.3 but now we have to introduce some pseudo-differential calculus. Pseudo-differential calculus is useful for at least three points

1. Estimating operators and commutators by simply counting exponents.
2. Before 1, ensuring the definition of commutators as operators well defined within a Sobolev scale. It is well-known (see for example [ReSi]) that a weak definition of commutators via bilinear forms can be misleading.
3. Providing estimates for operators constructed via functional analysis arguments.

3.1 Global Weyl-Hörmander pseudo-differential calculus.

An interest of the Weyl-Hörmander pseudo-differential calculus is that it provides directly global estimates. We give here a brief account of the general theory and then explain how it is applied (details may be found in [Hor2]-Chap XVIII, [BoLe], [BonChe], [NaNi], [HelNi]).

3.1.1 General review.

The introduction of a (small)-parameter $h > 0$ makes the asymptotic expansions more obvious, although we will essentially use the case $h = 1$ in these notes.

Notations : The generic phase-space coordinate will be denoted $Z = (z, \zeta) \in \mathbb{R}^{2n}$. On $\mathbb{R}^{2n}$, the symplectic form is $\sigma(Z_1, Z_2) = \zeta_1.z_2 - z_1.\zeta_2$. We shall use the notations

$$D_z = \frac{1}{i}\partial_z \quad \text{and} \quad D_Z = \frac{1}{i}\partial_Z$$

Weyl quantization : For $a \in \mathcal{S}'(\mathbb{R}^{2n})$, the kernel

$$\begin{aligned} K_a(z, z') &= \int_{\mathbb{R}^n} e^{i\frac{(z-z').\zeta}{h}} a(\frac{z+z'}{2}, \zeta)\ \frac{d\zeta}{(2\pi h)^n} \\ &= \frac{1}{h^n}(F_2^{-1}a)(\frac{z+z'}{2}, \frac{z-z'}{h}) \end{aligned}$$

belongs to $\mathcal{S}'(\mathbb{R}^{2n})$. Therefore

$$[a^W(z, hD_z)]u(z) = \int_{\mathbb{R}^n} K_a(z, z')u(z')\ dz'$$

defines the operator $a^W(z, hD_z)$ as a continuous operator $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

$$\big(a^W(z, hD_z) \text{ symmetric}\big) \Leftrightarrow (a \text{ real})\ .$$

Class of symbols : The standard class of symbols is defined as the set of $\mathcal{C}^\infty$ functions which fulfill

$$\left|\partial_z^\alpha \partial_\zeta^\beta a(z, \zeta)\right| \leq C_{\alpha,\beta} \langle\zeta\rangle^{m-|\beta|}\ .$$

This can be written as

$$\left|\partial_z^\alpha \left(\langle\zeta\rangle \partial_\zeta\right)^\beta a(z, \zeta)\right| \leq C_{\alpha,\beta} \langle\zeta\rangle^m$$

and one notes that ∂_{z_j} and $\langle\zeta\rangle \partial_{\zeta_j}$, $j = 1, \ldots, n$, are vector fieds T on $\mathbb{R}^{2n}$ such that $g(T) \leq 1$ in the metric $g = dz^2 + \frac{d\zeta^2}{\langle\zeta\rangle^2}$. Hence a general symbol class can be defined with a metric g on $\mathbb{R}^{2n}$ and a weight (here the function $\langle\zeta\rangle^m$).

Definition 3.1. *For a Riemannian metric g on $\mathbb{R}^{2n}$ and a weight $M : \mathbb{R}^{2n} \to (0, +\infty)$, the symbol class $S(M, g)$ is the space of $\mathcal{C}^\infty$ functions such that*

$$\forall N \in \mathbb{N}, \exists C_N > 0, \forall (z, \zeta) \in \mathbb{R}^{2n}, \qquad \max_{\substack{T_1, \ldots, T_N \\ g(T_i) \leq 1}} |T_1 \ldots T_N a(z, \zeta)| \leq C_N M(z, \zeta)\ .$$

Remark 3.2. *The best constant* C_N *provide a semi-norm* $p_N(a)$ *and* $S(M,g)$ *is a Fréchet space.*

Definition 3.3. *For a metric* g *on* $\mathbb{R}^{2n}$*, the dual metric* g^σ *is defined by*

$$g_Z^\sigma(T) = \max_{T\neq 0} \frac{(\sigma(T,X)^2)}{g_Z(X)}\,.$$

The gain function $\lambda : \mathbb{R}^{2n} \to (0,+\infty)$ *is then defined by*

$$\lambda(Z)^2 = \min_{T\neq 0} \frac{g^\sigma(T)}{g(T)}\,.$$

Exercise 3.4. *Check that for* $g = \sum_{j=1}^n \frac{dz_j^2}{a_i(Z)^2} + \frac{d\zeta_j^2}{b_j(Z)^2}$ *the dual metric is*

$$g^\sigma = \sum_{j=1}^n b_j(Z)^2 dz_j^2 + a_j^2(Z) d\zeta_j^2$$

and compute $\lambda(Z)$ *.*

The assumptions which lead to a good pseudo-differential calculus are :

(H1) Uncertainty principle : $\forall Z, T, \quad \frac{g_Z^\sigma(T)}{g_Z(T)} \geq \lambda^2(Z) \geq 1\,.$

(H2) Slowness : The metric varies slowly :

$$\exists C_0 > 0, \forall Z, Z', \quad \left(g_Z(Z-Z') \leq C_0^{-1}\right) \Rightarrow \max_{T\neq 0} \left(\left(\frac{g_Z(T)}{g_{Z'}(T)}\right)\right)^{\pm 1} \leq C_0\,.$$

(H3) Temperance : The metric is tempered, according to

$$\exists C_1 > 0, \exists N_1 \in \mathbb{N}, \max_{T\neq 0} \left(\left(\frac{g_Z(T)}{g_{Z'}(T)}\right)\right)^{\pm 1} \leq C_1 \left(1 + g_Z^\sigma(Z-Z')\right)^{N_1}\,.$$

(H4) Temperance of the weight : The weight M satisfies the g-dependent temperance estimate

$$\exists C_2 > 0, \exists N_2 \in \mathbb{N}, \quad \left(\frac{M(Z)}{M(Z')}\right)^{\pm 1} \leq C_2 \left(1 + g_Z^\sigma(Z-Z')\right)^{N_2}\,.$$

Remarks 3.5. a) *In a semiclassical setting* $h \to 0$ *and with an* h*-dependent metric* ***H1*** *can be replaced by* $\lambda^h(Z) \geq h$ *with different results.*

b) *The slowness assumption* ***H2*** *allows the construction of a partition of unity so that locally the metric* g_Z *can be replaced by a frozen metric* g_{Z_0} *.*

c) *The temperance assumptions **H3** and **H4** allow the summation of all the quantities localized after the partition of unity.*

The basic results under these assumptions are :
(R1) For any $a \in S(M, g)$, the $a^W(z, hD_z)$ defines a continuous operator from $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) into itself. The composition on the left or right with any continuous operator $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is possible, in particular with another pseudodifferential operator.
(R2) The algebra of pseudodifferential operators is transformed into an algebra of symbols after introducing the operation $\sharp^{W,h}$

$$(a\sharp^{W,h}b)^W(z, hD_z) = a^W(z, hD_z) \circ b^W(z, hD_z)\,.$$

For this operation one has the asymptotic expansion

$$a\sharp^{W,h}b(Z) = \left(e^{\frac{hi}{2}\sigma(D_{Z_1},D_{Z_2})}a(Z_1)b(Z_2)\right)\Big|_{Z_1=Z_2=Z} \tag{3.1}$$

$$= \sum_{j=0}^{J-1} \frac{\left(\frac{hi}{2}\sigma(D_{Z_1}, D_{Z_2})\right)^j}{j!} a(Z_1)b(Z_2)\Big|_{Z_1=Z_2=Z} \tag{3.2}$$

$$+ \int_0^1 \frac{(1-\theta)^{J-1}}{(J-1)!} e^{\frac{hi}{2}\theta\sigma(D_{Z_1},D_{Z_2})} \left(\frac{hi}{2}\sigma(D_{Z_1}, D_{Z_2})\right)^J a(Z_1)b(Z_2)\Big|_{Z_1=Z_2=Z} \tag{3.3}$$

$$= \sum_{j=0}^{J-1} h^j \frac{\left(\frac{i}{2}\sigma(D_{Z_1}, D_{Z_2})\right)^j}{j!} a(Z_1)b(Z_2)\Big|_{Z_1=Z_2=Z} + h^J R_J(a, b, h)(Z). \tag{3.4}$$

In this sum, every term

$$(a, b) \to \frac{\left(\frac{i}{2}\sigma(D_{Z_1}, D_{Z_2})\right)^j}{j!} a(Z_1)b(Z_2)\Big|_{Z_1=Z_2=Z}$$

defines a bilinear continuous mapping

$$S(M_1, g) \times S(M_2, g) \to S(M_1M_2\lambda^{-j}, g)$$

and the remainder

$$(a, b) \to R_J(a, b, h)$$

defines a bilinear continuous mapping

$$S(M_1, g) \times S(M_2, g) \to S(M_1M_2\lambda^{-J}, g)$$

with a uniform control with respect to $h \in (0, h_0)$.
(R3) Calderon-Vaillancourt Theorem :

$$(a \in S(1, g)) \Rightarrow \left(\left\|a^W(z, hD_z)\right\|_{\mathcal{L}(L^2)} \leq Cp_N(a)\right)$$

with $C > 0$ and $N \in \mathbb{N}$ independent of $h \in (0, h_0)$.
(R3') Compactness criterion : If $\lim_{Z\to\infty} M(Z) = 0$ then $a \in S(M, g)$ implies that $a^W(z, hD_z)$ is a compact operator in $L^2(\mathbb{R}^n)$.

A particular case is when $M(Z) = \lambda^k(Z)$. One gets for $a_i \in S(\lambda^{k_i}, g)$, $i = 1, 2$,

$$a_1 \sharp^{W,h} a_2 = a_1 a_2 + h R_1(a_1, a_2, h)$$

with $R_1(a_1, a_2, h)$ uniformly bounded in $S(\lambda^{k_1+k_2-1}, g)$. Hence in the limit $h \to 0$ or when $\lim_{Z\to\infty} \lambda(Z) = +\infty$, the $\sharp^{W,h}$ operation is asymptotically commutative. In any cases (even when $h = 1$ and $\lambda(Z) = 1$) one can compute

$$\frac{1}{ih}\left[a_1 \sharp^{W,h} a_2 - a_2 \sharp a_1\right] = \{a_1, a_2\} + \frac{h}{i} R_2(a_1, a_2, h),$$

where the Poisson bracket

$$\{a_1, a_2\} = \partial_\zeta a_1 . \partial_z a_2 - \partial_z a_1 . \partial_\zeta a_2 ,$$

belongs to $S(\lambda^{k_1+k_2-1}, g)$ and R_2 is uniformly bounded in $S(\lambda^{k_1+k_2-2}, g)$.

Exercise 3.6. *Compute g^σ, λ and check the assumptions **H1**... **H4** for the metrics*

$$\langle\zeta\rangle^{2\delta} dz^2 + \frac{d\zeta^2}{\langle\zeta\rangle^{2\varrho}} \, (0 \le \delta \le \varrho \le 1), \qquad \frac{dz^2}{\langle z\rangle^2} + \frac{d\zeta^2}{\langle\zeta\rangle^2}, \qquad \frac{dZ^2}{\langle Z\rangle^2}.$$

We end this review with the Beals criterion. We need and additional assumption (which can be weakened) :
(H5) Assume that the metric g is splitted :

$$g_Z(t_z, -t_\zeta) = g_Z(t_z, t_\zeta),$$

which means that g_Z has the block diagonal form

$$g_Z = \sum_{i,j} a_{ij} dz_i dz_j + b_{ij} d\zeta_i d\zeta_j .$$

With the assumptions **H1**... **H5**, the Beals criterion says :
(R4) $A = a^W(z, hD_z; h)$ with $a(h)$ uniformly bounded in $S(M, g)$ if and only if

$$\forall (\alpha, \beta) \in \mathbb{N}^d, \quad h^{-|\alpha|-|\beta|} \left\| \mathrm{ad}_{hD_z}^{\alpha} \mathrm{ad}_z^{\beta} A \right\|_{\mathcal{L}(H^h(M,g));L^2} \le C_{\alpha,\beta} .$$

The Sobolev spaces are defined such that $H^h(1, g) = L^2(\mathbb{R}^n)$ and $b^W(z, hD_z)$: $H^h(M, g) \to H^h(M/M_b, g)$ when $b \in S(M_b, g)$ with possibly h-dependent norms (see [BonChe] for $h = 1$ and [NaNi] for the h-dependent version).

3.1.2 Some specific metrics.

We work now with $h = 1$. Assume that the function $\Psi : \mathbb{R}^{2n} \to (0, +\infty)$ satisfies :

(Ψ1) $\Psi \geq 1$

(Ψ2) For some constant $c_0 > 0$, ($|z - z'| \leq c_0^{-1}$) and $|\zeta - \zeta'| \leq c_0^{-1}\Psi(z,\zeta)$ imply

$$\left(\frac{\Psi(Z)}{\Psi(Z')}\right)^{\pm 1} \leq c_0 \,.$$

(Ψ3) There exist two constants $c_1 > 0$ and $\nu > 0$ such that

$$\forall Z, Z' \in \mathbb{R}^{2n}, \quad \left(\frac{\Psi(Z)}{\Psi(Z')}\right)^{\pm 1} \leq c_1 \left(1 + \Psi(Z)^2 \left|z - z'\right|^2 + \left|\zeta - \zeta'\right|^2\right)^{\nu} \,.$$

Then the metric

$$g = dz^2 + \frac{d\zeta^2}{\Psi^2}$$

satisfies **H1**…**H5** and we have

$$g^\sigma = \Psi^2 dz^2 + d\zeta^2, \quad \lambda(Z) = \Psi(Z) \,.$$

Definition 3.7. *Under the assumptions ($\Psi 1$)($\Psi 2$)($\Psi 3$) the symbol class S_Ψ^m, $m \in \mathbb{R}$, equals $S(\Psi^m, dz^2 + \frac{d\zeta^2}{\Psi^2})$. The space of operators OpS_Ψ^m is defined as*

$$OpS_\Psi^m = \left\{a^W(z, D_z), \quad a \in S_\psi^m\right\} \,.$$

In [HelNi] we proved the next result for functions of bounded from below self-adjoint operators within such a class.

