On a Vlasov-Schrödinger-Poisson model

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Abstract

A Vlasov-Schrödinger-Poisson system is studied, modeling the transport and interactions of electrons in a bidimensional electron gas. The particles are assumed to have a wave behaviour in the confinement direction (z) and to behave like point particles in the directions parallel to the electron gas (x). For each fixed x and at each time t, the eigenfunctions and the eigen-energies of the Schrödinger operator in the z are computed. The occupation number of each eigenfunction is computed through the resolution of a Vlasov equation in the x direction, the force field being the gradient of the eigen-energy. The whole system is coupled to the Poisson equation for the electrostatic interaction. Existence of weak solutions is shown for boundary value problems in the stationary and time-dependent regimes.

1 Introduction and main results

1.1 The Quantum-Kinetic Subband Model

Classical motion of charged particles (say electrons) can be described by kinetic equations (Vlasov, Boltzmann) coupled to Poisson equation for the electrostatic forces [1, 6, 12, 30, 36, 42, and references therein]. For ultrasmall electron systems, like nanostructures, quantum effects such as tunneling become important [17, 52, 23]. The system is then well described by the Schrödinger-Poisson system [15, 32, 33, 37, 39, 44, 45, 46, 47] or by its phase-space counterpart, the Wigner-Poisson system [3, 4, 14, 25, 35, 38]. In various situations, like in resonant tunneling diodes [16, 24, 43], quantum effects occur in some parts of the electron ensemble while other parts exhibit purely classical behaviour. A sound description of such systems requires the use of the Schrödinger equation when necessary and kinetic (or fluid) equations otherwise. This leads to spatial coupling strategies between quantum and classical models [7, 8, 18].

In partially confined electron systems like two dimensional electron gases (2DEG), nanotubes or nanowires, the quantum-classical coupling has different features. Indeed, the width of a two-dimensional electron gas lying at a heterojunction (like

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Silicon-Oxide junctions in field effect transistors, or GaAs-AlGaAs junctions in modulation doped structures) is a few nanometers. As this length is comparable to the electron de Broglie length, the description of transport phenomena necessitates the use of the Schrödinger equation. In the direction(s) parallel to the heterojunction, the lengthscale is usually several times higher, and a classical description for electron transport is suitable. This leads to a coupling between classical and quantum models in momentum space. The so-called subband models [2, 5, 17, 23] which rely on the Born-Oppenheimer approximation allow such coupling. They have been recently derived by the authors in [10] thanks to a partial semi-classical limit of the Schrödinger equation (see [29, 51] and references therein for a related approach in molecular dynamics). The aim of this paper is to analyze the subband model coupled to the Poisson equation.

All along the paper, the confined direction is denoted by $z \in (0,1)$ (the study is restricted to one-dimensional confinement), while the non-confined direction is called $x \in \omega$, where ω is a bounded regular domain of $\mathbb{R}^2$. We shall use the symbols Δ and ∇ for the Laplace and Gradient operators in the (x,z) variables and shall use the subscript x when the differentiations are done with respect to the x variable only. We set $\Omega = \omega \times (0,1)$ and denote by $\nu(x)$ the outward unit normal vector at $x \in \partial\omega$. The domain Ω represents the spatial domain where transport and interaction will be studied. The incoming/outgoing sets are defined by

$$\Sigma_\pm = \{(x,v) \in \Sigma;\ \pm v \cdot \nu(x) > 0\}, \qquad \text{where } \Sigma = \partial\omega \times \mathbb{R}^2,$$

and equipped with the measure

$$d\Sigma(x,v) = |v \cdot \nu(x)|\, d\sigma(x)\, dv,$$

$d\sigma$ being the surface measure on $\partial\omega$. In dimensionless variables, the problem consists in finding, for $t \in (0,T)$, $x \in \omega$, $z \in (0,1)$, $v \in \mathbb{R}^2$ and $p \in \mathbb{N}^*$, the functions $V(t,x,z)$, $\epsilon_p(t,x)$, $\chi_p(t,x,z)$, $f_p(t,x,v)$ solving

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0, \\ \gamma^- f_p = g_p \quad \text{on } (0,T) \times \Sigma_-\,, \qquad f_p(0,x,v) = f_{p,0}(x,v), \end{cases} \tag{1.1}$$

where $(\epsilon_p(t,x),\ \chi_p(t,x))_{p\geq 1}$ is the complete set of (increasing) eigenvalues and eigenfunctions of the quasistatic one-dimensional Schrödinger equation

$$\begin{cases} -\dfrac{1}{2}\,\partial^2_{zz}\chi_p + (V + V_{ext})\chi_p = \epsilon_p\,\chi_p\,, \\ \chi_p(t,x,\cdot) \in H^1_0(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{1.2}$$

V_{ext} being a given external potential and V the selfconsistent electrostatic potential which satisfies the Poisson equation

$$\begin{cases} -\Delta V = \displaystyle\sum_{p\geq 1} \left(\int_{\mathbb{R}^2} f_p\, dv \right) |\chi_p|^2\,, \\ V = 0 \quad \text{on } \partial\omega \times (0,1), \qquad \partial_z V = 0 \quad \text{on } \omega \times \{0\} \,\cup\, \omega \times \{1\}. \end{cases} \tag{1.3}$$

In this description, the electron ensemble is, for each given (t,x), in a mixed quantum state

$$\varrho(t,x,z,z') = \sum_{p\geq 1} \rho_p(t,x)\,\chi_p(t,x,z)\,\chi_p(t,x,z'),$$

whose occupation numbers $\rho_p(t,x) = \int_{\mathbb{R}^2} f_p(t,x,v)\,dv$ are deduced from the classical evolution (1.1). The functions ϵ_p, χ_p are the energy and wave function of the p-th subband. For a given potential V, the problem (1.2) is an eigenvalue problem in the z variable, parametrized by t and x. The study is restricted to one-dimensional confinement in order to avoid energy crossing and to ensure the regularity, with respect to V, of the wave functions and energies of subbands.

1.2 Main Result

The data of the problem are the initial distribution functions $f_{p,0}$, the injection distribution functions at the boundary g_p and the external potential V_{ext}. We shall assume that, for any $T > 0$,

(H-1) the function V_{ext} lies in $C([0,T], C^2(\overline{\Omega})) \cap C^1([0,T], C(\overline{\Omega}))$ and $V_{ext} \geq 0$;

(H-2) $f_{p,0} \geq 0, \qquad ((1+v^2+p^2)f_{p,0}) \in \ell^1(L^1(\omega\times\mathbb{R}^2))$;

(H-3) $(f_{p,0}) \in \ell^1(L^\infty(\omega\times\mathbb{R}^2))$;

(H-4) $g_p \geq 0, \qquad ((1+v^2+p^2)g_p) \in \ell^1(L^1((0,T), L^1(\Sigma_-, d\Sigma)))$;

(H-5) $(g_p) \in \ell^1(L^\infty((0,T)\times\Sigma_-))$,

where, for any Banach space E, we have denoted by $\ell^1(E)$ the space of sequences $(h_p)_{p\in\mathbb{N}^*}$ such that for all $p \geq 1$ we have $h_p \in E$ and such that $\sum_{p\geq 1}\|h_p\| < +\infty$, the last quantity being the norm of $(h_p)_{p\in\mathbb{N}^*}$ in $\ell^1(E)$. When there is no ambiguity, we shall shortly denote by $\|f_p\|_{L^q_{t,x,v}}$ the $L^q((0,T)\times\omega\times\mathbb{R}^2)$ norm of f_p and analogously by $\|g_p\|_{L^q_{t,x,v}}$ the $L^q((0,T)\times\Sigma_-, dt\,d\Sigma)$ norm of g_p. Similarly the notation $L^q_{x,v}$ will be used for $L^q(\omega\times\mathbb{R}^2)$ as well as for $L^q(\Sigma_-, d\Sigma)$.

The main result of this paper is the following

Theorem 1.1 *Let $T > 0$ and assume that* **(H-1)**–**(H-5)** *hold. For any $M_0 > 0$, there exists a constant δ (depending only on M_0, on $\|V_{ext}(0)\|_{C^2}$ and on $\int_0^T \|[\partial_t V_{ext}]^+\|_{L^\infty}\,dt$) such that if*

$$\sum_{p\geq 1}\Big(\|(v^2+p^2)\,f_{p,0}\|_{L^1_{x,v}} + \|(v^2+p^2)\,g_p\|_{L^1_{t,x,v}} + \|f_{p,0}\|_{L^\infty_{x,v}} + \|g_p\|_{L^\infty_{t,x,v}}\Big) < M_0,$$

and

$$\sum_{p\geq 2}\Big(\|f_{p,0}\|_{L^1_{x,v}} + \|g_p\|_{L^1_{t,x,v}}\Big) < \delta,$$

then the system (1.1)-(1.2)-(1.3) *admits a weak solution* $(V, (\epsilon_p, \chi_p, f_p)_{p\in\mathbb{N}^*})$ *on* $[0,T]$ *in the following functional spaces*

$$V \in C([0,T], H^2(\Omega)), \qquad \epsilon_p \in C([0,T], H^2(\omega)),$$

$$\chi_p \in C([0,T], H^2(\Omega)), \qquad f_p \in L^\infty((0,T), L^1 \cap L^\infty(\omega \times \mathbb{R}^2)).$$

Remark 1.2 *The solution constructed here satisfies the charge density and energy inequalities* (4.4) *and* (4.5) *(where* ε *is omitted).*

The case $f_{p,0} = 0$, $g_p = 0$ for $p \geq 2$ is referred to in physical literature [23] as the *electrical quantum limit.* Theorem 1.1 provides a solution for this case without smallness assumption on $f_{1,0}$ and g_1. In the general case, the smallness assumption only concerns higher subbands. This is not due to a failure of the energy estimate (which holds without the smallness hypothesis). The reason is that –as described below in Subsection 1.3– our strategy of proof uses the resolution of quasistatic Schrödinger-Poisson subsystems (1.2)-(1.3) and we are not able to prove that the solution of this system is unique for large data on higher subbands, nor to select a solution which continuously depends on the data (for large data). This results in a lack of time compactness which is necessary for the construction of a solution for the Vlasov-Schrödinger-Poisson system. We also refer to the discussion of the Schrödinger-Poisson system in Remark 3.7, which indicates that the smallness assumption might not be only technical.