Proposition 3.8. *[HelNi] Let $A = a^W(z, D_z) \in OpS_\Psi^m$ be a self-adjoint operator with*

- $A \geq c_0 \mathrm{Id}$, $c_0 > 0$
- $m \geq 1$
- $|a| \geq C^{-1}\Psi^m - R$, $R \in S^{m-\delta}$ *with* $C > 0$ *and* $\delta > 0$.

Then for any $s \in \mathbb{R}$, $A^s \in OpS_\Psi^{ms}$ and there exists $t_0 > 0$ such that

$$\forall s \in \mathbb{R}, \quad A^s - \left[(t_0 + a)^W(z, D_z)\right]^s \in OpS_\Psi^{ms-1} \,.$$

Sketch of the proof : We provide the outline of the proof in the simple case $\Psi = 1$ and $A \in OpS^0_\Psi$. We know

$$\begin{aligned} &A \in \mathcal{L}(L^2), \quad A \geq c_0, \quad A^{-1} \in \mathcal{L}(L^2) \\ &A = a^W(z, D_z), \quad a \in S(1, dz^2 + d\zeta^2)\,. \end{aligned}$$

Consider the vector of operators $L = (D_z, z) = (L_1, L_2, \ldots, L_{2n})$ and note (provided that it makes sense)

$$\mathrm{ad}\,_L^I B = [L_{i_1}[L_{i_2} \ldots [L_{i_p}, B]] \ldots], \quad I = (i_1, \ldots, i_p), p = |I|\;.$$

The commutators $\mathrm{ad}\,_{D_z}^\alpha \mathrm{ad}\,_z^\beta A^{-1}$ can be studied by induction

- Start with $[B, A^{-1}] = -A^{-1}[B, A]A^{-1}$ which makes no problem in this simple case.
- Iterate according to

$$\mathrm{ad}\,_{D_z}^\alpha \mathrm{ad}\,_z^\beta A^{-1} = \sum_{\substack{|I_1| + \ldots |I_N| = |\alpha| + |\beta| \\ |I_j| \neq 0}} c^{\alpha,\beta}_{I_1,\ldots I_N} A^{-1}(\mathrm{ad}\,_L^{I_1} A)A^{-1}\circ$$
$$(\mathrm{ad}\,_L^{I_2} A) \ldots A^{-1}(\mathrm{ad}\,_L^{I_N} A)A^{-1}\,.$$

More generally one can prove in this particular case

$$\left\|\mathrm{ad}\,_{D_z}^\alpha \mathrm{ad}\,_z^\beta (t + A)^{-1}\right\| \leq C_{\alpha,\beta,A}(1 + t)^{-1}\,.$$

$r \in \mathbb{Z}$**:** The Beals criterion implies $A^{-1} \in OpS^0_\Psi$ and $A^r \in OpS^0_\Psi$ for any $r \in \mathbb{Z}$.

$r \in (-1, 0)$**:** We write

$$A^r = -\frac{\sin(\pi r)}{\pi} \int_0^\infty t^r (t + A)^{-1}\, dt$$

and one integrates the previous t-dependent inequality. This leads to $A^r \in OpS^0_\Psi$.

A little more has to be done in order to check that the principal symbol is $(t_\delta + a)^r$ (case when Ψ is not 1). ▮

Remark 3.9. *The Beals criterion is a convenient tool to check that an operator provided by functional analysis is a pseudodifferential operator : Via an integral representation reduce the problem to estimating commutators with the resolvent* $(z - A)^{-1}$. *Another often used formula is the Dynkin-Helffer-Sjöstrand formula (see [HeSj]) for self-adjoint operators (written with* $\tilde{f}$ *almost analytic extension of* f*)*

$$f(A) = \frac{1}{2i\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{f}(z)(z - A)^{-1}\, dz \wedge d\overline{z}\,.$$

3.1.3 Application to the Fokker-Planck framework.

We now apply the previous results with $n = d$, $Z = (x, \xi)$ and $n = 2d$, $Z = (x, v, \xi, \eta)$.
Assume that the potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ satify one of the two assumptions

(V1) For $|\alpha| \geq 1$, there exists C_α such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle \partial_x V(x) \rangle \ .$$

There exists $C > 0$ and $M > 0$ such that

$$C^{-1} \langle x \rangle^{1/M} \leq \langle \partial_x V(x) \rangle \leq C \langle x \rangle^M \ .$$

(V1') For $|\alpha| \geq 1$, there exists C_α such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle \partial_x V(x) \rangle \ .$$

There are constants $C > 0$, $M > 0$ and $\varrho_0 > 0$ such that

$$\begin{array}{rl} & \langle \partial_x V(x) \rangle \leq C \langle x \rangle^M \\ \text{for } |\alpha| = 2 & |\partial_x^\alpha V(x)| \leq C \langle \partial_x V(x) \rangle \langle x \rangle^{-\varrho_0} \ . \end{array}$$

Proposition 3.10. *[HelNi] By assuming **V1** of **V1'**, the properties **Ψ1**, **Ψ2** and **Ψ3** are satisfied by*

$$\Psi(x, \xi) = \sqrt{1 + |\xi|^2 + \frac{1}{4} |\partial_x V(x)|^2} \quad on \quad \mathbb{R}^{2d} \qquad (3.5)$$

$$\Psi(x, \xi, v, \eta) = \sqrt{1 + |\xi|^2 + |\eta|^2 + \frac{1}{4} |v|^2 + \frac{1}{4} |\partial_x V(x)|^2} \quad on \quad \mathbb{R}^{4d} . \qquad (3.6)$$

Corollary 3.11. *Assume **V1** of **V1'** and take Ψ according to* (3.6) *on $\mathbb{R}^{4d}$. The operators involved in the analysis of the Fokker-Planck operator satisfy*

$$\begin{array}{rl} & \partial_{x_j}, \partial_{v_i}, v_j, \partial_{x_j} V(x) \in OpS^1_\Psi, \quad j = 1, \ldots, d \,, \\ & 1 + \Delta^0_{V/2} - \Delta_v + v^2/4 - d/2 = 1 + \Delta^{(0)}_{v^2/4 + V(x)/2} \in OpS^2_\Psi \\ & K_\pm = v.\partial_x - \partial_x V(x).\partial_v - \Delta_v + v^2/4 - d/2 \in OpS^2_\Psi \\ & \Lambda^s := (1 + \Delta^{(0)}_{v^2/4 + V(x)/2})^{s/2} \in OpS^s_\Psi \\ & \Lambda^s - [a^{s/2}]^W(x, v, D_x, D_V) \in OpS^{s-1}_\Psi \\ \textit{with} & a = t_0 + \xi^2 + \eta^2 + \frac{1}{4} |\partial_x V(x)|^2 - \frac{1}{2} \Delta V(x) + \frac{v^2}{4} \,. \end{array}$$

3.1.4 (Important) remarks

- In Proposition 3.10, the treatment of (3.6) can be reduced to the one of (3.5) after replacing $V(x)$ by $\Phi(x,v) = v^2/2 + V(x)$ which also fulfills **V1** or **V1'**.
- The proof of Proposition 3.10 is a bit technical although rather standard. This is often the case when one wants to use the pseudodifferential calculus. Part of the technicalities are contained in the study of the metric. Once it is solved, a lot of estimates come from the general theory.
- In [HerNi] the assumption was stronger

  $$C^{-1} \langle x \rangle^M \leq \langle \partial_x V(x) \rangle \leq C \langle x \rangle^M$$

  with the same exponent on both sides (ellipticity assumption for the Witten Laplacian). The hypotheses **V1** and **V1'** are weaker.
- **Important Remark :** Both assumptions **V1** and **V1'** imply that the Hessian $\mathrm{Hess}\, V(x)$ is controlled by the gradient $\partial_x V(x)$ at infinity. We shall come back to this point later.

3.2 Sufficient conditions for the compactness of $(1 + K_\pm)^{-1}$.

The compactness of $(1 + K_\pm)^{-1}$ will be deduced from hypoelliptic estimates where vector fields are replaced by (possibly deformed) creation-annihilation operators of the harmonic oscillator Hamiltonian. With such an approach and with a global pseudo-differential calculus the regularity and decay estimates are on the same level. This analysis carried out in [HerNi] and slightly improved in [HelNi] was inspired by a former work by Eckmann-Pillet-ReyBellet [EckPiRe-Be] .

3.2.1 Notations and result.

Let us introduce some convenient notations. We observe that the operator K defined in (2.2) and (2.3)) can be written

$$K = X_0 + b^* b \,, \tag{3.7}$$

where

$$X_0 = (b^* a - a^* b) \,. \tag{3.8}$$

with

$$b = \partial_v + \frac{v}{2} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad , \qquad a = \partial_x + \frac{1}{2}\partial_x V = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} . \tag{3.9}$$

The adjoint forms of a and b are

$$b^* = (b_1^*, \ldots, b_n^*) \quad \text{and} \quad a^* = (a_1^*, \ldots, a_n^*); . \tag{3.10}$$

With these notations the operator Λ already introduced in Corollary 3.11 is given by

$$\Lambda^2 = 1 + a^*a + b^*b \ ,$$

and provides a natural Sobolev scale for the problem. Its square

$$\Lambda^2 - 1 = a^*a + b^*b = \Delta^{(0)}_{\Phi/2} = \Delta^{(0)}_{V/2} \otimes \mathrm{Id}_v + \mathrm{Id}_x \otimes \Delta^{(0)}_{v^2/4}$$

is the phase-space Witten Laplacian associated to $\Phi/2$, with $\Phi = \frac{v^2}{2} + V(x)$.

Theorem 3.12.
*If the potential $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ verifies Assumption **V1** or **V1'**, then there exists a constant $C > 0$ such that*

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}) \, , \ \left\|\Lambda^{1/4}u\right\|^2 \leq C \left(\|Ku\|^2 + \|u\|^2\right) \ . \tag{3.11}$$

Remark 3.13.
As for Kohn's proof for the hypoellipticity in Proposition 1.2, the exponent $\frac{1}{4}$ in (3.11) is not optimal. The accurate analysis which can be easily carried out like in [HelNi] with a quadratic potential gives the exponent $2/3$. For the geometric Fokker-Planck equations on compact Riemannian manifold the exponent $2/3$ has also been proved in [Leb2].

Corollary 3.14.
*If the potential $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfies Assumption **V1** then the operator K has a compact resolvent.*
*If the potential $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfies Assumption **V1'** , then K has a compact resolvent if (and only if) the Witten Laplacian $\Delta^{(0)}_{V/2}$ has a compact resolvent.*

Proof:
The closure of K, initially defined on $\mathcal{C}_0^\infty(\mathbb{R}^n)$, is maximally accretive according to Proposition 2.4. Theorem 3.12 says that the first factor of

$$(1+K)^{-1} = \left[(1+K)^{-1}\Lambda^{1/4}\right]\Lambda^{-1/4}$$

is bounded, while the second one belongs to the class $\mathrm{Op}\, S_\Psi^{-1/4}$ owing to Corollary 3.11. Under Assumption **V1**, the function Ψ satisfies

$$\lim_{(x,v,\xi,\eta)\to\infty} \Psi(x,v,\xi,\eta) = +\infty$$

and $\Lambda^{-1/4}$ is compact. This last condition is not implied by Assumption **V1'** but the compactness of $\Lambda^{-1/4}$ is then a consequence of the compactness of $\Lambda^{-2}=\left(1+\Delta^{(0)}_{V/2}+\Delta^{(0)}_{v^2/2}\right)^{-1}$. The "only if" part was proved in Theorem 2.6. ∎

3.2.2 Algebraic properties of the Fokker-Planck operator. The bracket condition.

Before starting the proof of Theorem 3.12 let us recall the algebraic properties associated with the Fokker-Planck operator.

The Canonical Commutation Relations (CCR) of the annihilation-creation operators b_j, b_j^* are satisfied :

$$[b_j,b_k]=[b_j^*,b_k^*]=0\ ,\qquad [b_j,b_k^*]=\delta_{jk}\ . \tag{3.12}$$

More generally with $\partial_{x_j}\partial_{x_k}V=\partial_{x_k}\partial_{x_j}V$, we have :

$$[a_j,a_k]=[a_k,a_j]=0\ ,\qquad [a_j,a_k^*]=\partial^2_{x_jx_k}V\ . \tag{3.13}$$

The a's and b's commute with each other

$$[a_j^\dagger,b_k^\sharp]=0\ , \tag{3.14}$$

where $a^\dagger$ (resp. $b^\sharp$) equals a or a^* (resp. b or b^*).

The bracket condition is formally the same as in our initial example (1.1). Here it reads :

$$[b_j,X_0]=a_j\ ,\qquad [b_j^*,X_0]=a_j^*\ . \tag{3.15}$$

Similarly, the b_j's and b_j^*'s can be derived from the a_j's, a_j^*'s and X_0

$$[a_j,X_0]=-\sum_{k=1}^d\left(\partial^2_{x_jx_k}V\right)b_k\ ,\qquad [a_j^*,X_0]=-\sum_{k=1}^d b_k^*\left(\partial^2_{x_kx_j}V\right)\ . \tag{3.16}$$

For any $r,r'\in\mathbb{R}$, we have :

$$\left[\Lambda^r,(1+a^*a)^{r'}\right]=\left[\Lambda^r,(1+b^*b)^{r'}\right]=0\ . \tag{3.17}$$

The relations (3.15) and (3.16) are summarized by

$$\begin{array}{ll}[b,X_0]=a\ , & [b^*,X_0]=a^*\ ,\\ [a,X_0]=-\mathrm{Hess}\,V\,b\ , & [a^*,X_0]=-b^*\,\mathrm{Hess}\,V\end{array} \tag{3.18}$$

where we make use of the notations (3.9) and (3.10).We will often use this matricial notation where * refers to forms or line matrices. As an example, we also have by combination the formulas :

$$\left[\Lambda^2,X_0\right]=-b^*(\mathrm{Hess}\,V-\mathrm{Id})a-a^*(\mathrm{Hess}\,V-\mathrm{Id})b \tag{3.19}$$

and

$$b(b^*b)=(b^*b+1)b\ . \tag{3.20}$$

3.2.3 Hypoelliptic estimates : a basic lemma.

As a consequence of these relations combined with the estimates

$$\left\|\Lambda^{2\rho-2}a^*\right\| \leq 1 \tag{3.21}$$

and

$$\left\|\Lambda^{2\rho-2}b^*\right\| \leq 1\ , \tag{3.22}$$

for $\rho \leq 1/2$, one has the following result which is a variation of the beginning of the proof of Proposition 1.2.