1.3 Strategy of the Proof

In order to construct a solution, we shall take advantage of the structure of the whole coupled system: an evolution system of Vlasov equations (1.1) coupled to a quasistatic Schrödinger-Poisson problem (1.2)–(1.3). Therefore, the main ingredients of the existence proof which relies on a fixed-point argument will be the following ones: some *a priori* estimates following from physical conservations, and the well-posedness of the nonlinear Schrödinger-Poisson problem.

The analysis of this quasistatic Schrödinger-Poisson problem, close to the one studied by Nier in [44], is the object of Section 3. In section 4, the coupled time-dependent problem is tackled with a fixed point procedure. The *a priori* estimate is obtained in Subsection 4.1. The properties of the Vlasov equation (1.1) are recalled in Subsection 4.2, while Subsection 4.3 is devoted to the analysis of a regularized Vlasov-Schrödinger-Poisson problem whose unregularized limit, performed in Subsection 4.4, finishes the proof of Theorem 1.1. Section 5 is then devoted to the analysis of the stationary situation, which can be solved independently of Theorem 1.1 with no smallness assumption and in Section 6 we briefly mention how volume and boundary collisions can be incorporated in this model.

As mentioned above, we shall introduce a regularization of the overall problem. This is known for Vlasov-Poisson systems and is done in order to insure sufficient regularity of the driving forces ($\nabla_x \epsilon_p$ in our case) for the Vlasov equation to have a unique weak solution. Let us now make precise this regularized problem. First define the linear regularization operator by

$$\begin{array}{rrcl} R^\varepsilon : & L^1(\Omega) & \to & C^\infty(\overline{\Omega}) \\ & V & \mapsto & R^\varepsilon[V](x,z) = \left(\overline{V} *_x \xi_{\varepsilon,x} *_z \xi_{\varepsilon,z}\right)\big|_{\overline{\Omega}} \end{array} \tag{1.4}$$

where $\overline{V}$ is the extension of V by zero outside Ω and $\xi_{\varepsilon,x}$ and $\xi_{\varepsilon,z}$ are C^∞ nonnegative compactly supported even approximations of the unity, respectively on $\mathbb{R}^2$ and $\mathbb{R}$. Then the regularized system is written

$$\left\{ \begin{array}{l} \partial_t f_p^\varepsilon + v \cdot \nabla_x f_p^\varepsilon - \nabla_x \epsilon_p^\varepsilon \cdot \nabla_v f_p^\varepsilon = 0, \\ \\ \gamma^- f_p^\varepsilon = g_p \quad \text{on } (0,T) \times \Sigma_- , \qquad f_p^\varepsilon(0,x,v) = f_{p,0}(x,v), \end{array} \right. \tag{1.5}$$

$$\left\{ \begin{array}{l} -\dfrac{1}{2}\,\partial^2_{zz}\chi_p^\varepsilon + (R^\varepsilon[V^\varepsilon] + V_{ext})\chi_p^\varepsilon = \epsilon_p^\varepsilon \chi_p^\varepsilon , \\ \\ \chi_p^\varepsilon(t,x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p^\varepsilon\, \chi_q^\varepsilon\, dz = \delta_{pq} , \end{array} \right. \tag{1.6}$$

$$\left\{ \begin{array}{l} -\Delta V^\varepsilon = R^\varepsilon \left[\displaystyle\sum_{p \geq 1} \left(\int_{\mathbb{R}^2} f_p^\varepsilon \, dv \right) \left|\chi_p^\varepsilon\right|^2 \right], \\ V^\varepsilon = 0 \quad \text{on } \partial\omega \times (0,1), \qquad \partial_z V^\varepsilon = 0 \quad \text{on } \omega \times \{0\} \,\cup\, \omega \times \{1\}. \end{array} \right. \tag{1.7}$$

Remark 1.3 *When $\varepsilon = 0$, we have $R^0 = Id$ and the regularized problem* (1.5)–(1.7) *reduces to the unregularized system* (1.1)–(1.3)*.*

2 Notation and auxiliary lemmas

Anisotropic Sobolev embeddings.

The spatial directions x and z do not play symmetric roles in this problem, and we shall employ throughout this paper some anisotropic functional spaces. In this section, we consider an arbitrary space dimension d for the x variable which belongs to $\omega \subset \mathbb{R}^d$ (although only $d = 2$ will be used in the sequel), and we set $\Omega = \omega \times (0,1)$.

Definition 2.1 *(i) For $1 \leq p, q \leq +\infty$, we define*

$$L^p_x L^q_z = \left\{ u \in L^1_{loc}(\Omega), \;\; \|u\|_{L^p_x L^q_z} = \left(\int_\omega \|u(x,\cdot)\|^p_{L^q(0,1)}\, dx \right)^{1/p} < +\infty \right\} \tag{2.1}$$

(with an obvious generalization of this definition for $p = +\infty$).
(ii) Define for $1 \leq r \leq +\infty$

$$W^{1,r}_\omega = \left\{ u \in W^{1,r}(\Omega), \quad u = 0 \text{ on } \partial\omega \times (0,1) \right\}; \quad H^1_\omega = W^{1,2}_\omega. \tag{2.2}$$

(iii) If $1 \le r < +\infty$*, we define* $W_\omega^{-1,r'}$ *as the dual of* $W_\omega^{1,r}$*, where* $r' = \frac{r}{r-1}$*.*

By using Gagliardo-Nirenberg inequalities and some interpolation and Sobolev embedding results, one can prove the following anisotropic Sobolev embedding (which gives the standard result when $p = q$) (the proof, which is rather straightforward, is skipped here; a similar result with H^s spaces can be found in [31]):

Lemma 2.2 *Let* $s > 1$*. The following Sobolev embeddings hold true:*
(i) $\underline{1 < s < d}$*. Let* $1 \le p \le +\infty$*,* $s \le q \le +\infty$ *be such that*

$$\frac{d}{p(d+1)} + \frac{1}{q(d+1)} \ge \frac{1}{s} - \frac{1}{d+1}. \tag{2.3}$$

Then $W^{1,s}(\Omega) \subset L_x^p L_z^q$*. If* (2.3) *holds strictly then the embedding is compact.*
(ii) $\underline{1 < s = d}$*. Let* $1 \le p < +\infty$ *and* $1 \le q \le +\infty$*. Then* $W^{1,s}(\Omega)$ *is compactly embedded in* $L_x^p L_z^q$ *if*

$$\frac{d}{p} + \frac{1}{q} > \frac{1}{d}. \tag{2.4}$$

If $p \le q$ *and if* (2.4) *is an equality then* $W^{1,s}(\Omega)$ *is continuously embedded in* $L_x^p L_z^q$*.*
(iii) $\underline{d < s < d+1}$*. If* $1 \le p, q \le +\infty$ *satisfy* (2.3) *then* $W^{1,s}(\Omega) \subset L_x^p L_z^q$*. The embedding is compact if* (2.3) *holds strictly.*
(iv) $\underline{d/2 < s}$*. Then* $W^{2,s}(\Omega) \subset C^0(\overline{\omega}, L^s(0,1))$ *and the embedding is compact.*

Spectral properties of Sturm-Liouville operators.

We now present some basic properties satisfied by the eigenvalues and eigenfunctions of the one-dimensional Schrödinger operator. Most of these properties are standard and given without proof; they can be found for instance in [34, 44, 49, 50]. Let U be a real-valued function in $L^2(0,1)$ and let $H[U]$ be the Dirichlet Schrödinger operator

$$H[U] = -\frac{1}{2}\frac{d^2}{dz^2} + U(z) \tag{2.5}$$

defined on the domain $D(H[U]) = H^2(0,1) \cap H_0^1(0,1)$. The operator $H[U]$ is a selfadjoint operator on $L^2(0,1)$, bounded from below and with compact resolvent. There exists a strictly increasing sequence $(\epsilon_p[U])_{p\ge1}$ of real numbers tending to $+\infty$ and an orthonormal basis of $L^2(0,1)$ $(\chi_p[U])_{p\ge1}$ such that

$$\begin{cases} -\dfrac{1}{2}\dfrac{d^2}{dz^2}\chi_p + U\chi_p = \epsilon_p\chi_p\,, \\ \chi_p \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p(z)\,\chi_q(z)\,dz = \delta_{pq}\,. \end{cases} \tag{2.6}$$

The eigenvalues $\epsilon_p[U]$ are simple, while the corresponding eigenfunctions are uniquely defined by the convention $\frac{d}{dz}\chi_p(0) > 0$. Besides, it is readily seen from (2.6) that

$$\epsilon_p = \frac{1}{2}\int_0^1 \left|\frac{d}{dz}\chi_p(z)\right|^2 dz + \int_0^1 U(z)\,|\chi_p(z)|^2\,dz, \tag{2.7}$$

and for $U = 0$, we have

$$\epsilon_p[0] = \frac{1}{2}\pi^2 p^2 \quad ; \quad \chi_p[0](z) = \sqrt{2}\sin(\pi p z).$$

An immediate consequence of the Min-Max formula [34] is

$$\text{if} \qquad U \geq V \quad \text{a.e. on } (0,1) \qquad \text{then} \qquad \forall p \in \mathbb{N}^* \quad \epsilon_p[U] \geq \epsilon_p[V]. \tag{2.8}$$

Another consequence is that if U and V are two real-valued functions such that $U - V \in L^\infty(0,1)$, the corresponding eigenvalues verify

$$|\epsilon_p[U] - \epsilon_p[V]| \leq \|U - V\|_{L^\infty(0,1)}. \tag{2.9}$$

For Section 3, it is interesting to consider the Schrödinger operator with $L^1(0,1)$ potentials. By adapting the proofs of [49] (Chapt. 2) for such potentials, one can prove the following lemmas:

Lemma 2.3 *Let $U \in L^1(0,1)$. Then the eigenvalue problem* (2.6) *admits a unique solution $(\epsilon_p, \chi_p)_{p\geq 1}$. The sequence $(\epsilon_p)_{p\geq 1}$ is strictly increasing to $+\infty$. The sequence $(\chi_p)_{p\geq 1}$ is an orthonormal basis of $L^2(0,1)$.*

Lemma 2.4 *Let $U, V \in L^1(0,1)$. Then there exists a positive constant $C_{U,V}$ such that*

$$|\epsilon_p[U] - \epsilon_p[V]| + \|\chi_p[U] - \chi_p[V]\|_{L^\infty(0,1)} \leq C_{U,V}\, \|U - V\|_{L^1}. \tag{2.10}$$

Moreover the constant $C_{U,V}$ can be chosen independent of p and such that

$$C_{U,V} \leq C_1 \exp\left(C_2 \|U\|_{L^1}\right) + C_1 \exp\left(C_2 \|V\|_{L^1}\right),$$

the constants C_1 and C_2 being independent of U and V.

Lemma 2.5 *For any $p \in \mathbb{N}^*$, the mappings*

$$\begin{aligned} \epsilon_p[\cdot] : &\qquad V \in L^1(0,1) \;\mapsto\; \epsilon_p[V] \in \mathbb{R}, \\ \chi_p[\cdot] : &\qquad V \in L^1(0,1) \;\mapsto\; \chi_p[V] \in L^\infty(0,1), \end{aligned}$$

are Gâteaux differentiable and their derivatives are respectively given by

$$d\epsilon_p[V] \cdot W = \int_0^1 W\, |\chi_p[V]|^2\, dz,$$

$$d\chi_p[V] \cdot W = \sum_{q \neq p} \frac{1}{\epsilon_p[V] - \epsilon_q[V]} \left(\int_0^1 W\, \chi_p\, \chi_q\, dz \right) \chi_q.$$

The regularization operator R^ε

We end this section by stating the following lemma satisfied by the regularization operator defined in Subsection 1.3. It can be obtained straightforwardly by using convolution results:

Lemma 2.6 *(i) The operator R^ε is a bounded operator on $L^p_xL^q_z$ for $1 \le p, q \le +\infty$ and satisfies*

$$\forall V \in L^p_xL^q_z, \quad \|R^\varepsilon[V]\|_{L^p_xL^q_z} \le \|V\|_{L^p_xL^q_z},$$

$$\textit{if} \quad 1 \le p, q < +\infty \quad \textit{and } V \in L^p_xL^q_z, \quad \textit{then} \quad \lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^p_xL^q_z} = 0.$$

(ii) Assume that $V \in C^0((\overline{\omega}), L^q(0,1))$ for some $q < +\infty$. Then

$$\lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^\infty_x L^q_z(\omega'\times(0,1))} = 0$$

for any open ω' such that $\overline{\omega'} \subset \omega$. If in addition $V(x, \cdot) = 0$ on $\partial\omega$, then

$$\lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^\infty_x L^q_z} = 0.$$

(iii) The operator R^ε is selfadjoint on $L^2(\Omega)$.
(iv) Let $r \ge 1$ be given and let $V \in W^{1,r}_\omega$. Then

$$\nabla_x R^\varepsilon[V] = R^\varepsilon[\nabla_x V] \quad ; \quad \lim_{\varepsilon\to 0} \|\nabla_x R^\varepsilon[V] - \nabla_x V\|_{L^r(\Omega)} = 0.$$

3 Analysis of the Schrödinger-Poisson system

This section is devoted to the study of the “quasistatic part” of (1.5)–(1.7): the regularized Schrödinger-Poisson system

$$\begin{cases} -\dfrac{1}{2}\,\partial^2_{zz}\chi^\varepsilon_p + (R^\varepsilon[V^\varepsilon] + V_{ext})\chi^\varepsilon_p = \epsilon^\varepsilon_p\chi^\varepsilon_p\,, \\ \chi^\varepsilon_p(t,x,\cdot) \in H^1_0(0,1), \qquad \displaystyle\int_0^1 \chi^\varepsilon_p\,\chi^\varepsilon_q\,dz = \delta_{pq}\,, \end{cases} \tag{3.1}$$

$$\begin{cases} -\Delta V^\varepsilon = R^\varepsilon\left[\displaystyle\sum_{p\ge 1} \rho_p\,\left|\chi^\varepsilon_p\right|^2\right], \\ V^\varepsilon = 0 \quad \text{on } \partial\omega\times(0,1), \qquad \partial_z V^\varepsilon = 0 \quad \text{on } \omega\times\{0\}\,\cup\,\omega\times\{1\}, \end{cases} \tag{3.2}$$

where the occupation numbers ρ_p are assumed to be given nonnegative functions, and where the regularization parameter ε lies in $[0,1]$. Recall that in the case $\varepsilon = 0$ we recover the unregularized Schrödinger-Poisson system (1.2)-(1.3) (where we set $\rho_p = \int f_p\,dv$). The time t only appears as a parameter and is skipped for notational simplicity: $\rho_p = \rho_p(x)$. We shall denote $(\rho) = (\rho_p)_{p\in\mathbb{N}^*}$ and define $\ell^1(L^q(\omega))^+$ by

$$\ell^1(L^q(\omega))^+ = \{(\rho) \in \ell^1(L^q(\omega)) : \quad \rho_p \ge 0 \quad \forall p \ge 1\}.$$

If $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\ge1})$ solves (3.1)-(3.2) then we define the kinetic and potential energies by

$$\mathcal{E}^\varepsilon_{kin,z} = \sum_{p\ge 1} \iint_\Omega \frac{1}{2}\,|\partial_z\chi^\varepsilon_p|^2\,\rho_p\,dx\,dz, \tag{3.3}$$

$$\mathcal{E}^\varepsilon_{pot} = \sum_{p\geq 1} \iint_\Omega V_{ext}\,\rho_p\,dx\,dz + \iint_\Omega \frac{1}{2}|\nabla V^\varepsilon|^2\,dx\,dz. \tag{3.4}$$

The main results of this section are the following two theorems:

Theorem 3.1 (Existence and estimates) *Let* $\varepsilon \in [0,1]$, $1 < q < +\infty$ *and* $(\rho) \in \ell^1(L^q(\omega))^+$.
(i) Assume that $V_{ext} \in L^{q'}_x L^\infty_z \cap L^\infty_x L^1_z$ *where* $q' = \frac{q}{q-1}$. *Then* (3.1)-(3.2) *admits a solution* $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ *such that* $V^\varepsilon \in W^{2,q}(\Omega)$.
(ii) Assume that $V_{ext} \in L^\infty_x L^1_z$ *and let* $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ *be a solution of* (3.1)-(3.2) *such that* $V^\varepsilon \in W^{1,1}_\omega$. *Then* $V^\varepsilon \in W^{2,q}(\Omega)$ *and*

$$\|V^\varepsilon\|_{W^{2,q}} \leq C_{\rho,V_{ext}}\,\|\rho\|_{\ell^1(L^q)},$$

where $C_{\rho,V_{ext}}$ *is a generic constant continously depending on* $\|\rho\|_{\ell^1(L^q)}$ *and* $\|V_{ext}\|_{L^\infty_x L^1_z}$. *If moreover we have* $(p^2\rho_p)_{p\geq 1} \in \ell^1(L^1(\omega))$ *then the kinetic and potential energies defined by* (3.3) *and* (3.4) *satisfy the estimates*

$$\mathcal{E}^\varepsilon_{kin,z} \leq C_{\rho,V_{ext}} \sum_{p\geq 1} p^2\,\|\rho_p\|_{L^1} \quad ; \quad \mathcal{E}^\varepsilon_{pot} \leq C_{\rho,V_{ext}}\,\|\rho\|_{\ell^1(L^1)}. \tag{3.5}$$

Theorem 3.2 (Uniqueness and continuity) *Let* $\varepsilon \in [0,1]$ *and* $1 < q < +\infty$. *Assume that* $V_{ext} \in L^{q'}_x L^\infty_z \cap L^\infty_x L^1_z$. *There exists a neighborhood* $\mathcal{U} \subset \ell^1(L^q(\omega))$ *of the set* $\{(\rho) \in \ell^1(L^q(\omega)) : \rho_p \equiv 0 \ \forall p \geq 2\}$ *such that the following properties hold:*
(i) For any $(\rho) \in \mathcal{U}^+ := \mathcal{U} \cap \ell^1(L^q(\omega))^+$ *the solution* $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ *of* (3.1)-(3.2) *is unique and satisfies*

$$\|V^\varepsilon\|_{W^{2,q}} \leq C_{\rho_1}\,\|\rho\|_{\ell^1(L^q)},$$

where C_{ρ_1} *depends only on* ρ_1. *Moreover, as* $\varepsilon \to 0$, *the solution* V^ε *of the regularized problem* (3.1)-(3.2) *converges to the solution* V *of the unregularized problem in the* $W^{2,q}(\Omega)$ *topology and uniformly with respect to* $(\rho) \in \ell^1(L^q)$.
(ii) Let $(\rho) \in \mathcal{U}^+$ *and* $(\tilde\rho) \in \mathcal{U}^+$. *Then the corresponding solutions satisfy*

$$\|V^\varepsilon - \tilde{V^\varepsilon}\|_{W^{2,q}} \leq C_{\rho_1,\tilde\rho_1}\,\|\rho - \tilde\rho\|_{\ell^1(L^q)}, \tag{3.6}$$

where $C_{\rho_1,\tilde\rho_1}$ *depends only on* ρ_1 *and* $\tilde\rho_1$.

Remark 3.3 *This theorem expresses the fact that the first occupation number* ρ_1 *being arbitrary large, if* ρ_p *for* $p \geq 2$ *is small enough then the solution of* (3.1)-(3.2) *is unique. We shall construct the set* $\mathcal{U} \subset \ell^1(L^q(\omega))$ *as*

$$\mathcal{U} = \{(\rho) \in \ell^1(L^q), \qquad \sum_{p\geq 2} \|\rho_p\|_{L^q} < \mathcal{N}(\|\rho_1\|_{L^q})\},$$

where $\mathcal{N}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *is a decreasing function, to be defined.*

The proofs of these theorems are developed in the three following subsections.

3.1 Step 1 : Construction of a Solution in $H^1(\Omega)$

In order to construct a solution in H^1_ω of the regularized Schrödinger-Poisson system, we proceed analogously to [44] and notice that this problem has a variational structure. Indeed, we shall see that $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ is a weak solution of (3.1)-(3.2) if and only if V^ε is a critical point in H^1_ω of the functional

$$J_{\rho,\varepsilon}(V) = J^0(V) + J^1_{\rho,\varepsilon}(V), \tag{3.7}$$

where

$$J^0(V) = \frac{1}{2}\iint_\Omega |\nabla V|^2\, dx\, dz$$

and

$$J^1_{\rho,\varepsilon}(V) = \sum_{p\geq 1}\int_\omega \left(\epsilon_p[V_{ext}(x,\cdot)] - \epsilon_p[R^\varepsilon[V](x,\cdot) + V_{ext}(x,\cdot)]\right)\rho_p(x)\, dx\,.$$

The function $\epsilon_p[U]$ is the one defined in Section 2. We shall prove that, under slightly different hypotheses from those of Theorem 3.1, the above functional has a minimizer and that this minimizer defines a solution of (3.1)-(3.2).