Lemma 3.15.
Take $\rho \in [0, 1/4]$. The estimate

$$\begin{aligned}\|\Lambda^\rho u\|^2 \ \leq\ & \operatorname{Re}\langle Ku \mid (L+L^*)u\rangle - \operatorname{Re}\langle \mathcal{A}^*bu \mid u\rangle + \operatorname{Re}\langle LKu \mid Lu\rangle \\ & - \operatorname{Re}\langle \mathcal{A}^*bu \mid Lu\rangle + 3\,\|bu\|^2 + 3\,\|u\|^2\ ,\end{aligned} \tag{3.23}$$

holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, with

$$L = \Lambda^{2\rho-2}a^*b = \Lambda^{2\rho-2}(\sum_j a_j^* b_j)$$

and

$$\mathcal{A}^* = \left[\Lambda^{2\rho-2}a^*, X_0\right] = (\mathcal{A}_j^*)\ ,\ \mathcal{A}_j^* = \left[\Lambda^{2\rho-2}a_j^*, X_0\right]\ .$$

Proof: Step 1.
We first show that :

$$\|\Lambda^\rho u\|^2 \leq \operatorname{Re}\langle X_0 u \mid (L+L^*)u\rangle - \operatorname{Re}\langle \mathcal{A}^*bu \mid u\rangle + ||bu||\ ||u|| + \left\|\Lambda^{\rho-1}u\right\|^2\ . \tag{3.24}$$

The starting point is

$$\|\Lambda^\rho u\|^2 = \langle \Lambda^{2\rho-2}b^*bu \mid u\rangle + \langle \Lambda^{2\rho-2}a^*au \mid u\rangle + \langle \Lambda^{2\rho-2}u \mid u\rangle\ .$$

We obtain the result immediately from (3.22) and from the identity

$$\langle \Lambda^{2\rho-2}a^*au \mid u\rangle = \operatorname{Re}\langle X_0u \mid (L+L^*)u\rangle - \operatorname{Re}\langle \mathcal{A}^*bu \mid u\rangle\ ,$$

which simply results from $a = bX_0 - X_0 b$ (cf (3.18)).
Step 2.
We now show that

$$\operatorname{Re}\langle X_0u \mid (L+L^*)u\rangle \leq \operatorname{Re}\langle Ku \mid (L+L^*)u\rangle - 2\operatorname{Re}\langle b^*bu \mid Lu\rangle + ||bu||\ ||u||\ . \tag{3.25}$$

We start from

$$\operatorname{Re}\langle X_0u \mid (L+L^*)u\rangle = \operatorname{Re}\langle Ku \mid (L+L^*)u\rangle - \operatorname{Re}\langle b^*bu \mid Lu\rangle - \operatorname{Re}\langle b^*bu \mid L^*u\rangle\ ,$$

and work on the last term of the right hand side. We have

$$\operatorname{Re}\langle b^*bu \mid L^*u\rangle = \operatorname{Re}\langle bb^*bu \mid a\Lambda^{2\rho-2}u\rangle \,.$$

Using $bb^*b = (b^*b+1)\,b$ we get

$$\operatorname{Re}\langle b^*bu \mid L^*u\rangle = \operatorname{Re}\langle b^*bbu \mid a\Lambda^{2\rho-2}u\rangle + \operatorname{Re}\langle bu \mid a\Lambda^{2\rho-2}u\rangle \,.$$

The next point is to observe the commutation of b^*b with Λ

$$\begin{aligned}\operatorname{Re}\langle b^*bu \mid L^*u\rangle = {}& \operatorname{Re}\langle bu, \mid a\Lambda^{2\rho-2}b^*bu\rangle + \operatorname{Re}\langle bu \mid a\Lambda^{2\rho-2}u\rangle \\ &= \operatorname{Re}\langle b^*bu \mid Lu\rangle + \operatorname{Re}\langle bu \mid a\Lambda^{2\rho-2}u\rangle \,.\end{aligned}$$

We conclude by using (3.21) for the last term.

Step 3.

It remains to control $-2\operatorname{Re}\langle b^*bu \mid Lu\rangle$. We will show

$$-2\operatorname{Re}\langle b^*bu \mid Lu\rangle \leq \frac{3}{2}||bu||^2 + \operatorname{Re}\langle LKu \mid Lu\rangle - \operatorname{Re}\langle \mathcal{A}^*bu \mid Lu\rangle + \frac{1}{2}||u||^2 \,. \tag{3.26}$$

We start from

$$\begin{aligned}-2\operatorname{Re}\langle b^*bu \mid Lu\rangle &\leq ||bu||^2 + ||bLu||^2 \\ &\leq ||bu||^2 + \operatorname{Re}\langle KLu \mid Lu\rangle \\ &\leq ||bu||^2 + \operatorname{Re}\langle [K,L]u \mid Lu\rangle + \operatorname{Re}\langle LKu \mid Lu\rangle \,.\end{aligned}$$

We now observe that :

$$[K,L] = -L - \mathcal{A}^*b - \Lambda^{2\rho-2}a^*a \,.$$

The last term to control is

$$-\operatorname{Re}\langle \Lambda^{2\rho-2}a^*au \mid Lu\rangle = -\operatorname{Re}\langle a\Lambda^{4\rho-4}a^*au \mid bu\rangle \,.$$

Using again (3.21), it is controlled when $\rho \leq \frac{1}{4}$.

Putting together (3.24), (3.25) and (3.26) ends the proof of the lemma. ∎

3.2.4 Proof of Theorem 3.12.

We are now able to prove Theorem 3.12 by bounding each of the six terms in the right-hand side of (3.23) :

First term :

We write

$$|\langle Ku \mid Lu\rangle| \leq \|Ku\|\,\|Lu\| \leq \|Ku\|\,\left\|\Lambda^{2\rho-2}a^*\right\|\,\|bu\|$$

and recall $\|\Lambda^{2\rho-2}a^*\| \le 1$ for $\rho \le 1/2$. The simple inequality

$$\|bu\|^2 = \langle b^*bu \mid u\rangle = \operatorname{Re}\langle Ku \mid u\rangle \le \|Ku\|\,\|u\|$$

now gives

$$|\langle Ku \mid Lu\rangle| \le \|Ku\|^{3/2}\,\|u\|^{1/2}\,.$$

For the second part we write $|\langle Ku \mid L^*u\rangle| \le \|Ku\|\,\|L^*u\|$ and we use, observing the commutation of b^*b with a and Λ,

$$L^* = b^*a\Lambda^{2\rho-2}(1+b^*b)^{-1/2}(1+b^*b)^{1/2} = b^*(1+b^*b)^{-1/2}a\Lambda^{2\rho-2}(1+b^*b)^{1/2}\ .$$

From this we deduce

$$\begin{aligned}\|L^*u\| &\le \left\|b^*(1+b^*b)^{-1/2}\right\|\left\|a\Lambda^{2\rho-2}\right\|\left\|(1+b^*b)^{1/2}u\right\|\\ &\le C_n\left(\|Ku\|\,\|u\|+\|u\|^2\right)^{1/2}\ ,\end{aligned}$$

and we obtain

$$|\langle Ku \mid (L+L^*)u\rangle| \le C\left(\|Ku\|^2+\|u\|^2\right)\ .$$

Terms 5 and 6 :
They are all bounded by

$$C\left(\|bu\|^2+\|u\|^2\right) \le C'\left(\|Ku\|^2+\|u\|^2\right)\ .$$

Term 3 :
We write

$$\operatorname{Re}\langle LKu \mid Lu\rangle = \operatorname{Re}\langle \Lambda^{2\rho-2}a^*bKu \mid \Lambda^{2\rho-2}a^*bu\rangle = \operatorname{Re}\langle a\Lambda^{4\rho-4}a^*bKu \mid bu\rangle\ .$$

Since a, a^* and b belong to $\operatorname{Op}S^1_\Psi$, the operator $a\Lambda^{4\rho-4}a^*b$ is bounded for $\rho \le 1/4$, which is just the condition appearing in Theorem 3.12. We get

$$|\operatorname{Re}\langle LKu \mid Lu\rangle| \le C\,\|Ku\|\,\|bu\| \le C\left(\|Ku\|^2+\|u\|^2\right)\ ,$$

for $\rho \le 1/4$.

Term 4 :
We write

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}^*bu \mid Lu\rangle &= \operatorname{Re}\langle\left[\Lambda^{2\rho-2}a^*, X_0\right]bu \mid \Lambda^{2\rho-2}a^*bu\rangle\\ &= \operatorname{Re}\langle a\Lambda^{2\rho-2}\left[\Lambda^{2\rho-2}a^*, X_0\right]bu \mid bu\rangle\ .\end{aligned}$$

The Hamiltonian vector field X_0 belongs to $\operatorname{Op}S^2_\Psi$ (see (3.8)) and the pseudodifferential calculus for commutators gives :

$$a\Lambda^{2\rho-2}\left[\Lambda^{2\rho-2}a^*, X_0\right] \in \operatorname{Op}S_\Psi^{1+4\rho-4+1+2-1} = \operatorname{Op}S_\Psi^{4\rho-1} \subset \mathcal{L}(L^2)\ ,$$

for $\rho \leq 1/4$. We conclude like for the third term with

$$|\operatorname{Re}\langle \mathcal{A}^* bu \mid Lu\rangle| \leq C\left(\|Ku\|^2 + \|u\|^2\right).$$

Term 2 :
This term is the more delicate and we have to split the variables x and v while refining our pseudo-differential calculus with some exact commutator expressions. First we have

$$\begin{aligned}
\mathcal{A}^* &= \left[\Lambda^{2\rho-2}a^*, X_0\right] = \left[\Lambda^{2\rho-2}, X_0\right] a^* + \Lambda^{2\rho-2} b^* \operatorname{Hess} V \\
&= (b^*b+1)^{1/2}(b^*b+1)^{-1/2}\left[\Lambda^{2\rho-2}, X_0\right] a^* \\
&\quad + (b^*b+1)^{1/2}\Lambda^{2\rho-2}(b^*b+1)^{-1/2} b^* \operatorname{Hess} V \\
&= (b^*b+1)^{1/2}(A_1 + A_2)\,,
\end{aligned}$$

with

$$A_1 := (1+b^*b)^{-1/2}\left[\Lambda^{2\rho-2}, X_0\right] a^*\,,$$

and

$$A_2 := \Lambda^{2\rho-2}(b^*b+1)^{-1/2} b^* \operatorname{Hess} V\,.$$

If A_1 and A_2 are bounded, one obtains

$$|\operatorname{Re}\langle \mathcal{A}^* bu \mid u\rangle| \leq \left|\operatorname{Re}\langle (A_1+A_2)bu \mid (1+b^*b)^{1/2}u\rangle\right| \leq C\left(\|Ku\|^2 + \|u\|^2\right)\,.$$

The boundedness of A_2 is simple to verify. The coefficients $\Lambda^{2\rho-2}\partial^2_{x_i\,x_j}V$ belong to $\operatorname{Op} S_{\Psi}^{2\rho-2+1}$ and are bounded if $\rho \leq 1/2$. The boundedness of A_2 is then a consequence of the property that $(1+b^*b)^{-1/2}\, b^* \in \mathcal{L}(L^2)$.

Noting that $A_1 = \left((1+b^*b)^{-1/2}\left[\Lambda^{2\rho-2}, X_0\right]\Lambda\right)(\Lambda^{-1}a^*)$, the boundedness of A_1 is given by the following lemma applied with $r_1 = 0$, $r_2 = 2\rho - 2$ and $r_3 = 1$ (this requires $\rho \leq 1/2$).

Lemma 3.16.
For $r_1 + r_2 + r_3 \leq 0$, the operator $(1+b^*b)^{-1/2}\Lambda^{r_1}\left[\Lambda^{r_2}, X_0\right]\Lambda^{r_3}$ is bounded on $L^2(\mathbb{R}^{2n})$.

Proof :
Since $\Lambda^{r_1}\left[\Lambda^{r_2}, X_0\right] = \left[\Lambda^{r_1+r_2}, X_0\right] - \left[\Lambda^{r_1}, X_0\right]\Lambda^{r_2}$ we can simply consider the case $r_1 = 0$. Note that the vector field $X_0 = v\cdot\partial_x - \partial_x V(x)\cdot\partial_v$ is the sum of terms in the form $\ell(v, D_v)a(x, v, D_x, D_v)$ where ℓ is a linear symbol in (v, η) and $a \in S^1_{\Psi}$. We expand the commutator as

$$\begin{aligned}
&\left[\Lambda^{r_2}, \ell(v, D_v)a(x, v, D_x, D_v)\right] \\
&\qquad = \left[\Lambda^{r_2}, \ell(v, D_v)\right] a(x, v, D_x, D_v) + \ell(v, D_v)\left[\Lambda^{r_2}, a(x, v, D_x, D_v)\right] \\
&\qquad\qquad := B_1 + B_2.
\end{aligned}$$

Since the commutator $[\Lambda^{r_2}, a(x,v,D_x,D_v)]$ belongs to $\mathrm{Op}\, S_\Psi^{r_2}$, we have

$$(1+b^*b)^{-1}B_2\Lambda^{-r_2} \in \mathcal{L}(L^2)\ .$$

Let us now look at B_1. It is enough to show the

Sublemma 3.17.

$$[\Lambda^{r_2}, \ell(v,D_v)] \in \mathrm{Op}\, S_\Psi^{r_2-1}\ .$$

We first note that this is not a direct consequence of the previous pseudo-differential calculus which says only that this term is in $\mathrm{Op}\, S_\Psi^{r_2}$. But this calculus says that modulo $S_\Psi^{r_2-1}$ the symbol of the commutator is obtained by $\frac{1}{i}$ the Poisson bracket of the principal symbols of Λ^{r_2} (equal to Ψ^{r_2} according to Corollary 3.11) and of $\ell(v,D_v)$. An explicit computation then shows that this Poisson bracket is actually in $S_\Psi^{r_2-1}$. ∎

We proved

$$|\,\mathrm{Re}\,\langle \mathcal{A}^* b u \mid u\rangle| \leq C\left(\|Ku\|^2 + \|u\|^2\right)\ ,$$

which ends the proof of (3.11) in Theorem 3.12. ∎

4 Degenerate Witten Laplacian. Compactness criteria.

Here some compactness criteria of the Witten Laplacian will be studied with assumptions which are much weaker than the ones, **V1** or **V1'**, used in the analysis of the Fokker-Planck operator. For a stronger property than the compactness of the resolvent, a necessary and sufficient condition is provided by Helffer-Nourrigat results. We shall see also that a natural relationship exists between the qualitative results for Witten Laplacians (question **QW2**) and quantitative estimates (question **QW1**).

In the two first subsections the potential $V(x)$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^d_x$. In the other subsections we will focus on polynomial potentials $V=\sum_{|\alpha|\leq r} a_\alpha x^\alpha$, $x\in\mathbb{R}^d$.