Lemma 3.4 *Let* $\varepsilon \in [0,1]$*. Assume that* $(\rho) \in \ell^1(L^q(\omega))$ *for some* $q > 4/3$ *and that* $V_{ext} \in L^{q'}_x L^\infty_z$*. Then the functional* $J_{\rho,\varepsilon}$ *defined in* (3.7) *is continuous, locally Lipschitz and weakly lower semicontinuous on* H^1_ω*. It is coercive : there exists a constant* C_q *depending on* q *(and not on* ε*) such that*

$$J_{\rho,\varepsilon}(V) \geq \frac{1}{2}\|\nabla V\|^2_{L^2} - C_q\|\rho\|_{\ell^1(L^q)}\|\nabla V\|_{L^2}.$$

Proof. We have $J_{\rho,\varepsilon} = J^0 + J^1_{\rho,\varepsilon}$. The first functional J^0 is clearly continuous and weakly lower semicontinuous on $H^1(\Omega)$, while the second one $J^1_{\rho,\varepsilon}$ satisfies

$$|J^1_{\rho,\varepsilon}(U) - J^1_{\rho,\varepsilon}(V)| \leq \sum_{p\geq 1}\|\epsilon_p[R^\varepsilon[U] + V_{ext}] - \epsilon_p[R^\varepsilon[V] + V_{ext}]\|_{L^{q'}}\|\rho_p\|_{L^q}$$

where $q' < 4$ is the conjugate coefficient of q. Lemma 2.6 and (2.9) imply

$$\begin{aligned} |J^1_{\rho,\varepsilon}(U) - J^1_{\rho,\varepsilon}(V)| &\leq \|\rho\|_{\ell^1(L^q)}\|R^\varepsilon[U-V]\|_{L^{q'}_x L^\infty_z} \\ &\leq \|\rho\|_{\ell^1(L^q)}\|U-V\|_{L^{q'}_x L^\infty_z}. \end{aligned}$$

Therefore $J^1_{\rho,\varepsilon}$ is Lipschitz on $L^{q'}_x L^\infty_z$ (with an ε-independent Lipschitz constant). By Lemma 2.2 *(ii)*, we have $H^1(\Omega) \hookrightarrow L^4_x L^\infty_z$ and $H^1(\Omega)$ is compactly embedded in $L^{q'}_x L^\infty_z$. Hence $J^1_{\rho,\varepsilon}$ is Lipschitz and weakly continuous on $H^1(\Omega)$. Finally, the coercivity inequality on $J_{\rho,\varepsilon}$ can be deduced from $J^1_{\rho,\varepsilon}(0) = 0$ and Poincaré's inequality which is satisfied on H^1_ω. □

We are now able to prove the

Proposition 3.5 *Let* $\varepsilon \in [0,1]$ *and* $(\rho_p(x))_{p\geq 1}$ *be a set of occupation factors in* $\ell^1(L^q(\omega))$ *for some* $q > 4/3$. *Assume that* $V_{ext} \in L^{q'}_x L^\infty_z$, *where* q' *is the conjugate of* q. *Then the system* (3.1)-(3.2) *admits a solution* $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ *such that* $V^\varepsilon \in H^1(\Omega)$.

Proof. Lemma 3.4 yields the existence of a minimizer V^ε to $J_{\rho,\varepsilon}$. Since $J_{\rho,\varepsilon}(0) = 0$, the coercivity inequality implies $\|\nabla V^\varepsilon\|_{L^2} \leq 2C_q\|\rho\|_{\ell^1(L^q)}$. Besides, from Lemma 2.5, it is clear that $J_{\rho,\varepsilon}$ is Gâteaux differentiable and that for any $W \in H^1(\Omega)$ we have

$$\begin{aligned} dJ_{\rho,\varepsilon}[V] \cdot W = &\iint_\Omega \nabla V \cdot \nabla W \, dx\, dz \\ &- \iint_\Omega \sum_{p\geq 1} \rho_p(x) \left|\chi_p[(R^\varepsilon[V] + V_{ext})(x,\cdot)]\right|^2 (z)\, R^\varepsilon[W](x,z)\, dx\, dz. \end{aligned}$$

From the selfadjointness of R^ε on $L^2(\Omega)$, one deduces that any minimizer $V^\varepsilon \in H^1_\omega$ of $J_{\rho,\varepsilon}$ satisfies (3.1)-(3.2). □

In the special case where the ρ_p are decreasing with respect to p, the solution of (3.1)-(3.2) can be shown to be unique. The following result is independent of Theorems 3.1 and 3.2 and is true even for large data ρ_p :

Proposition 3.6 *Let* $\varepsilon \in [0,1]$ *and* $(\rho_p(x))_{p\geq 1}$ *be a set of occupation factors in* $\ell^1(L^q(\omega))$ *for some* $q > 4/3$. *Assume that* $V_{ext} \in L^{q'}_x L^\infty_x$, *where* q' *is the conjugate of* q. *If* $\rho_{p+1}(x) \leq \rho_p(x)$ *for all* $(p,x) \in \mathbb{N}^* \times \omega$ *then the system* (3.1)-(3.2) *admits a unique solution such that* $V \in H^1_\omega$.

Proof. Only the uniqueness has to be proved. To this aim, we proceed analogously to [44] and prove that $J_{\rho,\varepsilon}$ is strictly convex. But since J^0 is itself strictly convex, it is enough to show that $J^1_{\rho,\varepsilon}$ is convex. By a density argument it is enough to show that for $V_{ext} \in L^\infty(\Omega)$ the functional $J^1_{\rho,\varepsilon}$ is convex on $L^\infty(\Omega)$. To this aim, we apply Lemma 2.5 to deduce that $J^1_{\rho,\varepsilon}$ is twice Gateaux differentiable on $L^\infty(\Omega)$ and satisfies

$$\begin{aligned} d^2J^1_{\rho,\varepsilon}[V] \cdot W \cdot W &= -2\sum_p \sum_{q\neq p} \int_\omega \frac{\rho_p}{\epsilon_p - \epsilon_q} \left(\int_0^1 \chi_p \chi_q \, R^\varepsilon[W] \, dz \right)^2 dx \\ &= \sum_p \sum_{q\neq p} \int_\omega \frac{\rho_p - \rho_q}{\epsilon_q - \epsilon_p} \left(\int_0^1 \chi_p \chi_q \, R^\varepsilon[W] \, dz \right)^2 dx \geq 0. \end{aligned}$$

□

Remark 3.7 *The result of Proposition* 3.6 *is related to the work of F. Nier [44, 45, 46] where the electron system is assumed at thermodynamical equilibrium: in* *this case, $\rho_p = F(\epsilon_p)$, where F is a given decreasing function (typically related to Boltzmann or Fermi-Dirac statistics). The classical counterpart of this Schrödinger-Poisson system is the semi-linear elliptic equation*

$$-\Delta V = F(V + V_{ext}) \qquad \textit{(with boundary conditions)} \tag{3.8}$$

which is known to be well-posed if F is monotone decreasing. This suggests that the solution of the Schrödinger-Poisson system (3.1)-(3.2) *with non decreasing (ρ_p) might exhibit similar difficulties as* (3.8) *when F is not decreasing. When we couple the Schrödinger-Poisson system* (3.1)-(3.2) *to the Vlasov equations* (1.1)*, the ρ_p's are computed through classical dynamics with different force fields $(\nabla_x \epsilon_p)$. Therefore, in the general case it is not possible to ensure that (ρ_p) stays (pointwise in (t, x)) decreasing with respect to p. Nevertheless, the electrical quantum limit (ρ_1 arbitrary and $\rho_p \equiv 0$ for $p \geq 2$) satisfies the decay property and the smallness assumption in* Theorem 1.1 *ensures that (ρ_p) is a small perturbation of this case.*

3.2 Step 2 : Proof of Theorem 3.1

$W^{2,q}$ estimate. Let $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon)_{p\geq 1})$ be a solution of (3.1)-(3.2) such that $V^\varepsilon \in W_\omega^{1,1}$. Since the ρ_p's are nonnegative, by the maximum principle V^ε is nonnegative. Therefore, the function

$$W^\varepsilon(x) = \|V^\varepsilon(x, \cdot)\|_{L^1_z(0,1)} = \int_0^1 V^\varepsilon(x, z)\, dz$$

satisfies the equation

$$\begin{aligned} -\Delta_x W^\varepsilon(x) &= \int_0^1 R^\varepsilon \left[\sum_{p\geq 1} \rho_p(x)\ |\chi_p(x, \cdot)|^2 \right] dz \\ &\leq \sum_{p\geq 1} \left[\overline{\rho_p}(x)\ \|\chi_p(x, \cdot)\|^2_{L^2_z(0,1)} \right] * \xi_{\varepsilon,x} = \sum_{p\geq 1} \overline{\rho_p} * \xi_{\varepsilon,x} \in L^q(\omega). \end{aligned}$$

Standard elliptic regularity results insure that $W^\varepsilon \in W^{2,q}(\omega)$ and that its norm depends only on $\|\rho\|_{\ell^1(L^q)}$. Besides, since $q > 1$, $W^{2,q}(\omega)$ is embedded in $L^\infty(\omega)$. Hence $V^\varepsilon \in L^\infty_x L^1_z$. Since $V_{ext} \in L^\infty_x L^1_z$, we deduce from Lemmas 2.4 and 2.6 that the χ_p^ε's are bounded in $L^\infty(\Omega)$ independently of p:

$$\|\chi_p^\varepsilon\|_{L^\infty} \leq C_{\rho,V_{ext}} = C_1 \exp\left(C_2\, \|\rho\|_{\ell^1(L^q)}\right),$$

where the constants are independent of p and ε. Therefore the right-hand side $R^\varepsilon[\sum_p \rho_p |\chi_p^\varepsilon|^2]$ of the Poisson equation (3.2) is in $L^q(\Omega)$ and its norm is bounded by $(C_{\rho,V_{ext}})^2\, \|\rho\|_{\ell^1(L^q)}$. Elliptic regularity results show that $V \in W^{2,q}(\Omega)$ and finishes the proof the first part of Theorem 3.1, Item *(ii)*.