4.1 Structures associated with $\Delta^{(0)}_{V,h}$.

We already introduced the parameter dependent ($h>0$) Witten Laplacian on 0-form as a deformed Hodge Laplacian in Subsubsection 1.2.2 :

$$\begin{aligned}\Delta^{(0)}_{V,h} &= d^*_{V,h}d_{V,h} \qquad (d_{V,h}=e^{-V/h}(hd)e^{V/h} \quad d^*_{V,h}=e^{V/h}(hd^*)e^{-V/h})\\ &= -(h\partial_x-\partial_xV(x)).(h\partial_x+\partial_xV(x))\\ &= -h^2\Delta+|\partial_xV(x)|^2-h\Delta V(x)\,.\end{aligned}$$

It is non negative and Proposition 1.5 says that it is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n)$.

Another writing of the parameter dependence

$$\frac{1}{h^2}\Delta_{V,h}^{(0)} = -\Delta + \frac{1}{h^2}\left|\partial_x V(x)\right|^2 - \frac{1}{h}\Delta V(x) = \Delta_{\tau_0 V}^{(0)}$$

is possible with $\tau_0 = h^{-1} \in (0, +\infty)$. We will see that it is interesting to consider more generally the Witten Laplacian

$$\Delta_{\tau_0 V}^{(0)} = (\Delta_{\tau_0 V,1}^{(0)}), \qquad \tau_0 \in \mathbb{R}^*\,,$$

that is with possibly $\tau_0 < 0$.
Owing to the H^2_{loc} regularity when solving $\Delta_{\tau_0 V}^{(0)} u = f \in L^2$, the compactness of $\Delta_{\tau_0 V}^{(0)}$ is a consequence (see also Person's Lemma in [CFKS]) of :

$$\begin{array}{ll} \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^n) & \left\|R(x)u\right\|^2 = \left\langle u\,,\, R(x)^2 u\right\rangle \leq C\left[\left\langle u, \Delta_{\tau_0 V}^{(0)} u\right\rangle + \left\|u\right\|^2\right] \\ \quad\text{with} & \lim\limits_{x\to\infty} R(x) = +\infty\,. \end{array} \tag{4.1}$$

We end this paragraph with the introduction of the Lie-algebra structure hidden in $\Delta_{\tau_0 V}^{(0)}$ and which will be explored further.
Consider in $\mathbb{R}^{d+1}_{x,t}$, the Lie algebra generated by the vector fields

$$X_j = \partial_{x_j}\,, \quad Y_j = \partial_{x_j} V(x)\partial_t \quad j = 1, \ldots, d \tag{4.2}$$

For any $\tau_0 \in \mathbb{R}^*$, we consider the unitary representation Π_{V,τ_0} of this Lie-algebra in $L^2(\mathbb{R}^d)$ given by

$$\Pi_{V,\tau_0}(X_j) = \partial_{x_j}\,, \quad \Pi_{V,\tau_0}(Y_j) = \partial_{x_j} V(x) i\tau_0$$

After setting

$$L_j = X_j - iY_j = \partial_{x_j} - i\partial_{x_j} V(x)\partial_t \quad j = 1, \ldots, d\,, \tag{4.3}$$

the Witten Laplacian $\Delta_{\tau_0 V}^{(0)}$ can be written

$$\begin{aligned} \Delta_{\tau_0 V}^{(0)} &= \sum_j \Pi_{V,\tau_0}(L_j)^* \Pi_{V,\tau_0}(L_j) \\ &= \sum_j \Pi_{V,\tau_0}(-X_j^2 - Y_j^2 + i[X_j, Y_j]) \\ &= \Pi_V(L^* L)\,. \end{aligned}$$

Hence finding an lower bound like (4.1) becomes now related to the hypoellipticity of the overdetermined system $L = (L_1, \ldots, L_d)$, that is the question whether for some neighborhood $\omega_{x_0,t}$ of $(x_0, t) \in \mathbb{R}^{d+1}$

$$\left(L_j u \in \mathcal{C}^\infty(\omega_{x_0,t}),\ \forall j \in \{1, \ldots, d\}\right) \Rightarrow \left(u \in \mathcal{C}^\infty(\omega_{x_0,t})\right)\,.$$

4.2 Examples with polyhomogeneous potentials.

We shall work here with a simple comparison principle which was extensively used in [HelNi1].

Lemma 4.1. *For two potentials $V_1, V_2 \in \mathcal{C}^\infty(\mathbb{R}^d)$, the inequalities*

$$\begin{aligned} \Delta^{(0)}_{V_1+V_2} &= \Delta^{(0)}_{V_1} + 2\partial_x V_1(x).\partial_x V_2(x) + |\partial_x V_2(x)|^2 - \Delta V_2(x) \\ &\geq 2\partial_x V_1(x).\partial_x V_2(x) + |\partial_x V_2(x)|^2 - \Delta V_2(x) \end{aligned}$$

holds in the sense of quadratic forms on $\mathcal{C}_0^\infty(\mathbb{R}^d)$.

Consider the examples $\pm V_1 + \varepsilon V_2$ built with the two potentials in $\mathbb{R}^2$

- $V_1(x_1, x_2) = x_1^2 x_2^2$
- $V_2(x_1, x_2) = (x_1^2 + x_2^2)^{(1+\delta)/2}$, for $|x| \geq 1$, with $V_2 \in \mathcal{C}^\infty(\mathbb{R}^2)$ and $0 < \delta \leq 1$.

With the polar coordinates,

$$(x_1, x_2) = (r\cos\theta, r\sin\theta), \quad r \in \mathbb{R}_+^*, \theta \in [-\pi, \pi)\,,$$

we have

$$V_1 = r^4\varphi(\theta), \quad \varphi(\theta) = \frac{\sin^2(2\theta)}{4}, \quad V_2 = r^{1+\delta} \text{ for } r \geq 1\,.$$

Example 1: Take $V = V_1 + V_2$. Lemma 4.1 says

$$\Delta^{(0)}_V \geq 8(1+\delta)r^{3+\delta}\varphi(\theta) + (1+\delta)^2 r^{2\delta} - C \geq (1+\delta)^2 r^{2\delta} - C$$

and $\Delta^{(0)}_V$ has a compact resolvent.
The Hessian in the direction $\theta = 0$, $x = (x_1, 0)$ increases like x_1^2 that is like r^2 while the gradient in this direction is bounded by $r^\delta \leq r$.

Conclusion 1: *This is an example of potential for which $\Delta^{(0)}_V$ has a compact resolvent while Assumptions **V1** and **V1'** are not true.*
Let us look a bit further. A complete computation of the symbol of $\Delta^{(0)}_{V_1}$ gives

$$|\xi|^2 + |\partial_x V_1(x)|^2 - \Delta V_1(x) = |\xi|^2 + 16r^6\varphi(\theta)^2 + r^6\varphi'(\theta)^2 - 16r^2\varphi(\theta) - r^2\varphi''(\theta)$$

By taking $|\theta| \leq C^{-1}r^{-2}$), $C > 0$ large enough, we get

$$|\xi|^2 + |\partial_x V_1(x)|^2 - \Delta V_1(x) \leq |\xi|^2 - r^2/2 - C$$

Even after adding the correction

$$2\partial_x V_1(x).\partial_x V_2(x) + |\partial_x V_2(x)|^2 - \Delta V_2(x)$$

one gets

$$|\xi|^2 + |\partial_x V_1(x)|^2 - \Delta V_1(x) \leq\leq |\xi|^2 - r^2/4 - C$$

if $|\theta| \leq C^{-1} r^{-2}$ and $\delta < 1$.

Conclusion 2: *This example with $\delta < 1$, shows that the lower bound (4.1) cannot be obtained from a naive application of pseudo-differential calculus (the symbol goes to $-\infty$ in some directions). One has to consider more accurately the uncertainty principle.*

Example 2: $V = \pm V_1$. For $V = -V_1$, one gets at once

$$\Delta^{(0)}_{-V_1} = -\Delta + |\partial_x V_1|^2 + \Delta V_1(x) \geq \Delta V_1 = 2(x_2^2 + x_1^2) \overset{x\to\infty}{\rightarrow} +\infty$$

and $\Delta^{(0)}_{-V_1}$ has a compact resolvent.

For $V = +V_1$, 0 actually belongs to the essential spectrum of $\Delta^{(0)}_{V_1}$ (the resolvent cannot be compact). It suffices to note that for $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\left\langle \chi e^{-V}\, ,\, \Delta^{(0)}_V (\chi e^{-V}) \right\rangle = \left\| (\partial_x \chi) e^{-V} \right\|^2_{L^2}.$$

We take a sequence of cut-off functions

$$\chi_n(x) = \psi(2^{-n} r)\psi_0(\theta), \quad n \in \mathbb{N}$$

and we set $u_n = \chi_n e^{-V}$, with ψ supported around $\theta = 0$ with $\psi_0 \equiv 1$ in a smaller neighborhood and ψ supported in $(3/4, 3/2)$ with $\varphi \equiv 1$ around 1. An simple asymptotic expansion of Laplace integrals (around $\theta = 0$) ensures that the Rayleigh quotient

$$\frac{\left\langle u_n\, ,\, \Delta^{(0)}_V u_n \right\rangle}{\|u_n\|^2} = \frac{\int_{3/4}^{3/2}\int_{-\pi}^{\pi} [|\psi'|^2 |\psi_0|^2 + r^{-2} |\psi|^2 |\psi_0'|^2] e^{-2^{8n}\varphi(\theta)}\ r dr\ d\theta}{\int_{3/4}^{3/2}\int_{-\pi}^{\pi} [|\psi|^2 |\psi_0|^2] e^{-2^{8n}\varphi(\theta)}\ 2^{2n} r dr\ d\theta}$$

goes to 0 as $n \to \infty$ (and the u_n's are orthogonal).

Conclusion 3: *The compactness of the resolvent of $\Delta^{(0)}_{\tau_0 V}$ depends on the sign of τ_0.*

4.3 Maximal microhypoellipticity for systems.

We work here with a polynomial potential $V \in \mathbb{R}[X_1, \ldots, X_d]$, of degree $r \in \mathbb{N}$.

Definition 4.2. *For $r \in \mathbb{N}$, let E_r denote the set of polynomials with degree not greater than r :*

$$E_r = \{P \in \mathbb{R}[X_1, \ldots, X_d], \quad d^\circ P \leq r\}$$

A natural quantity will often be used.

Definition 4.3. *For a polynomial $P \in E_r$, the function $R_P : \mathbb{R}^d \to \mathbb{R}$ is defined by*

$$R_P(x) = \sum_{1 \leq |\alpha| \leq r} |\partial_x^\alpha P(x)|^{1/|\alpha|} \,. \tag{4.4}$$

Exercise 4.4. *By using the Taylor formula for polynomials*

$$P(x_0 + t) = \sum_{|\alpha| \leq d^\circ P} \frac{\partial_x^{(\alpha)} P(x_0)}{\alpha!} t^\alpha$$

check that

1. *the family of polynomials $(P(x_0 + \frac{x}{R_P(x_0)}) - P(x_0))_{x_0 \in \mathbb{R}^d, P \in E_r}$ is compact in $\mathbb{R}[X_1, \ldots, X_d]$;*
2. *for a fixed $P \in E_r$, the metric $R_P(x)^2 dx^2$ is slow : There exists $C > 0$ such that*

$$\left(|x - x_0| \leq R_P(x_0)^{-1}\right) \Rightarrow \left(\left(\frac{R_P(x)}{R_P(x_0)}\right)^{\pm 1} \leq C\right) .$$

The link with Rothschild-Stein theory involving nilpotent Lie algebras is immediate for polynomial potentials

Proposition 4.5. *When V belongs to E_r, the Lie algebra $\mathcal{G}$ generated by the vector fields $(X_j, Y_j)_{j=1,\ldots,d}$ defined on $\mathbb{R}^{d+1}_{x,t}$ by (4.2) is nilpotent with rank less than r (indeed the rank of $\mathcal{G}$ equals here $d^\circ V$).*

Definition 4.6. *The system $L_1, \ldots, L_d$ introduced in (4.3) on $\mathbb{R}^{d+1}_{x,t}$ is said maximally hypoelliptic at a point (x_0, t_0) if there exists $C > 0$ such that*

$$\sum_{j=1}^{d} (\|X_j u\|^2 + \|Y_j u\|^2) \leq C \left[\sum_{j=1}^{d} \|L_j u\|^2 + \|u\|^2\right] .$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ supported in a neighborhood ω_{x_0,t_0} of (x_0, t_0).

After taking the representation Π_{V,τ_0}, the maximal hypoellipticity of the system (4.3) globally satisfied on all $\mathbb{R}^{d+1}$ would imply

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d), \sum_{j=1}^{d}(\left\|\partial_{x_j} u\right\|^2 + |\tau_0|^2 \left\|\partial_{x_j} V(x) u\right\|^2) \le C \left[\left\langle u\,,\, \Delta^{(0)}_{\tau_0 V} u\right\rangle + \|u\|^2\right]\,,$$

with no condition on the sign of τ_0. Actually we will see below that such an inequality would imply the compactness of the resolvent (for example for $V = \pm x_1^2 x_2^2$) with no sign condition, in contradiction with **Conclusion 3**. Hence we have to treat separately the positive and negative frequency variables τ_0, that is to work microlocally.

Definition 4.7. *The system* $L_1, \ldots, L_d$ *introduced in* (4.3) *on* $\mathbb{R}^{d+1}_{x,t}$ *is said maximally microhypoelliptic around* x_0 *in the direction* $\tau > 0$, *if there exists* $C > 0$ *such that the estimate*

$$\sum_{j=1}^{d}(\|\Pi_{V,\tau_0}(X_j)u\|^2 + \|\Pi_{V,\tau_0}(Y_j)u\|^2) \le C\left[\sum_{j=1}^{d}\|\Pi_{V,\tau_0}(L_j)u\|^2 + \|u\|^2\right]$$

holds for all $\tau_0 > 0$ *and all* $u \in \mathcal{C}_0^\infty(\omega_{x_0})$.

4.4 A criterion for maximal microhypoellipticity.

4.4.1 Canonical sets.

The notion of maximal microhypoellipticity of the system (4.3) leads us to consider the global inequalities

$$\begin{aligned}\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad & \sum_{j=1}^{d}(\|\Pi_{V,\tau_0}(X_j)u\|^2 + \|\Pi_{V,\tau_0}(Y_j)u\|^2) \\ & \le C_\delta\left[\sum_{j=1}^{d}\|\Pi_{V,\tau_0}(L_j)u\|^2 + \delta R_V(0)^2\|u\|^2\right], \quad (\tau_0 > 0) \quad (4.5)\end{aligned}$$

for $\delta \in \{0, 1\}$. Actually the case $\delta = 1$ is a rewriting of a global version of Definition 4.7 and the case $\delta = 0$ has to be introduced in the analysis relying on some induction on $r = d^\circ V$.