Energy estimates. Let us now end the proof of Item *(ii)*. From the $L^\infty_x L^1_z$ bound of $R^\varepsilon[V^\varepsilon]+V_{ext}$ and (2.10) we deduce

$$\|\chi^\varepsilon_p\|_{L^\infty}\leq C_{\rho,V_{ext}}\quad ;\quad \|\epsilon^\varepsilon_p\|_{L^\infty}\leq p^2\,C_{\rho,V_{ext}}$$

($C_{\rho,V_{ext}}$ being a generic constant depending on ρ and V_{ext}). Consequently (2.7) implies the uniform estimate

$$\|\partial_z\chi^\varepsilon_p\|^2_{L^\infty_x L^2_z}\leq p^2\,C_{\rho,V_{ext}}.$$

This gives the estimate of $\mathcal{E}^\varepsilon_{kin,z}$, while the estimate of $\mathcal{E}^\varepsilon_{pot}$ is obtained by multiplying (3.2) by V^ε and integrating on Ω:

$$\begin{aligned}\iint|\nabla V^\varepsilon|^2\,dx\,dz\;&=\iint V^\varepsilon\,R^\varepsilon\left[\sum_{p\geq 1}\rho_p\,\left|\chi^\varepsilon_p\right|^2\right]\,dx\,dz\\ &\leq\|V^\varepsilon\|_{L^\infty_x L^1_z}\,\|\rho\|_{\ell^1(L^1)}\,\sup_{p\geq 1}\|\chi^\varepsilon_p\|^2_{L^\infty}\leq C_{\rho,V_{ext}}\,\|\rho\|_{\ell^1(L^1)}.\end{aligned}$$

Proof of Item *(i)*. For $q>4/3$, Proposition 3.5 insures the existence of a solution. The fact that V^ε belongs to $W^{2,q}(\Omega)$ is a consequence of Item *(ii)* of Theorem 3.1. In the case $1<q<4/3$, we set $\rho^n_p=\dfrac{\rho_p}{1+\frac{1}{n}p^2\,|\rho_p|}$ for $n>0$ and define V^n as the corresponding solution of the regularized Schrödinger-Poisson system constructed in Proposition 3.5. It is clear that $(\rho^n)\in\ell^1(L^\infty(\omega))$ and that

$$\|\rho^n\|_{\ell^1(L^q)}\leq\|\rho\|_{\ell^1(L^q)}.$$

Item *(ii)* implies that $\|V^n\|_{W^{2,q}}\leq C$, where C is independent of n. Hence by Lemma 2.2 we can extract a subsequence which converges as $n\to\infty$ in the $L^\infty_x L^q_z$ strong topology and in the $W^{2,q}(\Omega)$ weak topology. The inequality (2.10) enables to deduce that for any fixed p the sequences ϵ^n_p and χ^n_p also converge as $n\to\infty$ respectively in $L^\infty(\omega)$ and $L^\infty(\Omega)$ (uniformly with respect to p). Therefore we can pass to the limit in (3.1)-(3.2) and the limit V is a $W^{2,q}(\Omega)$ solution of this system. The proof of Theorem 3.1 is complete. □

3.3 Step 3 : Proof of Theorem 3.2

Let us start with Item *(ii)*. We will define $\mathcal{U}$ as in Remark 3.3. Consider $M>0$ and let (ρ) and $(\tilde\rho)$ in $\ell^1(L^q(\omega))$ such that $\|\rho_1\|_{L^q}\leq M$ and $\|\tilde\rho_1\|_{L^q}\leq M$. The aim is to find a constant $\mathcal{N}=\mathcal{N}(M)>0$, depending on M, such that (3.6) holds if we have

$$\sum_{p\geq 2}\|\rho_p\|_{L^q}<\mathcal{N}\quad ;\quad \sum_{p\geq 2}\|\tilde\rho_p\|_{L^q}<\mathcal{N}.$$

Let V^ε and $\tilde V^\varepsilon$ be two solutions corresponding to (ρ) and $(\tilde\rho)$. We have

$$-\Delta(V^\varepsilon-\tilde V^\varepsilon)=R^\varepsilon\left[\sum_{p\geq 1}(\rho_p-\tilde\rho_p)\,|\tilde\chi^\varepsilon_p|^2\right]+R^\varepsilon\left[\sum_{p\geq 1}\rho_p\left(|\chi^\varepsilon_p|^2-|\tilde\chi^\varepsilon_p|^2\right)\right].\tag{3.9}$$

By Item *(ii)* of Theorem 3.1 and the embedding $W^{2,q}(\Omega) \hookrightarrow C^0(\overline{\omega}, L^q(0,1))$ we have

$$\|V^\varepsilon\|_{L^\infty_x L^q_z} + \|\tilde{V}^\varepsilon\|_{L^\infty_x L^q_z} \leq C_{\rho,\tilde{\rho}}$$

where $C_{\rho,\tilde{\rho}}$ denotes a generic constant depending on $\|\rho\|_{\ell^1(L^q)}$, $\|\tilde{\rho}\|_{\ell^1(L^q)}$ (and also on $\|V_{ext}\|_{L^\infty_x L^1_z}$), uniform in ε. From (2.10) and Lemmas 2.2, 2.6 we deduce that

$$\|\chi_p^\varepsilon\|_{L^\infty} + \|\tilde{\chi}_p^\varepsilon\|_{L^\infty} \leq C_{\rho,\tilde{\rho}}$$

$$\forall r < +\infty \qquad \|\chi_p^\varepsilon - \tilde{\chi}_p^\varepsilon\|_{L^r_x L^\infty_z} \leq C_{\rho,\tilde{\rho}}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1}\,,$$

thus for any $s < q$ we have

$$\begin{aligned}\left\| R^\varepsilon \left[\sum_{p\geq 2} \rho_p \left(|\chi_p^\varepsilon|^2 - |\tilde{\chi}_p^\varepsilon|^2 \right) \right] \right\|_{L^s_x L^\infty_z} &\leq C_{\rho,\tilde{\rho}}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1} \sum_{p\geq 2} \|\rho_p\|_{L^q} \\ &\leq C_{\rho,\tilde{\rho}}\, \mathcal{N}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1}\,.\end{aligned}$$

Next, multiplying (3.9) by $V^\varepsilon - \tilde{V}^\varepsilon$ and integrating on Ω, we obtain

$$\begin{aligned}\iint_\Omega \left|\nabla(V^\varepsilon - \tilde{V}^\varepsilon)\right|^2 dx\,dz - \iint_\Omega R^\varepsilon \left[\rho_1 \left(|\chi_1^\varepsilon|^2 - |\tilde{\chi}_1^\varepsilon|^2 \right)\right] (V^\varepsilon - \tilde{V}^\varepsilon)\, dx\,dz \\ \leq C_{\rho,\tilde{\rho}}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1}\, \|\rho - \tilde{\rho}\|_{\ell^1(L^q)} + C_{\rho,\tilde{\rho}}\, \mathcal{N}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|^2_{H^1}\,,\end{aligned} \tag{3.10}$$

where we used the embedding $H^1(\Omega) \hookrightarrow L^{s'}_x L^1_z$. We will now prove that the second term in the left-hand side of (3.10) is nonnegative, *i.e.* (thanks to the selfadjointness of R^ε) that

$$-\iint_\Omega \rho_1 \left(|\chi_1^\varepsilon|^2 - |\tilde{\chi}_1^\varepsilon|^2 \right)\, R^\varepsilon \left[V^\varepsilon - \tilde{V}^\varepsilon \right] dx\,dz \geq 0. \tag{3.11}$$

For any U and $\tilde{U}$ in $L^1(0,1)$, consider the application from $[0,1]$ to $\mathbb{R}$ defined by

$$\lambda \mapsto \epsilon(\lambda) = \epsilon_1[U + \lambda(\tilde{U} - U)].$$

We also denote $\chi(\lambda) = \chi_1[U + \lambda(\tilde{U} - U)]$. By Lemma 2.5, the application ϵ is differentiable and

$$\frac{d\epsilon}{d\lambda}(\lambda) = \int_0^1 |\chi(\lambda)|^2\, (\tilde{U} - U)\, dz.$$

Furthermore, thanks to the min-max formula which defines the first eigenvalue by

$$\epsilon_1[U] = \min_{\substack{\phi \in H^1_0(0,1) \\ \|\phi\|_{L^2} = 1}} \left(\int_0^1 \frac{1}{2} \left| \frac{d\phi}{dz}(z) \right|^2 dz + \int_0^1 U(z)\, |\phi(z)|^2 dz \right),$$

it is clear that the mapping $U \mapsto \epsilon_1[U]$ is concave, thus ϵ is also concave. Consequently

$$\frac{d\epsilon}{d\lambda}(0) \geq \frac{d\epsilon}{d\lambda}(1),$$

which means that

$$\int_0^1 (|\chi_1|^2 - |\tilde{\chi_1}|^2)(U - \tilde{U})\, dz \leq 0.$$

Applying this inequality to $U = R^\varepsilon[V^\varepsilon] + V_{ext}$ and $\tilde{U} = R^\varepsilon[\tilde{V}^\varepsilon] + V_{ext}$ leads to (3.11). Therefore Poincaré's inequality together with (3.10) give

$$\|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1} \leq C_{\rho,\tilde{\rho}}\, \|\rho - \tilde{\rho}\|_{\ell^1(L^q)} + C_{\rho,\tilde{\rho}}\, \mathcal{N}\, \|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1}.$$

Consequently, if $\mathcal{N}$ is small enough, we can deduce the estimate

$$\|V^\varepsilon - \tilde{V}^\varepsilon\|_{H^1} \leq C_{\rho,\tilde{\rho}}\, \|\rho - \tilde{\rho}\|_{\ell^1(L^q)}.$$

The $W^{2,q}$ estimate of Item *(ii)* is then obtained from (3.9) after a bootstrap argument.