Definition 4.8. *The estimate* (4.5) *is called a (global) maximal estimate (with remainder term when* $\delta = 1$*).*

A simple change of variable $u(\frac{x-x_0}{\lambda})$ shows that the maximal estimate (4.5) for the potential $V \in E_r$, implies the same estimate for the potential

$y \to V(x_0 + \lambda y)$. Moreover by writing

$$C_\delta(V,\tau_0) = \sup_{\substack{u \in \mathcal{C}_0^\infty \\ \|u\| = 1}} \frac{\sum_{j=1}^d (\|\Pi_{V,\tau_0}(X_j)u\|^2 + \|\Pi_{V,\tau_0}(Y_j)u\|^2)}{\left[\sum_{j=1}^d \|\Pi_{V,\tau_0}(L_j)u\|^2 + \delta R_V(0)^2 \|u\|^2\right]}$$

we see [4] that the set of potentials V for which $C_\delta(V,\tau_0) \leq C_\delta$ is closed. Note finally that the value $V(0)$ can be set to 0 because only derivatives of V are involved in the problem. We are thus led to introduce the notion of canonical sets.

Definition 4.9. *A subset $\mathcal{L}$ of E_r will be said canonical if the three next properties are true :*

1. *If $P \in \mathcal{L}$ and $y \in \mathbb{R}^d$, then the polynomial defined by*

   $$Q(x) = P(x+y) - P(y), \forall x \in \mathbb{R}^d\,,$$

   is also in $\mathcal{L}$.

2. *If $P \in \mathcal{L}$ and $\lambda > 0$, then $Q(x) = P(\lambda x)$ is also in $\mathcal{L}$.*

3. *$\mathcal{L}$ is a closed subset of E_r.*

Definition 4.10. *For a potential $V \in E_r$, the smallest canonical set which contains V is denoted $\mathcal{L}_V$.*

The set $\mathcal{L}_V$ is made of limits with respect the parameters (x,λ) of polynomials

$$V(x+\lambda y) - V(x) = \sum_{1\leq|\alpha|\leq r} \frac{\lambda^\alpha \partial_x^\alpha V(x)}{\alpha!} y^\alpha$$

and all the possible limits

$$\left(\lambda_n^{|\alpha|}\partial_x^\alpha V(x_n)\right)_{|\alpha|\geq 1} \to (\ell^\alpha)_{|\alpha|\geq 1} = (\partial_x^\alpha P(0))_{|\alpha|\geq 1}$$

have to be considered.

Exercise 4.11. *Assume $\lim_{x\to\infty} R_V(x) = +\infty$. Check that there are two possible kinds of limits (after the extraction of subsequences) :*

1. $\lim_{n\to\infty}\lambda_n = \lambda_0 \in (0,+\infty)$, $\lim_{n\to\infty} x_n = x_0 \in \mathbb{R}^d$ *and* $P(y) = V(x_0 + \lambda_0 y) - V(x_0)$.

2. $\lim_{n\to\infty}\lambda_n = 0$ *and* $d^\circ P \leq r - 1$.

---

[4] The case $\delta = 0$ will be clarified further.

If the inequality holds for one $\tau_0 > 0$, then it is true for any $M > 0$ and $\tau_0 \in (0, M]$ with a constant $C_\delta(M)$. For the compactness criterion, we can work with $\tau_0 > 0$ fixed. But when one considers really the microhypoellipticy question the limit $\tau_0 \to +\infty$ has to be considered. We are thus led to introduce for this analysis the polynomials which are limits of

$$\tau_n[V(x_n + \lambda_n y) - V(x_n)] = \sum_{1\leq|\alpha|\leq r} \frac{\tau_n \lambda_n^{|\alpha|} \partial_x^\alpha V(x_n)}{\alpha!} y^\alpha \,.$$

Definition 4.12.
We denote by $\mathcal{L}_{V,x_0}$ the canonical set which contains all the polynomials P of degree less or equal to r ($P \in E_r$) vanishing at x_0 such that there exists a sequence $x_n \to x_0$, $\tau_n \to +\infty$ and $d_n \to 0$ such that :

$$d_n^{|\alpha|} \tau_n (\partial_x^\alpha V)(x_n) \to \partial_x^\alpha P(x_0) \,. \tag{4.6}$$

Exercise 4.13. *Check that when $\lim_{x\to\infty} R_V(x) = +\infty$ the limit process* (4.6) *can lead to a polynomial of degre r only if $V(x - x_0)$ is homogeneous and $P = cV$.*

4.4.2 Maximal estimate and compactness of $\Delta_V^{(0)}$.

We start with a Lemma which is a rather straightforward consequence of the Baker-Campbell-Hausdorff formula (see [Hor1] and [Nou] for other versions).

Lemma 4.14. *In a Hilbert space $\mathcal{H}$ assume that $A_1, \ldots, A_n$ is a collection of self-adjoint operators with a common core D and such that the commutators*

$$iA_I = [iA_{i_1}[iA_{i_2}, \ldots [iA_{i_{p-1}}, iA_{ip}]] \ldots] \,, \quad I = (i_1, \ldots, i_p)$$

form a nilpotent Lie algebra $\mathcal{G}_A$ with rank $r \in \mathbb{N}$

$$iA_I = 0 \quad (when) \quad p = |I| > r \,,$$

all the A_I's being essentially self-adjoint on D [5]*. Then the estimate*

$$\forall u \in D, \quad \left\| |A_I|^{1/|I|} u \right\|^2 \leq C \left[\sum_{j=1}^n \|A_j u\|^2 + \|u\|^2 \right]$$

holds for all I, $|I| \leq r$.

---

[5] Such an assumption is redundant in the nilpotent setting but it is immediately checked in our case.

Proof: For a self-adjoint operator B the norm in $D(\langle B\rangle^{\nu})$ is equivalent to

$$\int_0^\infty \left\|e^{-itB}u-u\right\|^2 \frac{dt}{t^{2\nu+1}} = \int_0^\infty \left\|e^{it^{1/\nu}B}u-u\right\|^2 \frac{\nu dt}{t^3} \qquad (4.7)$$

for any $\nu \in (0,1]$[6].
For a self-adjoint operator A, the Hilbert space $D(\langle A\rangle^{\nu})$ is denoted by H_A^{ν} for $\nu\mathbb{R}$.
Now it is more convenient to work with "vector fields" by setting $X_I = -iA_I$.
For $|I_1| \leq r$ and $|I_2| \leq r$, the Baker-Campbell-Hausdorff formula implies :

$$e^{t^{|I_1|}X_{I_1}}e^{t^{|I_2|}X_{I_2}}e^{-t^{|I_1|}X_{I_1}}e^{-t^{|I_2|}X_{I_2}} = e^{t^{|I_1|+|I_2|}[X_{I_1},Y_{I_2}]+Z(I_1,I_2,t)}$$

where $Z(I_1,I_2,t)$ is a finite ($\mathcal{G}_A$ is nilpotent) series

$$Z(I_1,I_2,t) = \sum_{|J|>|I_1|+|I_2|} c_J t^{|J|} X_J\,.$$

From this one can show by induction that for $|I| \leq r$, there exist $Z_I(t)$ and sequences $(j_1,\ldots,j_{N_I})$, $(\varepsilon_1,\ldots,\varepsilon_{N_I})$, with $j_k \in \{1\ldots,n\}$ and $\varepsilon_{j_k} = \pm 1$ for all $k \leq N_I$, such that

$$e^{t^{|I|}X_I} = e^{\varepsilon_1 t_1 X_{j_1}}e^{\varepsilon_2 t X_{j_2}}\ldots e^{\varepsilon_{N_I} t X_{j_{N_I}}}e^{Z_I(t)}$$

with

$$Z_I(t) = \sum_{|J|>|I|} c_{I,J} t^{|J|} X_J\,.$$

This leads to

$$\begin{aligned}\left(e^{t^{|I|}X_{|I|}}-\mathrm{Id}\right) &= e^{\varepsilon_1 t_1 X_{j_1}}e^{\varepsilon_2 t X_{j_2}}\ldots e^{\varepsilon_{N_I} t X_{j_{N_I}}}\left(e^{Z_I(t)}-\mathrm{Id}\right)+\cdots \\ &\quad e^{\varepsilon_1 t_1 X_{j_1}}e^{\varepsilon_2 t X_{j_2}}\ldots\left(e^{\varepsilon_{N_I} t X_{j_{N_I}}}-\mathrm{Id}\right)+\cdots+\left(e^{\varepsilon_1 t_1 X_{j_1}}-\mathrm{Id}\right)\,.\end{aligned}$$

We obtain

$$\left\|e^{t^{|I|}X_I}u-u\right\| \leq \left\|\left(e^{Z_I(t)}-\mathrm{Id}\right)u\right\| + C_I(\langle t\rangle^{-1}|t|\sum_{j=1}^{n}\|X_j u\| + \|u\|)\,.$$

Another application of Baker-Campbell-Hausdorff formula, with the same decomposition as above, leads to

$$\left\|e^{Z_I(t)}u-u\right\| \leq C_I' \sum_{|J|>|I|}\left\|e^{d_{I,J}t^{|J|}X_J}u-u\right\|\,.$$

---

[6] Simply use the spectral theorem and Fubini theorem with $\int_0^\infty \sin^2(tx)\frac{dt}{t^{2\nu+1}} = C_\nu x^{2\nu}$.

We make an integration with respect to t with the measure dt/t^3 according to (4.7) :

$$\left\|A_I^{1/|I|}u\right\| \leq C_I'' \left[\|u\| + \sum_{j=1}^n \|A_j u\| + \sum_{|J|>|I|} \left\|A_J^{1/|J|}u\right\|\right] .$$

We conclude by reverse induction starting from $|I| = r$. ∎

A consequence is a lower bound for the Witten Laplacian when the system (4.3) is maximally microhypoelliptic.

Proposition 4.15. *If the maximal estimate* (4.5) *holds for* τ_0 *then there exists a constant* $C > 0$ *such that*

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ \|R_{\tau_0 V}(x)u\|^2 \leq C\left[\left\langle u\,, \Delta_{\tau_0 V}^{(0)} u\right\rangle + \|u\|^2\right] \tag{4.8}$$

with $R_{\tau_0 V}$ *defined in* (4.4).
If moreover the condition

$$\lim_{x\to\infty} R_{\tau_0 V}(x) = +\infty \tag{4.9}$$

is satisfied then $\Delta_{\tau_0 V}^{(0)}$ *has a compact resolvent.*

Proof: It suffices to apply Lemma 4.14 with $n = 2d$, $D = \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$A_{j\leq d} = i\partial_{x_j}, \quad A_{j>d} = (\partial_{x_{j-d}} V(x))\tau_0$$

for which the higher order commutators are

$$\{A_J,\ |J| = p\} = \{\partial_x^\alpha V(x)\tau_0,\ |\alpha| = p\}\,, \quad (2 \leq p \leq r)\,.$$

∎

Remark 4.16. *The condition* (4.9) *for a polynomial actually means that* V *depend on all variables. (see the exercise below).*

Exercise 4.17. *Let* $V(Y)$ *belong to* $\mathbb{R}[Y_1, \ldots, Y_d]$ *. Prove that one can find an affine change of coordinate* $Y = AX + B$, $(X_1, \ldots, X_d)$ *on* $\mathbb{R}^d$*, and* $n(V) \in \mathbb{N}$ *such that*

1. *for all* $j \in \{1, \ldots, n(V)\}$ $\partial_{X_j} V \equiv 0$ *;*
2. *for* $j \in \{n(V)+1, \ldots, d\}$*, there exists* $\Phi_j \in \mathbb{R}[X_{j+1}, \ldots, X_d]$ *such that*

$$X_j + \Phi_j(X_{j+1}, \ldots, X_d) \in \operatorname{Span}\{\partial_X^\alpha V,\ \alpha \in \mathbb{N}^d\}\,.$$

Indication: Consider the linear mappings ∂_{Y_j}, $j = 1, \ldots, d$, *which send the finite dimensional vector space* $F_p = \mathrm{Span}\,\{\partial_Y^\alpha V, |\alpha| \geq p\}$ *into* F_{p+1} *and construct the coordinates* X_j*'s by reverse induction in a Jordan approach.*
Prove that when $n(V) = 0$, *there exists a polynomial* $Q_P \in \mathbb{R}[(T^\alpha)_{\alpha\in\mathbb{N}^d,|\alpha|\leq r}]$ *such that*

$$Q_P((\partial_x^\alpha V(Y))_{\alpha\in\mathbb{N}^d,|\alpha|\leq r]}) = \sum_{j=1}^d Y_j^2\,.$$

Conclude that $\lim_{y\to\infty} R_V(y) = +\infty$ *if and only if there is no* $e \in \mathbb{R}^d$ *such that* $(e.\partial_Y)V \equiv 0$ *and that moreover* $R_V(y) \geq (1 + |y|)^{\alpha_P}$ *for some* $\alpha_P > 0$.