The first part of Item *(i)* (the uniqueness property) follows from *(ii)*. To prove the second part, consider $(\rho) \in \mathcal{U}$ and let V^ε (resp. V) be the solution of the regularized (resp. unregularized) Schrödinger-Poisson problem. We have

$$-\Delta(V^\varepsilon - V) = (R^\varepsilon - Id)\left[\sum_{p\geq 1} \rho_p\, |\chi_p|^2\right] + R^\varepsilon \left[\sum_{p\geq 1} \rho_p\, \left(|\chi_p^\varepsilon|^2 - |\chi_p|^2\right)\right]. \tag{3.12}$$

Since $\left\|\sum_{p\geq 1} \rho_p\, |\chi_p|^2\right\|_{L^q_x L^2_z} \leq C$, Lemma 2.6 implies the convergence to zero (as $\varepsilon \to 0$, and in $L^q_x L^2_z$) of the first term of the right-hand side of (3.12). In order to show the convergence $V^\varepsilon \to V$ in $W^{2,q}(\Omega)$, we first prove this convergence in $H^1(\Omega)$, then a bootstrap argument can be used (left to the reader). Multiply (3.12) by $V^\varepsilon - V$ and integrate on Ω. By using (3.11) (with $\tilde{V}^\varepsilon$ replaced by V), Poincar's inequality and Lemma 2.6, we get (with any $s \in (1, q)$)

$$\|V^\varepsilon - V\|_{H^1} \leq C \left\|(R^\varepsilon - Id)\left[\sum_{p\geq 1} \rho_p\, |\chi_p|^2\right]\right\|_{L^q_x L^2_z} + C \left\|\sum_{p\geq 2} \rho_p\, \left(|\chi_p^\varepsilon|^2 - |\chi_p|^2\right)\right\|_{L^s_x L^2_z}.$$

(we also used the fact that $H^1(\Omega) \hookrightarrow L^r_x L^2_z$ for any $r < +\infty$). Moreover, the uniform L^∞ bound of χ^ε and Lemma 2.4 imply

$$\begin{aligned}
\left\|\sum_{p\geq 2} \rho_p\, \left(|\chi_p^\varepsilon|^2 - |\chi_p|^2\right)\right\|_{L^s_x L^2_z} &\leq C\mathcal{N}\, \|R^\varepsilon[V^\varepsilon] - V\|_{L^q_x L^2_z} \\
&\leq C\mathcal{N}\, \left(\|R^\varepsilon[V^\varepsilon] - R^\varepsilon[V]\|_{L^q_x L^2_z} + \|R^\varepsilon[V] - V\|_{L^q_x L^2_z}\right) \\
&\leq C\mathcal{N}\, \left(\|V^\varepsilon - V\|_{H^1} + \|(R^\varepsilon - Id)[V]\|_{L^q_x L^2_z}\right).
\end{aligned}$$

Using the convergence of R^ε to Id and choosing $\mathcal{N}$ small enough, we obtain the convergence of V^ε towards V in $H^1(\Omega)$ strong. □

4 Analysis of the Vlasov-Schrödinger-Poisson system

The aim of this section is the proof of the main Theorem 1.1. As we said in the Introduction, the strategy relies on a fixed point argument. Remark that the Poisson equation provides compactness with respect to position variables but not with respect to time. Therefore, we shall use the averaging lemmas of the Vlasov equation presented in Subsection 4.2, coupled to the results of Section 3 (Theorems 3.1 and 3.2), which requires the smallness of the distribution functions for subbands $p \geq 2$.

4.1 The Energy Estimate

We present here some *a priori* estimates which are satisfied by the solutions of the regularized Vlasov-Schrödinger-Poisson system (1.5)–(1.7). Let us first introduce a few notations. For any solution of this system, the $p-$th subband surface charge density and surface current density are defined by

$$\rho_p^\varepsilon = \int_{\mathbb{R}^2} f_p^\varepsilon \, dv \quad ; \quad j_p^\varepsilon = \int_{\mathbb{R}^2} v \, f_p^\varepsilon \, dv \tag{4.1}$$

while the total (volume) charge density is

$$n^\varepsilon = \sum_{p \geq 1} \left(\int_{\mathbb{R}^2} f_p^\varepsilon \, dv \right) |\chi_p^\varepsilon|^2 = \sum_{p \geq 1} \rho_p^\varepsilon \, |\chi_p^\varepsilon|^2. \tag{4.2}$$

The kinetic energy and the potential energy of the system are defined by

$$\mathcal{E}_{kin}^\varepsilon(t) = \sum_{p \geq 1} \iint \frac{v^2}{2} f_p^\varepsilon \, dx \, dv + \sum_{p \geq 1} \iiint \frac{1}{2} \left| \partial_z \chi_p^\varepsilon \right|^2 f_p^\varepsilon \, dx \, dz \, dv$$

$$\mathcal{E}_{pot}^\varepsilon(t) = \iint n^\varepsilon V_{ext} \, dx \, dz + \iint \frac{1}{2} \left| \nabla V^\varepsilon \right|^2 dx \, dz$$

and the total energy is

$$\mathcal{E}_{tot}^\varepsilon(t) = \mathcal{E}_{kin}^\varepsilon(t) + \mathcal{E}_{pot}^\varepsilon(t). \tag{4.3}$$

Proposition 4.1 (Energy estimate)
Under Assumptions (**H-1**)–(**H-5**)*, let* $\varepsilon \in (0, 1]$ *and* $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon, f^\varepsilon)_{p \geq 1})$ *be a weak solution of* (1.5)–(1.7)*. Then the charge density and energy respectively defined by* (4.2) *and* (4.3) *satisfy the inequalities*

$$\|n^\varepsilon(t)\|_{L^1} \leq \|n^\varepsilon(0)\|_{L^1} + \sum_{p \geq 1} \int_0^t \iint_{\Sigma_-} g_p \, d\Sigma \, ds, \tag{4.4}$$

$$\begin{aligned} \mathcal{E}_{tot}^\varepsilon(t) \leq \mathcal{E}_{tot}^\varepsilon(0) &+ \int_0^t \|n^\varepsilon(s)\|_{L^1} \, \|[\partial_t V_{ext}(s)]^+\|_{L^\infty} \, ds \\ &+ \sum_{p \geq 1} \int_0^t \iint_{\Sigma_-} \left(\frac{v^2}{2} + \epsilon_p^\varepsilon \right) g_p \, d\Sigma \, ds. \end{aligned} \tag{4.5}$$

Proof. For notational simplicity, we skip the exponent ε in this proof. The integration of (1.5) with respect to v yields the charge conservation equation

$$\partial_t \rho_p + \mathrm{div}_x\, j_p = 0. \tag{4.6}$$

Inequality (4.4) is obtained by integrating this equation with respect to x and t and summing up with respect to p. Besides, multiplying (1.5) by $\frac{v^2}{2}$ and integrating with respect to x and v leads, after some algebra, to

$$\begin{aligned} \frac{d}{dt} \iint \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, dx\, dv - \iint f_p\, \partial_t \epsilon_p\, dx\, dv & \\ = - \iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, v \cdot \nu\, d\sigma\, dv. & \end{aligned} \tag{4.7}$$

Identity (2.7) gives

$$\sum_{p\geq 1} \iint \epsilon_p\, f_p\, dx\, dv = \sum_{p\geq 1} \iiint \frac{1}{2} \left|\partial_z \chi_p\right|^2 f_p\, dx\, dz\, dv + \iint (R^\varepsilon[V] + V_{ext})\, n\, dx\, dz$$

and Lemma 2.5 implies $\partial_t \epsilon_p = \int_0^1 |\chi_p|^2\, \partial_t(R^\varepsilon[V] + V_{ext})\, dz$. Therefore, we have

$$-\sum_{p\geq 1} \iint f_p\, \partial_t \epsilon_p\, dx\, dv = -\iint n\, \partial_t (R^\varepsilon[V] + V_{ext})\, dx\, dz.$$

Besides, it is readily seen from the regularized Poisson equation (1.7) and from Lemma 2.6 that

$$\iint R^\varepsilon[V]\, n\, dx\, dz = \iint |\nabla V|^2\, dx\, dz \ \ ; \ \iint n\, \partial_t R^\varepsilon[V]\, dx\, dz = \frac{1}{2}\frac{d}{dt} \iint |\nabla V|^2\, dx\, dz.$$

Consequently, after a summation on p, (4.7) becomes

$$\frac{d\mathcal{E}_{tot}}{dt} = \iint n\, \partial_t V_{ext}\, dx\, dz - \sum_{p\geq 1} \iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, v\cdot \nu\, d\sigma\, dv.$$

Finally, (4.5) is obtained from the above inequality after having noticed that ϵ_p is positive in virtue of the positivity of the potential.

The above formal proof is fully justified if the f_p^ε's are classical solutions of the Vlasov equation and compactly supported in the v variable. However, the final result is still valid for weak solutions under the hypotheses (**H-1**)–(**H-5**). Indeed, thanks to the uniqueness of the solution of the linear Vlasov equation (see Lemma 4.2 below), the function f_p^ε can be approximated by truncating the initial and boundary data (keeping the ϵ_p^ε unchanged). The corresponding solution is compactly supported in v. With an adequate choice of test-function, we obtain (4.7) in the distribution sense (see for instance [6]). The energy inequality is then obtained for the cut-off problem (up to remainders which vanish at the limit); it is conserved in the limit. □

4.2 The Linear Vlasov Equation

Let us now give some results about the collection of Vlasov equations (1.1). We shall assume that the force fields $F_p = -\nabla_x \epsilon_p$ are known and solve the equations indexed by p

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p + F_p \cdot \nabla_v f_p = 0, \\ \gamma^- f_p = g_p \quad \text{on } (0,T) \times \Sigma_- , \qquad f_p(0,x,v) = f_{p,0}(x,v). \end{cases} \tag{4.8}$$

The following lemma states the existence and uniqueness of the weak solution for each Vlasov equation (4.8):

Lemma 4.2 *Let $T > 0$ and assume that the initial and boundary data $f_{p,0}$ and g_p satisfy*

$$f_{p,0} \geq 0, \quad (1+v^2) f_{p,0} \in L^1(\omega \times \mathbb{R}^2), \quad f_{p,0} \in L^\infty(\omega \times \mathbb{R}^2),$$
$$g_p \geq 0, \quad (1+v^2) g_p \in L^1((0,T), L^1(\Sigma_-, d\Sigma)), \quad g_p \in L^\infty((0,T) \times \Sigma_-).$$

Assume that $F_p \in L^1((0,T), W^{1,1}(\omega) \cap L^\infty(\omega))$. Then (4.8) admits a unique weak solution $f_p \in L^\infty((0,T), L^1 \cap L^\infty(\omega \times \mathbb{R}^2))$. Moreover, this solution lies in $C([0,T], L^q(\omega \times \mathbb{R}^2))$ for any $q < +\infty$ and satisfies the following estimates:

$$0 \leq f_p(t,x,v) \leq \max(\|g_p\|_{L^\infty}, \|f_{p,0}\|_{L^\infty}) \quad a.e., \tag{4.9}$$

$$\|f_p(t,\cdot,\cdot)\|_{L^1(\omega\times\mathbb{R}^2)} \leq \|f_{p,0}\|_{L^1(\omega\times\mathbb{R}^2)} + \int_0^t \|g_p\|_{L^1(\Sigma_-,d\Sigma)}\, ds. \tag{4.10}$$

If we additionally assume that $((1+v^2)f_{p,0}) \in \ell^1(L^1_{x,v})$, $(f_{p,0}) \in \ell^1(L^\infty_{x,v})$, $((1+v^2)g_p) \in \ell^1(L^1_{t,x,v})$, $(g_p) \in \ell^1(L^\infty_{t,x,v})$ and F_p is bounded in $L^1((0,T), L^\infty_x)$ independently of p, then the kinetic energy

$$\mathcal{E}_{kin}(t) := \sum_{p\geq 1} \iint_{\omega\times\mathbb{R}^2} \frac{v^2}{2} f_p(t,x,v)\, dx\, dv$$

satisfies the inequality

$$\mathcal{E}_{kin}(t) \leq \mathcal{E}_{kin}(0) + \sum_{p\geq 1} \int_0^t \iint_{\Sigma_-} \frac{v^2}{2} g_p(s,x,v)\, d\Sigma\, ds + \Phi(t) + \Phi(t)^4, \tag{4.11}$$

where

$$\Phi(t) = C \left(\sum_{p\geq 1} \|f_p\|_{L^\infty} \right)^{1/4} \int_0^t \sup_{q\geq 1} \|F_q(s,\cdot)\|_{L^\infty}\, ds$$

and C is a constant of the data.

Proof. Since the results of this lemma are standard [1, 6, 13, 19, 30, 41], we shall only detail the proof of (4.11) which is more specific. Like in the proof of Proposition 4.1, we shall give a formal proof which would be valid for classical solutions. The final results is however valid for weak solutions (see the end of the proof of 4.1). Multiplying (4.8) by $\frac{v^2}{2}$ then integrating on $\omega \times \mathbb{R}^2$ and summing over p, one obtains

$$\frac{d\mathcal{E}_{kin}}{dt} = -\sum_{p\geq 1} \iint_{\Sigma} \frac{v^2}{2} f_p(t,x,v)\, v\cdot \nu \, d\sigma dv + \int_{\omega} F_p(t,x)\cdot j_p(t,x)\, dx, \tag{4.12}$$

where $j_p = \int v f_p \, dv$. Like in the proof of Proposition 4.1, we have

$$-\iint_{\Sigma} \frac{v^2}{2} f_p \, v\cdot\nu \, d\sigma dv \leq \iint_{\Sigma_-} \frac{v^2}{2} g_p \, v\cdot \nu d\Sigma dv.$$

Besides, classical interpolation results for kinetic equations [13] lead to

$$|j_p(t,x)| \leq C \, \|f_p\|_{L^\infty}^{1/4} \left(\int \frac{v^2}{2} \, f_p(t,x,v)\, dv \right)^{3/4},$$

and Hölder inequality leads to

$$\int_{\omega} F_p(t,x)\cdot j_p(t,x)\, dx \leq C \, \|F_p(t,\cdot)\|_{L^\infty} \, \|f_p\|_{L^\infty}^{1/4} \left(\iint \frac{v^2}{2} \, f_p(t,x,v)\, dv\, dx \right)^{3/4}$$

(the domain ω is bounded). Inserting this inequality in (4.12), one obtains

$$\frac{d\mathcal{E}_{kin}}{dt}(t) \leq C \, \sup_{p\geq 1} \|F_p(t,\cdot)\|_{L^\infty} \left(\sum_{p\geq 1} \|f_p\|_{L^\infty} \right)^{1/4} (\mathcal{E}_{kin}(t))^{3/4} + \sum_{p\geq 1} \iint_{\Sigma_-} \frac{v^2}{2} \, g_p(t,x,v)\, d\Sigma.$$

This leads to the result after a Gronwall argument. □

The following lemma given without proof generalizes the standard interpolation inequalities [13] for the Vlasov equation.

Lemma 4.3 *Assume that* $(f_p)_{p\geq 1}$ *lies in* $\ell^1(L^\infty(\omega\times\mathbb{R}^2))$ *and* $(v^2 f_p)_{p\geq 1} \in \ell^1(L^1(\omega \times \mathbb{R}^2))$. *Then* $\rho_p = \int f_p \, dv$ *belongs to* $L^2(\omega)$ *and satisfies*

$$\sum_{p\geq 1} \|\rho_p\|_{L^2} \leq C \left(\sum_{p\geq 1} \|f_p\|_{L^\infty} \right)^{1/2} \left(\sum_{p\geq 1} \|v^2 f_p\|_{L^1} \right)^{1/2}. \tag{4.13}$$

It is well-known that the average quantities of f with respect to the velocity satisfy compactness properties. The following Lemma makes them precise in the case of series (f_p):

Lemma 4.4 *Let* $F_p^n \in L^1((0,T)\times\omega)$ *for any* p *and* n*, such that for any fixed* p $\|F_p^n\|_{L^1}$ *is bounded independently of* n*. Let* (f_p^n) *be a sequence of weak solutions of* (4.8) *such that for all* $n \in \mathbb{N}$

$$\sum_{p\geq 1}\left(\|f_p^n\|_{L^\infty_{t,x,v}} + \|(v^2+p^2)f_p^n\|_{L^1_{t,x,v}}\right) \leq C, \tag{4.14}$$

with a uniform bound with respect to n*. Then the sequence of charge densities* $(\rho_p^n)_{p\geq 1} = (\int f_p^n\,dv)_{p\geq 1}$ *is compact (with respect to* n*) in the* $\ell^1(L^q((0,T)\times\omega))$ *topology for any* $q<2$*.*

Proof. Since Lemma 4.3 yields the boundedness of (ρ^n) in $\ell^1(L^2((0,T)\times\omega))$, it is enough to prove the compactness of (ρ^n) in $\ell^1(L^1((0,T)\times\omega))$.

The first step of this proof is the standard mean compactness result from [26, 27, 20] (more precisely, we refer to Theorem 1.8 of the book [13]): for any function $\psi \in C_c^\infty(\mathbb{R}^2)$ and for any fixed p, the sequence indexed by n

$$\int_{\mathbb{R}^2} f_p^n(t,x,v)\psi(v)\,dv$$

is compact in $L^1_{loc}((0,T)\times\omega)$.

Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that $0\leq\psi\leq 1$ and $\psi(v)=1$ if $|v|\leq 1$. For any $R>0$ we denote by $\Lambda_R \subset (0,T)\times\omega$ the set of all (t,x) whose distance to the boundary of $(0,T)\times\omega$ is larger than $1/R$. Then for any $R>0$, the charge density can be written

$$\rho_p^n = \rho_p^{n,R} + r_p^{n,R}$$

with

$$\rho_p^{n,R} = \mathbb{1}_{p\leq R}\,\mathbb{1}_{(t,x)\in\Lambda_R}\int_{\mathbb{R}^2} f_p^n(t,x,v)\psi(v/R)\,dv.$$

After a diagonal extraction procedure, for any $R\in\mathbb{N}^*$ (a subsequence of) the sequence $(\rho^{n,R})$ converges in $\ell^1(L^1((0,T)\times\omega))$ as $n\to+\infty$. Inequalities (4.14) and (4.13) enable to show that $\sum_{p\geq 1}\|r_p^{n,R}\|_{\ell^1(L^1((0,T)\times\omega))}$ is arbitrarily small when R is large, uniformly in n. This is enough to conclude the proof. □

We end this section by a stability result whose proof is skipped (its second part is a consequence of Lemma 4.4).

Lemma 4.5 *Let* $F_p^n \in L^1((0,T)\times\omega)$ *be a collection of force fields and let* f_p^n *be a corresponding sequence of weak solutions of* (4.8)*. (i) If for all* $p\geq 1$

$$F_p^n \overset{n\to\infty}{\longrightarrow} F_p \quad in \quad L^1_{t,x} \quad , \quad f_p^n \overset{n\to\infty}{\rightharpoonup} f_p \quad in\ L^\infty_{t,x,v}\ weak-\star \tag{4.15}$$

then the limit f_p *is a weak solution of* (4.8) *with the force field* F_p*.*
(ii) If additionally we have

$$\sum_{p\geq 1}\left(\|f_p^n\|_{L^\infty_{t,x,v}} + \|(v^2+p^2)f_p^n\|_{L^1_{t,x,v}}\right) \leq C, \tag{4.16}$$

where C is a constant independent of n, then for any $q < 2$

$$(\rho^n) = \left(\int f_p^n \, dv\right)_{p\geq 1} \overset{n\to\infty}{\longrightarrow} (\rho) = \left(\int f_p \, dv\right)_{p\geq 1} \quad \text{in } \ell^1(L^q((0,T)\times\omega)).$$

4.3 Weak Solutions for the regularized VSP Problem

In this subsection, we shall construct weak solutions of the regularized problem. To this aim we shall use the following notations

$$\begin{aligned} L_1(t) &= \sum_{p\geq 1} \|(v^2+p^2)\, f_p(t)\|_{L^1_{x,v}} + \sum_{p\geq 1} \|f_p(t)\|_{L^\infty_{x,v}} \\ &\quad + \int_t^T \sum_{p\geq 1} \|(v^2+p^2)\, g_p(s)\|_{L^1_{x,v}} ds + \sum_{p\geq 1} \sup_{s\in(t,T)} \|g_p(s)\|_{L^\infty_{x,v}} \qquad (4.17) \\ L_2(t) &= \sum_{p\geq 2} \|f_p(t)\|_{L^1_{x,v}} + \sum_{p\geq 2} \int_t^T \|g_p(s)\|_{L^1_{x,v}} ds \qquad (4.18) \end{aligned}$$

Let us first start with the following two technical lemmas

Lemma 4.6 *Let $T > 0$ be given and let $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon, f_p^\varepsilon)_{p\geq 1})$ be a weak solution of the regularized system* (1.5)–(1.7) *on the interval* $[0,T]$. *We have the following properties:*
(i) The function L_2 defined in (4.18) *is nondecreasing.*
(ii) For any $M_0 > 0$, there exists M_1 and ε_{M_0} depending on M_0, T and $\int_0^T \|[\partial_t V_{ext}]^+\|_{L^\infty} dt$ such that

$$L_1(0) < M_0 \quad \Longrightarrow \quad \forall t \in [0,T], \ \forall \varepsilon \in (0, \varepsilon_{M_0}), \ \ \mathcal{E}^\varepsilon_{tot}(t) < M_1,$$

where $\mathcal{E}^\varepsilon_{tot}$ is given by (4.3).
(iii) For any $M_0 > 0$, there exists $M > M_0$ and ε_{M_0} depending on M_0, T and $\int_0^T \|[\partial_t V_{ext}]^+\|_{L^\infty} dt$ such that

$$L_1(0) < M_0 \quad \Longrightarrow \quad \forall t \in [0,T], \ \forall \varepsilon \in (0, \varepsilon_{M_0}), \ \ L_1(t) < M.$$