4.4.3 Reduction of the number of variables.

We saw that the maximal estimates have to be considered for all $P \in \mathcal{L}_V$. While doing so, we are led to consider polynomials P which do not depend on all the $(x_1, \ldots, x_d)$ variables. After a suitable affine change of variables $x \mapsto t$, we get an integer $n(P)$ such that $\partial_{t_j} P = 0$ for all $j = 1, \ldots n(P)$. The linear change of variable $x \to t$ can be implemented by a unitary transformation on $L^2(\mathbb{R}^d)$. After taking the Fourier transform with respect to the $(t_1, \ldots, t_{n(P)})$ variable the maximal estimates (4.5) with P becomes

$$\begin{aligned}|\tau'|^2 \|u\|^2 &+ \sum_{j=n(P)+1}^{d} (\|\partial_{t_j} u\|^2 + \|\tau_0 \partial_{t_j} P u\|^2) \\ &\leq C_\delta \left[|\tau'|^2 \|u\|^2 + \sum_{j=n(P)+1}^{d} \|(\partial_{t_j} + \tau_0 \partial_{t_j} P)u\|^2 + \delta R_P(0)^2 \|u\|^2 \right]\end{aligned}$$

with $\tau' = (\tau_1, \ldots, \tau_{n(P)})$. We set $k = k(P) = d - n(P)$ and $\tau = (\tau_0, \tau') \in (0, +\infty) \times \mathbb{R}^{n(P)}$. The previous inequality for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is equivalent to the inequality for all $\tau' \in \mathbb{R}^{n(P)}$ and all $u \in \mathcal{C}_0^\infty(\mathbb{R}^k)$. We are led to introduce the unitary representation $\Pi_{P,\tau}$ of the Lie algebra $\mathcal{G}$ generated by (4.2) on $L^2(\mathbb{R}^{k(P)})$ [7] defined by

$$\begin{aligned} &\Pi_{P,\tau}(\partial_{t_j}) = \tau_j\ , \ \text{for } j = 1, \ldots, n(P)\ , \\ &\Pi_{P,\tau}(\partial_{t_j}) = \partial_{t_j}\ , \ \text{for } j = n(P) + 1, \ldots, d\ , \\ &\Pi_{P,\tau}((\partial_{t_j} P)\partial_t) = 0\ , \ \text{for } j = 1, \ldots, n(P)\ , \\ &\Pi_{P,\tau}((\partial_{t_j} P)\partial_t) = i(\partial_{t_j} P)\tau_0\ , \ \text{for } j = n(P) + 1, \ldots, d\ . \end{aligned} \tag{4.10}$$

This representation is irreductible in the sense that

$$\lim_{t\to\infty} R_P(t) = +\infty \quad \text{in} \quad \mathbb{R}^{k(P)}\,.$$

---

[7] Kirillov's theory of nilpotent Lie algebra says that all the irreducible induced representations of $\mathcal{G}$ are obtained in this way.

Hence it suffices to prove the estimate

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{k(P)}),\ \sum_{j=1}^{d}(\|\Pi_{P,\tau}(X_j)u\|^2 + \|\Pi_{P,\tau}(Y_j)u\|^2) \\ \leq C_\delta \left[\sum_{j=1}^{d} \|\Pi_{P,\tau}(L_j)u\|^2 + \delta R_P(0)^2 \|u\|^2\right]$$

for any $P \in \mathcal{L}_V$ (with $C_\delta \geq 1$). Only the case $\tau' = 0$ has to be considered. We end here with a remark about the relationship between the maximal estimates with remainder ($\delta = 1$) and the maximal estimate ($\delta = 0$): If the maximal estimate with remainder, (4.5) with $\delta = 1$, holds then Proposition 4.15 implies that the Witten Laplacian $\Delta^{(0)}_{\tau_0 P} = \sum_j \Pi_{P,\tau}(L_j)^*\Pi_{P,\tau}(L_j)$ has a compact resolvent in $L^2(\mathbb{R}^{k(P)})$. The control of the remainder term $R_P(0)^2 \|u\|^2$ by $\left\langle u\,,\, \Delta^{(0)}_{\tau_0 P} u\right\rangle$ is then equivalent to the injectivity of $\Delta^{(0)}_{\tau_0 P}$ (see Step **d)** below).

4.4.4 Helffer-Nourrigat result applied to the Witten Laplacian.

The result by Helffer-Nourrigat on polynomials of vector fields take the following form in the framework of Witten Laplacians. This presentation is partly inspired by the Lecture Notes [Nou] by J. Nourrigat and by an application to Schrödinger operators with magnetic fields in [HelMo] by B. Helffer and A. Morame.

Theorem 4.18. *(**Helffer-Nourrigat [HeNo]**) The system* (4.3) *is maximally microhypoelliptic around* x_0 *if and only if no nonzero polynomial* $P \in \mathcal{L}_{V,x_0} \cap E_{r-1}$ *admits a local minimum.*

A variation of it which answers the question of the compactness of $(1 + \Delta^{(0)}_V)^{-1}$ is

Theorem 4.19.
Let $V \in E_r$ *and let us assume that :*

1. $\lim_{x\to\infty} R_V(x) = +\infty$
2. *No non zero polynomial of the canonical set* $\mathcal{L}_V \cap E_{r-1}$ *admits a local minimum.*

Then the Witten Laplacian $\Delta^{(0)}_V$ *has compact resolvent. Moreover the maximal estimate* (4.5) *with* $\delta = 1$ *and the lower bound* (4.8) *hold.*

4.5 Helffer-Nourrigat induction argument.

We shall give a complete proof of Theorem 4.19 by mimicking Helffer-Nourrigat induction for more general problems. We will focus on $\tau_0 = +1$.

4.5.1 Partition of unity.

Since the metric $R_P(x)^2 dx^2$ is slow one can introduce a partition of unity associated with R_P (see [Hor2] for example).

Proposition 4.20. *For any $P \in E_r$ there exists a partition of unity $\sum_{j\in\mathbb{N}} \psi_j^2 \equiv 1$ on $\mathbb{R}^d$ such that*

1. *For any $x \in \mathbb{R}^d$, $\sharp\left\{j, \psi_j(x) \neq 0\right\}$ is uniformly bounded.*
2. $\operatorname{supp}\psi_j \subset B(x_j, aR_P(x_j)^{-1})$ *and $\psi_j \equiv 1$ in $B(x_j, bR_P(x_j)^{-1})$ for some $x_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$.*
3. *For all $\alpha \in \mathbb{N}^d \setminus \{0\}$, there exists $C_\alpha > 0$*

$$\sum_{j\in\mathbb{N}} \left|\partial_x^\alpha \psi_j\right|^2 \leq C_\alpha R_P(x)^{2|\alpha|} .$$

Moreover the constant a, b and C_α can be chosen uniformly with respect to $P \in E_r$, once $r \in \mathbb{N}$ and the dimension $d \in \mathbb{N}$ are fixed.

Remark 4.21. *The uniformity of the constants a, b, C_α is due to the uniformity of the slowness constant. It is proved after noticing that for any $V \in E_r$ and $x_0 \in \mathbb{R}^d$, the polynomial $V(x_0 + R_V(x_0)^{-1}y) - V(x_0)$ belongs to the compact familly of polynomials $\{P \in E_r, P(0) = 0$ and $R_P(0) = 1\}$.*

We can combine this with the IMS localization formula ([CFKS])

$$-\Delta_x = \sum_{j\in\mathbb{N}} \psi_j(-\Delta_x)\psi_j - \left|\partial_x \psi_j\right|^2(x) .$$

By taking $V \in E_r$ and $P = \varepsilon V$ with $\varepsilon > 0$ the inequality

$$R_{\varepsilon V}(x) \leq \varepsilon^{1/r} R_V(x)$$

leads to

$$\Delta_V^{(0)} \geq \sum_{j\in\mathbb{N}} \psi_j \Delta_V^{(0)} \psi_j - C\varepsilon^{2/r} R_V(x)^2 . \tag{4.11}$$

4.5.2 Proof of Theorem 4.19

Assume that $\mathcal{L}$ is a canonical set of E_r such that no non zero $P \in \mathcal{L} \cap E_{r-1}$ admits a local minimum.

We shall prove by induction on $r \in \mathbb{N}^*$ the ordered sequence of results (with $\tau_0 = +1$) :

a) There is a constant $c_0 > 0$ such that the maximal estimate (4.5) holds uniformly with $\delta = 0$ for all $P \in \mathcal{L} \cap E_{r-1}$.

b) For any $V \in \mathcal{L}$ there exists a constant $C_1(V) \geq 1$ such that the maximal estimate (4.5) with remainder (i.e. with $\delta = 1$) is true.

c) For $V \in \mathcal{L}$ such that $\lim_{x\to\infty} R_V(x) = +\infty$, $\Delta_V^{(0)} = \Pi_{1,V}(L^*L)$ has a compact resolvent.

d) If $V \in \mathcal{L}$ has no local minimum, then the maximal estimate (4.5) is true with $\delta = 0$ and a constant $C_0(V) \geq 1$.

e) When no nonzero $V \in \mathcal{L}$ admits a local minimum the constant $C_0(V)$ can be chosen uniformly on $\mathcal{L}$

$\mathbf{r = 1}$ The case $r = 1$ is solved because the only irreducible case is $k(P) = 0$ with scalar representations.

Assume the result at step $\mathbf{r - 1}$.

a) is contained in step $r - 1$.

b) Let V be a polynomial with $d^\circ V = r$. By possibly reducing the number of variables we can assume $\lim_{x\to\infty} R_V(x) = +\infty$ (For the sake of simplicity we still note d the dimension $k(V)$). Assume that there exists a sequence $(u_n)_{n\in\mathbb{N}}$

$$\|\partial_x u_n\|^2 + \|\partial_x V(x) u_n\|^2 \ \geq C_n \left[\left\langle u_n, \Delta_V^{(0)} u_n\right\rangle + R_V(0)^2 \|u_n\|^2\right]$$

such that $\lim_{n\to\infty} C_n = +\infty$. After the introduction of the partition of unity associated with $P = \varepsilon V$ we obtain

$$\begin{aligned}\sum_{j\in\mathbb{N}} &\left[\|\partial_x(\psi_j u_n)\|^2 + \|\partial_x V(x)\psi_j u_n\|^2\right] \\ &\geq C_n \sum_{j\in\mathbb{N}} \Big[\left\langle \psi_j u_n, \Delta_V^{(0)} \psi_j u_n\right\rangle - C\varepsilon^{2/r} \|R_V(x)\psi_j u_n\|^2 \\ &\qquad\qquad + R_V(0)^2 \|\psi_j u_n\|^2 \Big] .\end{aligned}$$

By referring to Lemma 4.14 with $n = 2d$, $A_{j\leq d} = i\partial_{x_j}$ and $A_{j>d} = \partial_{x_j}V(x)$, this implies

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} \left[\|\partial_x(\psi_j u_n)\|^2 + \|\partial_x V(x)\psi_j u_n\|^2\right] \\
&\quad \geq C_n \sum_{j\in\mathbb{N}} \Big[\left\langle \psi_j u_n, \Delta_V^{(0)}\psi_j u_n \right\rangle - C'\varepsilon^{2/r}(\|\partial_x(\psi_j u_n)\|^2 + \|\partial_x V(x)\psi_j u_n\|^2) \\
&\qquad\qquad\qquad\qquad + (R_V(0)^2 - C'\varepsilon^{2/r})\, \|\psi_j u_n\|^2 \Big] .
\end{aligned}$$

Hence we can find a sequence of functions $v_{n_q} = \psi_{j_{n_q}} u_{n_q} \neq 0$ supported in $B(x_{j_{n_q}}, aR_{\varepsilon V}(x_{j_{n_q}})^{-1}) \subset B(x_{j_{n_q}}, a\varepsilon^{-1}R_V(x_{j_{n_q}})^{-1})$ such that

$$\begin{aligned}
&\left[\left\|\partial_x v_{n_q}\right\|^2 + \left\|\partial_x V(x) v_{n_q}\right\|^2\right] \\
&\qquad \geq C'_{n_q} \Big[\left\langle v_{n_q}, \Delta_V^{(0)} v_{n_q} \right\rangle - C'\varepsilon^{2/r}(\left\|\partial_x v_{n_q}\right\|^2 + \left\|\partial_x V(x) v_{n_q}\right\|^2) \\
&\qquad\qquad\qquad\qquad + (R_V(0)^2 - C'\varepsilon^{2/r})\, \left\|v_{n_q}\right\|^2 \Big] .
\end{aligned}$$

with $\lim_{q\to\infty} x_{j_{n_q}} = \infty$ and $\lim_{q\to\infty} R_V(x_{j_{n_q}}) = \lim_{q\to\infty} C'_{n_q} = +\infty$. After taking q large enough and after making the change of variable $x = x_j + R_V(x_j)^{-1}y$, there exists a sequence $(w_{n_q})_{q\in\mathbb{N}}$, $w_{n_q} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with $w_{n_q} \neq 0$, $\operatorname{supp} w_{n_q} \subset B(0, a\varepsilon^{-1})$ and such that

$$\begin{aligned}
0 \geq C'_{n_q} \Big[&\left\langle w_{n_q} , \Delta_{\Phi_{n_q}}^{(0)} w_{n_q} \right\rangle - 2C'\varepsilon^{2/r}(\left\|\partial_y w_{n_q}\right\|^2 + \left\|\partial_y \Phi_{n_q}(y) w_{n_q}\right\|^2) \\
&\qquad\qquad + \frac{(R_V(0)^2 - C'\varepsilon^{2/r})}{R_V(x_{j_{n_q}})^2} \left\|w_{n_q}\right\|^2 \Big] .
\end{aligned}$$

with $\left(\Phi_{n_q}\right)_{q\in\mathbb{N}}$ compact in $\mathbb{R}[X_1, \ldots, X_d]$. Moreover since $\lim_{q\to\infty} R_V(x_{j_{n_q}}) = +\infty$ the only possible limits belong to E_{r-1}. The constant $\varepsilon > 0$ is chosen such that

$$C'\varepsilon^{2/r} \leq \min\left\{\frac{c_0}{4}, \frac{R_V(0)^2}{2}\right\} .$$

This implies

$$0 \geq \left\langle w_{n_q} , \Delta_{\Phi_{n_q}}^{(0)} w_{n_q} \right\rangle - \frac{c_0}{2}(\left\|\partial_y w_{n_q}\right\|^2 + \left\|\partial_y \Phi_{n_q}(y) w_{n_q}\right\|^2)$$

which is in contradiction with **a)** after considering a subsequence such that $\Phi_{n_q} \to \Phi \in E_{r-1} \cap \mathcal{L}$ (Note that $\operatorname{supp} w_{n_q}$ remains in the fixed bounded set $B(0, a\varepsilon^{-1})$) .

c) Once the maximal estimate with remainder (4.5) holds with a constant $C_1 = C_1(V) \geq 1$ possibly depending on V, the compactness of the resolvent of $\Delta_V^{(0)}$ is deduced from Proposition 4.15 .

d) Assume that $V \in E_r$ has no local minimum. After reducing the number of variables V has no local minimum in $\mathbb{R}^{k(V)}$ while the estimate with remainder implies that the non negative operator $\Delta_V^{(0)}$, acting on $\mathbb{R}^{k(V)}$, has a compact resolvent. Hence the maximal estimate without remainder is true except if $\Delta_V^{(0)}$ has a non trivial kernel. This occurs only if $e^{-V} \in L^2(\mathbb{R}^{k(V)})$, which implies that e^{-V} has a local maximum and V has a local minimum!! Hence the maximal estimate (4.5) is true with $\delta = 0$ and a possibly V-dependent constant.
e) The uniformity of the constant

$$C_0(V) = \sup_{u \in \mathcal{C}_0^\infty} \frac{\sum_{j=1}^{d} \left\|\partial_{x_j} u\right\|^2 + \left\|\partial_{x_j} V(x) u\right\|^2}{\left\langle u \,,\, \Delta_V^{(0)} u \right\rangle} \tag{4.12}$$

which allows the induction via **a)** is obtained in two steps, with a second one which refers to a more general framework. We need two specific notions for this part.

Definition 4.22. *With a sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ of $\mathcal{L}$ is associated the subset $\mathcal{L}_{\mathcal{V}}$ of $\mathcal{L}$ made of all possible limits*

$$P = \lim_{q\to\infty} V_{n_q}(x_q + \lambda_q y) - V_{n_q}(x_q)$$

where $(V_{n_q})_{q\in\mathbb{N}}$ is a subsequence of $\mathcal{V}$ ($n_q < n_{q+1}$) and $((\lambda_q, x_q))_{q\in\mathbb{N}}$ is a sequence of $\mathbb{R}_+^* \times \mathbb{R}^d$.

Definition 4.23. *We say that a function $f : \mathcal{L} \to \mathbb{R}_+$ satisfies the orbital semicontinuity property (denoted by OSCP) if any sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ such that $\sup_{P\in\mathcal{L}_{\mathcal{V}}} f(P) = K_{\mathcal{V}} < +\infty$ satisfies:*

$$\forall \delta > 0, \exists N_\delta \in \mathbb{N}, \forall n \geq N_\delta, \quad f(V_n) \leq K_{\mathcal{V}} + \delta\,.$$

Step 1: The function $V \to C_0(V)$ defined by (4.12) satisfies the orbital semicontinuity property (OSCP).

Assume on the contrary that there exists a sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ with $\sup_{P\in\mathcal{L}_{\mathcal{V}}} C_0(P) = K_{\mathcal{V}} < +\infty$, $\delta > 0$ and a subsequence $(V_{n_q})_{q\in\mathbb{N}}$ such that

$$\forall q \in \mathbb{N}, \quad C_0(V_{n_q}) \geq K_{\mathcal{V}} + \delta\,.$$

By definition of $C_0(V_{n_q})$, there exists a sequence $(u_q)_{q\in\mathbb{N}}$ of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\left\|\partial_x u_q\right\|^2 + \left\|(\partial_x V_{n_q}) u_q\right\|^2 \geq (K_{\mathcal{V}} + \frac{\delta}{2}) \left\langle u_q \,,\, \Delta_{V_{n_q}}^{(0)} u_q \right\rangle\,.$$

We now consider like before, but for every $q \in \mathbb{N}$, a partition of unity $\sum_{j\in\mathbb{N}} \psi_{j,q}^2 \equiv 1$ associated with the polynomial εV_{n_q} where $\varepsilon > 0$ will be fixed further. We obtain for all $q \in \mathbb{N}$

$$\sum_{j\in\mathbb{N}} \|\partial_x(\psi_{j,q}u_q)\|^2 + \left\|(\partial_x V_{n_q})(\psi_{j,q}u_q)\right\|^2$$
$$\geq (K_{\mathcal{V}} + \delta/2) \sum_{j\in\mathbb{N}} \left[\left\langle \psi_{j,q}u_q\,,\, \Delta^{(0)}_{V_{n_q}}\psi_{j,q}u_q\right\rangle - C\varepsilon^{2/r}\left\|R_{V_{n_q}}\psi_{j,q}u_q\right\|^2\right]$$

Hence there is a sequence $(j_q)_{q\in\mathbb{N}}$ such that $v_q = \psi_{j_q,q}u_q$ satisfies

$$v_q \in \mathcal{C}_0^\infty(B(x_{j_q,q}, aR_{\varepsilon V_{n_q}}(x_{j_q,q}))^{-1}), \quad v_q \neq 0$$
$$\|\partial_x v_q\|^2 + \left\|(\partial_x V_{n_q})v_q\right\|^2 \geq (K_{\mathcal{V}} + \delta/2)\left[\left\langle v_q\,,\, \Delta^{(0)}_{V_{n_q}}v_q\right\rangle - C\varepsilon^{2/r}\left\|R_{V_{n_q}}v_q\right\|^2\right].$$

By setting

$$w_q(y) = v_q(x_{j_q,q} + R_{V_{n_q}}(x_{j_q,q})^{-1}y) \quad \text{and} \quad \Phi_q(y) = V_{n_q}(x_{j_q,q} + R_{V_{n_q}}(x_{j_q,q})^{-1}y)\,,$$

the sequence $(\Phi_q)_{q\in\mathbb{N}}$ is compact in $\mathcal{L}$ with limit points in $\mathcal{L}_{\mathcal{V}}$ and all the w_q's belong to $\mathcal{C}_0^\infty(B(0, a\varepsilon^{-1}))$ with

$$\|\partial_y w_q\|^2 + \|(\partial_y\Phi_q)w_q\|^2$$
$$\geq (K_{\mathcal{V}} + \delta/2)\left[\left\langle w_q\,,\, \Delta^{(0)}_{\Phi_q}w_q\right\rangle - C'\varepsilon^{2/r}\|\partial_y w_q\|^2 - C'\varepsilon^{2/r}\|(\partial_y\Phi_q)w_q\|^2\right].$$

or equivalently

$$\|\partial_y w_q\|^2 + \|(\partial_y\Phi_q)w_q\|^2 \geq \frac{K_{\mathcal{V}} + \delta/2}{1 + (K_{\mathcal{V}} + \delta/2)C'\varepsilon^{2/r}}\left\langle w_q\,,\, \Delta^{(0)}_{\Phi_q}w_q\right\rangle.$$

We first fix $\varepsilon > 0$ such that the right-hand side is greater than $(K_{\mathcal{V}} + \delta/4)\left\langle w_q\,,\, \Delta^{(0)}_{\Phi_q}w_q\right\rangle$. By extracting a subsequence we can assume $\lim_{k\to\infty}\Phi_{q_k} = P \in \mathcal{L}_{\mathcal{V}}$. With the uniform control of the support $\operatorname{supp} w_{q_k} \subset B(0, a\varepsilon^{-1})$, this contradicts the bound $C_0(P) \leq K_{\mathcal{V}}$.
Step 2: The last step is actually the application of the next Proposition of which the proof is deferred to the next paragraph.

Proposition 4.24. *Let $\mathcal{L}$ be a canonical set of E_r and let $f : \mathcal{L} \to \mathbb{R}_+^*$ which satisfies the next three conditions:*

Invariance: *$f(Q) = f(P)$ if $Q(y) = P(x_0 + \lambda y) - P(x_0)$ for some $(x_0, \lambda) \in \mathbb{R}^d \times \mathbb{R}_+^*$.*

OSCP: *f satisfies the orbital semicontinuity property of Definition 4.23.*

Induction: *f is bounded on $\mathcal{L} \cap E_{r-1}$.*

Then f is bounded on $\mathcal{L}$.

The function C_0 satisfies these three conditions: the invariance after using the change of variables $x = x_0 + \lambda y$ the **OSCP** according to Step 1 and the last one is simply given by the induction assumption. Hence C_0 is bounded on $\mathcal{L}$ ∎

4.5.3 Orbital semicontinuity is enough.

Here is given as a final step the proof of Proposition 4.24, which says that the the orbital semicontinuity property combined with the two other assumptions provides the same uniform bound on compact set as the usual upper semicontinuity. It is actually the most intimately connected point with the representation theory of nilpotent Lie groups. A more general version of it which enters in the Helffer-Nourrigat general approach of polynomials of vector fields is given in the paragraph VIII-4 (pp 172–176) of [HeNo].

Some notations: Let G be the abelian additive group $\mathbb{R}^d$ with Lie algebra $\mathcal{G} = \left\{\partial_{x_j}, j = 1, \ldots, d\right\}$. There is a natural action of G on $E_r/\mathbb{R}$ given by

$$(\tau_{x_0} P)(x) = P(x_0 + x) - P(x_0)$$

which can be generalized as follows. Introduce the linearly free family $(e_\alpha)_{\alpha\in\mathbb{N}^d, 1\leq|\alpha|\leq r}$ and consider for any $j \in \{1, \ldots, r\}$ the vector space

$$L^j = \left\{(\partial_x^\alpha P \otimes e_\alpha)_{\alpha\in\mathbb{N}^d,\ j\leq|\alpha|\leq r}, P \in E_r\right\} \subset \oplus_{1\leq|\alpha|\leq r}[E_{r-|\alpha|} \otimes \mathbb{R}e_\alpha]\,.$$

We shall use the notation for any $P \in E_r$ and $j \geq 1$:

$$\partial^{\geq j} P = (\partial^\alpha P \otimes e_\alpha)_{|\alpha|\geq j} \in L^j\,,$$

while on the contrary $\mathcal{Q}_\alpha$ denotes the e_α-component of $\mathcal{Q} \in L^j$. The action of $G = \mathbb{R}^d$ on L^j is given by the Taylor formula

$$\begin{aligned}
[U^j(x_0).(\partial^{\geq j} P)]_\alpha &= \partial_x^\alpha P(x_0 + .) = \partial_x^\alpha P + \sum_{1\leq|\beta|} \frac{x_0^\beta}{\beta!} \partial_x^{\alpha+\beta} P \\
&= (\partial^{\geq j} P)_\alpha + \sum_{1\leq|\beta|} \frac{x_0^\beta}{\beta!} (\partial^{\geq j} P)_{\alpha+\beta}\,.
\end{aligned}$$

This defines a unipotent representation of $G = \mathbb{R}^d$ on the finite dimensional space L^j, $j \geq 1$, that is for any x_0, $U^j(x_0) - \mathrm{Id}$ is a nilpotent linear mapping on L^j.
For $j \in \{1, \ldots, r\}$ and $Q \in E_r$, the orbit of $\partial^{\geq j} Q$ in L^j is denoted by

$$O^j(\partial^{\geq j} Q) = \left\{\partial^{\geq j} P \quad \text{s.t.}\exists x_0 \in \mathbb{R}^d, \forall\alpha \in \mathbb{N}^d, j \leq |\alpha| \leq r, \partial_x^\alpha P(0) = \partial_x^\alpha Q(x_0)\right\}\,.$$

Lemma 4.25. *Take $j \in \{1, \ldots, r\}$ and Q in E_r. The orbit $O^j(Q)$ is closed in L^j. Moreover, for any sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of $O^j(Q)$ which admits a limit $\mathcal{P}_\infty$ in L^j, there exists a sequence $(x^n)_{n\in\overline{\mathbb{N}}}$ such that $\lim_{n\to\infty} x^n = x^\infty$ and*

$$\forall n \in \mathbb{N} \cup \{\infty\}\,, \quad \mathcal{P}_n = U^j(x^n)(\partial^{\geq j} Q)\,.$$

Proof: The integer $j \in \{1, \ldots, r\}$ and the polynomial $Q \in E_r$ are fixed. With the same method as in Exercise 4.17, an affine change of coordinates such that in the new Q-dependent coordinates system $(x_1, \ldots, x_d)$ the polynomial Q satisfies :

- For all $k \in \{1, \ldots, n(V,j)\}$ and for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \geq j$, $\partial_{x_k} \partial_x^\alpha Q \equiv 0$.
- For $k \in \{n(V,j)+1, \ldots, d\}$, there exists $\Phi_k \in \mathbb{R}[x_{k+1}, \ldots, x_d]$ such that

$$x_k + \Phi_k(x_{k+1}, \ldots, x_d) \in \mathrm{Span}\left\{\partial_x^\alpha Q, \alpha \in \mathbb{N}^d, |\alpha| \geq j\right\} .$$

By definition, any $\mathcal{P}_n \in O^j(Q)$ can be written $(\partial_x^\alpha Q(x^n + .) \otimes e_\alpha)_{|\alpha| \geq j}$. According to the first condition of the coordinate system, the polynomials $(\partial_x^\alpha Q(x^n + .))_{|\alpha| \geq j}$ depend only on the variables $\left(x_{n(V,j)+1}, \ldots, x_d\right)$ and on the parameters $\left(x^n_{n(V,j)+1}, \ldots, x^n_d\right)$. Moreover the convergence and the second condition provide the existence of $(x^\infty_{n(V,j)}, \ldots, x^\infty_d) \in \mathbb{R}^{d-n(V,j)}$ such that

$$\lim_{n \to \infty} (x^n_{n(V,j)+1}, \ldots, x^n_d) = (x^\infty_{n(V,j)+1}, \ldots, x^\infty_d) .$$

It suffices to take in this coordinate system $x^n = (0, \ldots, 0, x^n_{n(V,j)+1}, \ldots, x^n_d)$ for any $n \in \mathbb{N} \cup \{\infty\}$. ∎

Remark 4.26. *In [HeNo]-ChapVIII-Lemma 4.4 a more general result for unipotent representations of nilpotent Lie groups is stated. It refers to the general description of orbits in finite dimensional unipotent representations of nilpotent Lie groups (see [Puk]-Part.II-Chap.I-Section 3 for example).*

Proof of Proposition 4.24: The proof uses the homogeneity with respect to the transformation $P \to P(\lambda y)$, $\lambda > 0$, in a different way than the core of Helffer-Nourrigat induction. Moreover an additional induction process will be used.

We set

$$B = \left\{P \in E_r, \quad \sum_{|\alpha|=r} |\partial_x^\alpha P(0)|^{1/r} = 1\right\}$$

and for any $j \in \{1, \ldots, r\}$ and any $Q \in B$:

$$F(Q,j) = \left\{P \in \mathcal{L} \cap B, \exists x_0 \in \mathbb{R}^d, \forall \alpha \text{ s.t. } |\alpha| \geq j, \quad \partial_x^\alpha P(0) = \partial_x^\alpha Q(x_0)\right\} ,$$

with the additional convention

$$F(Q, r+1) = \mathcal{L} \cap B .$$

Indeed the case $\mathcal{L}\cap B=\emptyset$ is solved for it corresponds to the case $\mathcal{L}=\mathcal{L}\cap E_{r-1}$ on which the function f is assumed to be bounded.
Note also that

$$O^j(\partial^{\geq j}Q)=\left\{\partial^{\geq j}P, \quad P\in F(Q,j)\right\}$$

is nothing but the orbit of $\partial^{\geq j}Q$ under the action of $\mathbb{R}^d$ on L^j.
We prove by induction on $j\in\{1,\ldots,r+1\}$, that f is bounded on $F(Q,j)$ for any $Q\in\mathcal{L}\cap B$.
1) For $j=1$, the set $F(Q,1)$ is nothing but

$$\left\{P\in\mathcal{L},\ \exists x_0\in\mathbb{R}^d, P(x)=Q(x_0+x)-Q(x_0)\right\}$$

and $f(P)=f(Q)$ for any $P\in F(Q,1)$.
2) Assume that the result is true for $j-1\in\{1,\ldots,r\}$. Let $Q\in B$ and assume that there exists a sequence $\mathcal{V}=(V_n)_{n\in\mathbb{N}}$ in $F(Q,j)\subset\mathcal{L}\cap B$ such that $\lim_{n\to\infty}f(V_n)=+\infty$. The boundedness of f on $\mathcal{L}\cap E_{r-1}$ and the **OSCP** assumption implies $\mathcal{L}_{\mathcal{V}}\not\subset E_{r-1}$. Hence the set $\mathcal{L}_1=\mathcal{L}_{\mathcal{V}}\cap B$ is not empty. Moreover the convergence

$$V_{n_q}(x_q+\lambda_q y)-V_{n_q}(x_q)\overset{q\to\infty}{\to}P\in\mathcal{L}_1$$

combined with $V_{n_q}\in B$ implies $\lambda_q=1$, for all $q\in\mathbb{N}$. We set $\tilde{V}_q=V_{n_q}(x_q+.)-V_{n_q}(x_q)$ and we have

$$\partial^{\geq p}\tilde{V}_q\in O^p(\partial^{\geq p}V_{n_q})\quad\text{and}\quad\lim_{q\to\infty}\partial^{\geq p}\tilde{V}_q=\partial^{\geq p}P,\quad\forall p\geq 1\,.$$

This is true in particular for $p=j$ and Lemma 4.25 can be applied in L^j for this converging sequence: there exists a converging sequence $(x_q)_{q\in\mathbb{N}}$, with $\lim_{q\to\infty}x_q=x_\infty$ such that

$$\partial^{\geq j}\tilde{V}_q=U^j(x_q)\partial^{\geq j}Q\quad\text{and}\quad\partial^{\geq j}P=U^j(x_\infty)\partial^{\geq j}Q\,.$$

Then the sequence $(\partial^{\geq j-1}Q_q)_{q\in\mathbb{N}}$ and the element $\partial^{\geq j-1}Q'$ defined by

$$\partial^{\geq j-1}Q_q=U^{j-1}(-x_q)\tilde{V}_q\quad\text{and}\quad\partial^{\geq j-1}Q'=U^{j-1}(-x_\infty)P\,,$$

satisfy

$$\lim_{q\to\infty}\partial^{\geq j-1}Q_q=\partial^{\geq j-1}Q'\,.$$

A more accurate version says

$$\partial_x^\alpha(Q_q-Q)=0\quad\text{for}\quad|\alpha|\geq j \tag{4.13}$$

$$\lim_{q\to\infty}\partial_x^\beta(Q_q-Q')=0\quad\text{for}\quad|\beta|=j-1\,. \tag{4.14}$$

By setting $\mathcal{V}'=(V_{n_q})_{q\in\mathbb{N}}$ we shall check

$$\mathcal{L}_{\mathcal{V}'}\cap B\subset F(Q',j-1)\,.$$

Assume $R \in \mathcal{L}_{\mathcal{V}'} \cap B$. There is a sequence $(\hat{V}_{q_k})_{k\in\mathbb{N}}$, with

$$\partial^{\geq p}\hat{V}_{q_k} \in O^p(\partial^{\geq p}V_{n_{q_k}}) = O^p(\partial^{\geq p}\tilde{V}_{q_k}), \quad \text{and} \quad \lim_{k\to\infty} \partial^{\geq p}\hat{V}_{q_k} = \partial^{\geq p}R, \quad \forall p \geq 1\,.$$

Owing to $\partial^{\geq j-1}\tilde{V}_{q_k} = U^{j-1}(x_{q_k})Q_{q_k}$, there exists for all $k \in \mathbb{N}$ an element $z_k \in \mathbb{R}^d$ such that

$$\partial^{\geq j-1}\hat{V}_k = U^{j-1}(z_k)\partial^{\geq j-1}Q_{q_k}\,.$$

The fact that the representation U^{j-1} is unipotent combined with (4.13) and (4.14) implies

$$\lim_{k\to\infty} U^{j-1}(z_k)\left(\partial^{\geq j-1}Q_{q_k} - \partial^{\geq j-1}Q'\right) = 0\,.$$

One gets

$$\partial^{\geq j-1}R = \lim_{k\to\infty} U^{j-1}(z_k)\partial^{\geq j-1}Q_{q_k} = \lim_{k\to\infty} U^{j-1}(z_k)\partial^{\geq j-1}Q'\,.$$

We refer again to Lemma 4.25 applied on $O^{j-1}(Q') \subset L^{j-1}$ which says that $O^{j-1}(\partial^{j-1}Q')$ is closed:

$$\partial^{\geq j-1}R \in O^{j-1}(\partial^{\geq j-1}Q', j-1)\,.$$

We have proved $\mathcal{L}_{\mathcal{V}'} \cap B \subset F(Q', j-1)$.
By the induction assumption, the function f is bounded on $F(Q', j-1)$ and therefore on $\mathcal{L}_{\mathcal{V}'}$. The orbital semicontinuity property then implies that the sequence $(f(V_{n_q}))_{q\in\mathbb{N}}$ is bounded, in contradiction with the initial assumption $\lim_{n\to\infty} f(V_n) = +\infty$. ∎

5 Comments and open problems.

5.1 Comments.

1. The microhypoellipticity problem requires the additional analysis of the dependence with respect the parameter $\tau_0 \to +\infty$. Hence the qualitative analysis (**QW1**) and the quantitative problem (**QW2**) are related with this point of view. Actually our proof already requires a parameter dependent analysis with respect to τ' but it is completely trivial in our specific case. The general theory requires an analysis with respect to the full parameter $\tau = (\tau_0, \tau')$ and some microlocalization in cones in the τ-variable. The partition of the unity has to be made with pseudodifferential cut-off in the phase-space, which makes the general presentation significantly more sophisticated.

2. There are known examples (see [HelNi1]) where there is no maximal estimate but the Witten Laplacian $\Delta_V^{(0)}$ still has a compact resolvent. Such examples rely on the examples by Maire in [Mai][Mai4] of subelliptic differential operators which are not maximally hypoelliptic operators. Another more sophisticated example was given recently in [JoTr].

3. The Witten Laplacian with a polynomial potential, in addition to exhibiting a third interesting structure on Witten Laplacian (link with SDE, deformed Hodge theory), has the advantage of providing a pedagogical example where the whole induction approach by Helffer and Nourrigat in [HeNo] can be done completely without requiring all the material of Kirillov theory nor subtle microlocal surgery.

4. One interest of an Helffer-Nourrigat type criterion provided by Theorem 4.19 is that it provides an algorithm to study possibly very degenerate cases. Several examples were analyzed with this process in [HelNi] .

5. There is still a big gap between our understanding of the Witten Laplacian $\Delta_V^{(0)}$ and of the Fokker-Planck operator. Actually for $\Delta_V^{(0)}$ the compactness criteria work with an arbitrarily large rank r in the Hörmander condition. For the Fokker-Planck operator, the condition $|\partial_x^\alpha V(x)| \leq C_\alpha \left<\partial_x V(x)\right>$ essentially limits the result and the method to rank 2.

6. In the example of Witten Laplacians, one see that the hypoellipticity comes from a uniform estimate with respect to the parameter $\tau_0 > 0$. Actually for more general maximally hypoelliptic operators, a uniform control with respect to "frequency parameters" is required within the induction process in order to get the compactness of the resolvent (step **c)**). Hence the analysis of the (maximal) hypoellipticity of operators contains a lot of quantitative and accurate estimates. With respect to this and from an historical point of view, it was interesting to hear B. Helffer explaining during the workshop held in Rennes in February 2003 (see [HelNi]) how he naturally turned in the early eighties from the analysis of hypoelliptic operators to the semiclassical analysis.

5.2 Open problems.

The conjecture

$$((1 + K_+) \text{ compact}) \overset{?}{\Leftrightarrow} \left((1 + \Delta_{V/2}^{(0)})^{-1} \text{ compact}\right) \tag{5.1}$$

is far from being completely solved. It can be decomposed into several questions which have received no answer up to know :

1. Is it possible to prove the equivalence for a class of potential (or even an example) for which the second derivative $\mathrm{Hess}\, V(x)$ is not dominated at infinity by $\langle \partial_x V(x) \rangle$.

2. Are there examples for the Fokker-Planck operator for which the compactness of the resolvent shows a microlocal nature (i.e. depends on the sign $\pm V(x)$) ? (The case $V(x_1, x_2) = \pm x_1^2 x_2^2$ is not really understood up to now.)

3. In the case of a polynomial potential, is it possible to show that the Fokker-Planck operator $K_\pm$ has a compact resolvent when no nonzero polynomial in $\mathcal{L}_V \cap E_{r-1}$ has a local minimum ?

4. Is the equivalence (5.1) true for general $V \in \mathcal{C}^\infty(\mathbb{R}^d)$?

Warning : Below is a rather short list of references. A more complete bibliography may be found in [HelNi].

References

[BonChe] J.M. Bony and J.Y. Chemin. Espaces fonctionnels associés au calcul de Weyl-Hörmander. Bull. Soc. Math. France 122, n^0 1, p. 77-118 (1994).

[BoLe] J.M. Bony and N. Lerner. Quantification asymptotique et microlocalisation d'ordre supérieur I. Ann. Scient. Ec. Norm. Sup., 4^e série 22, p. 377-433 (1989).

[Bi] J.M. Bismut. The hypoelliptic laplacian on the cotangent bundle. J. Amer. Math. Soc. 18 (2), p. 379-476 (2005).

[BiLe] J.M. Bismut and G. Lebeau. The hypoelliptic Laplacian and Ray-Singer metrics. preprint.

[CFKS] H.L Cycon, R.G Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Text and Monographs in Physics. Springer-Verlag (1987).

[EckPiRe-Be] J.P. Eckmann, C.A. Pillet, and L. Rey-Bellet. Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. Comm. Math. Phys. 208 (2), p. 275-281 (1999).

[DaLi] R. Dautray and J.L. Lions. *Mathematical analysis and numerical methods for science and technology.* Springer Verlag (1992).

[Dav1] E.B. Davies. *One-parameter semigroups.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, (1980).

[Ev] L.C. Evans. *An introduction to Stochastic Differential Equations* http://math.berkeley.edu/ evans/

[HeNo] B. Helffer and J. Nourrigat. *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteur.* Progress in Mathematics, Birkhäuser, Vol. 58 (1985).

[HelNi] B. Helffer and F. Nier. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians.* Lect. Notes in Maths 1862, (2005).

[HelNi1] B. Helffer and F. Nier. Criteria for the Poincaré inequality associated with Dirichlet forms in $\mathbb{R}^d$, $d \geq 2$. Int. Math. Res. Notices 22, p. 1199-1224 (2003).

[HelKlNi] B. Helffer, M. Klein and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Mathematica Contemporanea (Brasilean Mathematical Society), Vol. 26 p. 41-86 (2005)

[HelMo] B. Helffer and A. Mohamed. Sur le spectre essentiel des opérateurs de Schrödinger avec champ magnétique. Ann. Inst. Fourier 38(2), p. 95-113 (1988).

[Her] F. Hérau. Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. to appear in Asymptotic Analysis.

[HerNi] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to the equilibrium for the Fokker-Planck equation with high degree potential. Archive for Rational Mechanics and Analysis 171 (2), p. 151-218 (2004).

[Hor1] L. Hörmander. Hypoelliptic second order differential equations. Acta Mathematica 119, p. 147-171 (1967).

[HeSj] B. Helffer and J. Sjöstrand. Equation de Schrödinger avec champ magnétique et équation de Harper. Lecture Notes in Phys., 345, Springer (1989) pp 118–197.

[Hor2] L. Hörmander. *The analysis of linear partial differential operators.* Springer Verlag (1985).

[JoTr] J.L. Journé and J.M. Trépreau. Hypoellipticité sans sous-ellipticité: le cas des systèmes de n champs de vecteurs complexes en $(n+1)$ variables. Séminaire Equations aux Dérivées Partielles de l'Ecole Polytechnique. 2005–2006, Exp. No. XIV, 19 pp.

[Ki] A. Kirillov. Unitary representations of nilpotent groups. Russian Math. Surveys 17, p. 53-104 (1962) .

[Ko] J. Kohn. Lectures on degenerate elliptic problems. Pseudodifferential operators with applications, C.I.M.E., Bressanone 1977, p. 89-151 (1978).

[Kol] A.N. Kolmogorov. Zufällige Bewegungen. Ann. of Math. (2) 35, p. 116-117 (1934).

[Leb1] G. Lebeau. Geometric Fokker-Planck equations. Port. Math. (N.S.) 62 (2005), no. 4, 469–530.

[Leb2] G. Lebeau. Equations de Fokker-Planck géométriques. II. Estimations hypoelliptiques maximales. Ann. Inst. Fourier (Grenoble) 57 (2007), no. 4, 1285–1314.

[Mai] H.M. Maire. Hypoelliptic overdetermined systems of partial differential equations. Comm. Partial Differential Equations 5 (4), p. 331-380 (1980).

[Mai4] H.M. Maire. Régularité optimale des solutions de systèmes différentiels et du Laplacien associé: application au $\Box_b$. Math. Ann. 258, p. 55-63 (1981).

[NaNi] F. Nataf and F. Nier. Convergence of domain decomposition methods via semi-classical calculus. Comm. Partial Differential Equations 23 (5-6), p. 1007–1059 (1998).

[Nou] J. Nourrigat. *Subelliptic estimates for systems of pseudo-differential operators.* Course in Recife (1982). University of Recife.

[Puk] L. Pukanszky. *Leçons sur les représentations des groupes.* Monographies de la Société Mathématique de France Vol 2, Dunod (1967).

[ReSi] M. Reed and B. Simon. *Method of Modern Mathematical Physics.* Academic press, (1975).

[Ris] H. Risken. *The Fokker-Planck equation. Methods of solution and applications.* Springer-Verlag, Berlin, second edition (1989).

[RoSt] L.P. Rothschild and E.M. Stein. Hypoelliptic differential operators and nilpotent groups. Acta Mathematica 137, p. 248-315 (1977).

[Sima] C.G. Simader. Essential self-adjointness of Schrödinger operators bounded from below. Math. Z. 159, p. 47-50 (1978).

[Tr1] F. Trèves. An invariant criterion of hypoellipticity. Amer. J. Math., Vol. 83, p. 645-668 (1961).

[Tr2] F. Trèves. Study of a model in the theory of complexes of pseudo-differential operators. Ann. of Maths (2) 104, p. 269-324 (1976). See also erratum: Ann. Math. (2) 113, p. 423 (1981).

[Vil] C. Villani. Hypocoercive diffusion operators. Proceedings for the 2006 ICM.

[Wi] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, p. 661-692 (1982).