Proof. Item *(i)* is immediate in view of (4.4) while *(iii)* follows easily from *(i)* and *(ii)*. In order to prove *(ii)*, we proceed in two steps. We first give a formal proof in the case $\varepsilon = 0$, although the result is not true in this case, and then treat the case $\varepsilon > 0$ which involves additional technicalities.
The unregularized system (1.1)–(1.3).
For simplicity, we start this proof with the case $\varepsilon = 0$ (the results of Proposition 4.1 will be used formally). By Lemma 4.3, we have $(\rho_p(0,\cdot)) \in \ell^1(L^2(\omega))$, thus an application of Theorem 3.1 (with $q = 2$) leads to

$$\mathcal{E}_{tot}(0) \leq C_{M_0}.$$

For $t > 0$ and $(x,z) \in \partial\omega \times (0,1)$ we have $V(t,x,z) = 0$. Thus, with the notations of Section 2, we have $\epsilon_p(t,x) = \epsilon_p[V_{ext}(t,x,\cdot)]$ and from (2.8) and (2.9) we deduce that

$$\begin{aligned}\epsilon_p &\leq \frac{1}{2}\pi^2 p^2 + \|V_{ext}(t)\|_{L^\infty} \\ &\leq \frac{1}{2}\pi^2 p^2 + \|V_{ext}(0)\|_{L^\infty} + \int_0^t \|[\partial_t V_{ext}(s)]^+\|_{L^\infty}\, ds.\end{aligned}$$

Consequently, (4.4) leads to

$$\int_0^t \|n(s)\|_{L^1}\, \|[\partial_t V_{ext}(s)]^+\|_{L^\infty}\, ds + \sum_{p\geq 1} \int_0^t \iint_{\Sigma_-} \left(\frac{v^2}{2} + \epsilon_p\right) g_p \, d\Sigma\, ds \leq C_{M_0},$$

This inequality, together with the estimate of the initial energy and (4.5), leads to Item *(ii)*.

The regularized system.
If $\varepsilon > 0$, the formal above proof is rigorous, except for one modification which concerns the estimate of ϵ_p^ε at the boundary, since we do not have $R^\varepsilon[V^\varepsilon] = 0$ on $\partial\omega \times (0,1)$. Nevertheless we can use the fact that this quantity is small. Let $x \in \partial\omega \times (0,1)$. Since $\epsilon_p^\varepsilon(t,x) = \epsilon_p[R^\varepsilon[V^\varepsilon] + V_{ext}](t,x)$, we deduce from (2.10) that

$$\left|\epsilon_p^\varepsilon - \epsilon_p[V_{ext}]\right|(t,x) \leq C e^{C_1 \|V^\varepsilon\|_{L^\infty_x L^1_z}} \, \|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1(0,1)}\,.$$

Therefore we have

$$\epsilon_p^\varepsilon(t,x) \leq \frac{1}{2}\pi^2 p^2 + \|[V_{ext}(t)]\|_{L^\infty} + C e^{C_1 \|V^\varepsilon\|_{L^\infty_x L^1_z}} \, \|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1_z(0,1)}\,. \tag{4.19}$$

We recall that by Theorem 3.1 and (4.13)

$$\|V^\varepsilon\|_{L^\infty_x L^1_z}(t) \leq C\|V^\varepsilon\|_{H^2}(t) \leq C_{M_0}\left(1 + \mathcal{E}^\varepsilon_{tot}(t)\right) \tag{4.20}$$

and that $\|R^\varepsilon[V] - V\|_{L^\infty_x L^1_z} \to 0 \quad \text{as} \quad \varepsilon \to 0$ for any function $V \in C^0(\overline{\omega}, L^1(0,1))$ such that $V(x,z) = 0$ on $\partial\omega \times (0,1)$ (see Lemma 2.6). Since the embedding of $H^2(\Omega)$ in $C^0(\overline{\omega}, L^1(0,1))$ is compact, it is not difficult to show by contradiction the existence of a constant $C(\varepsilon) > 0$ such that $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$ and

$$\|R^\varepsilon[V] - V\|_{L^\infty_x L^1_z} \leq C(\varepsilon)\|V\|_{H^2}. \tag{4.21}$$

Since $V^\varepsilon = 0$ on $\partial\omega \times (0,1)$, (4.20) implies that for $x \in \partial\omega$

$$\|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1_z(0,1)} \leq C_{M_0}\, C(\varepsilon)(1 + \mathcal{E}^\varepsilon_{tot}(t)).$$

Inserting this inequality and (4.20) in (4.19), then using (4.5), leads to

$$\mathcal{E}^\varepsilon_{tot}(t) \leq C_{M_0} + C_{M_0}\, C(\varepsilon) \int_0^t (1 + \mathcal{E}^\varepsilon_{tot}(s))\, e^{C_1 \mathcal{E}^\varepsilon_{tot}(s)}\, ds.$$

A standard perturbation argument coupled with a Gronwall lemma shows that for any $T > 0$ there exists $\varepsilon_{M_0} > 0$ and M_1 such that $\mathcal{E}^\varepsilon_{tot}(t) \leq M_1$ for $\varepsilon < \varepsilon_{M_0}$ and $t \in (0,T)$. □

Lemma 4.7 *Let $M > 0$ and let $(f_p(x,v))_{p\geq 1}$ be a sequence of nonnegative distribution functions. Then there exists a constant $\delta_M > 0$ such that if $(f_p)_{p\geq 1}$ verifies*

$$\sum_{p\geq 1} \|f_p\|_{L^\infty} + \sum_{p\geq 1} \|(v^2+1)f_p\|_{L^1} < 4M, \qquad \sum_{p\geq 2} \|f_p\|_{L^1} < \delta_M, \tag{4.22}$$

then the collection of charge densities defined by $\rho_p = \int f_p\, dv$ satisfies $(\rho) \in \mathcal{U}$, where $\mathcal{U}$ is the set defined in Theorem 3.2 with $q = 3/2$.

Proof. This Lemma is a direct consequence of the interpolation Lemma 4.3 and of Remark 3.3. Indeed, we deduce from (4.13) and (4.22) that there exists a constant C_0 such that

$$\sum_{p\geq 1} \|\rho_p\|_{L^2} \leq C_0 M.$$

This leads to

$$\sum_{p\geq 2} \|\rho_p\|_{L^{3/2}} \leq \left(\sum_{p\geq 2} \|\rho_p\|_{L^2}\right)^{2/3} \left(\sum_{p\geq 2} \|\rho_p\|_{L^1}\right)^{1/3} \leq (C_0^2 M^2\, \delta_M)^{1/3},$$

while

$$\|\rho_1\|_{L^{3/2}} \leq \|\rho_1\|_{L^2}^{2/3} \|\rho_1\|_{L^1}^{1/3} \leq 4^{1/3} C_0^{2/3} M.$$

Applying Remark 3.3 to define $\mathcal{N}(\cdot)$, one can see that a convenient δ_M is given by

$$\delta_M = \frac{\left(\mathcal{N}(4^{1/3} C_0^{2/3} M)\right)^3}{C_0^2\, M^2}.$$

□

Proposition 4.8 *Let $T > 0$ and $M_0 > 0$ be given and let Assumptions* (**H-1**)–(**H-5**) *be satisfied. Let ε_{M_0} and M be defined in terms of M_0 thanks to Lemma* 4.6 *(iii) and δ_M be defined in terms of M like in Lemma* 4.7*. Assume that the initial data verify*

$$L_1(0) < M_0 \tag{4.23}$$

and that

$$L_2(0)) < \delta_M, \tag{4.24}$$

where L_1 and L_2 are defined by (4.17),(4.18)*. Then for any $\varepsilon \in (0, \varepsilon_{M_0})$ the regularized VSP problem* (1.5)-(1.7) *admits a global weak solution $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon, f_p^\varepsilon)_{p\geq 1})$ on the interval* $[0,T]$.

Proof.
The proof relies on the Schauder fixed point theorem. Since we have to check that the occupation numbers $(\rho_p)_{p\geq 1}$ stay in $\mathcal{U}^+$ during time evolution, we will only be able to prove existence of solutions on a small time interval. The idea is then to

prove that this solution can be extended to a larger time interval and to check that the time increment does not become small. To this aim, the energy estimate will be of great use.

Let $T_1 \geq 0$ be given and assume that we have already constructed a weak solution on the interval $[0, T_1]$, such that $f_p^\varepsilon \in C([0,T], L^q)$ and let us prove that we can extend this solution to the interval $[0, T_1 + T_\varepsilon]$, where $T_\varepsilon > 0$ does not depend on T_1 but only on M. To this aim, we shall apply the Schauder fixed point theorem on the unknown V^ε for $t \in [T_1, T_1 + T_\varepsilon]$. Namely, we define the mapping $\mathcal{S}^\varepsilon$ as follows : starting from $V \in L^{3/2}((T_1, T_1 + T_\varepsilon) \times \Omega)$, we set $\epsilon_p = \epsilon_p[R^\varepsilon[V] + V_{ext}]$. The regularity of V_{ext} (assumption **(H-1)**) implies that $R^\varepsilon[V] + V_{ext}$ belongs to $L^{3/2}((T_1, T_1 + T_\varepsilon), C^2(\overline{\Omega}))$. Then the properties of the eigenvalues problem (2.6) insure that $\epsilon_p \in L^{3/2}((T_1, T_1 + T_\varepsilon), C^2(\overline{\omega}))$. This allows to construct $(f_p)_{p\geq 1}$ by solving the Vlasov equations (1.5) on the interval $[T_1, T_1 + T_\varepsilon]$ where the initial condition at time $t = T_1$ is provided by the value at $t = T_1$ of the already constructed solution on $[0, T_1]$. Finally, we compute $(V^*, (\epsilon_p^*, \chi_p^*)_{p\geq 1})$ as the solution of the regularized Schrödinger-Poisson system (3.1)-(3.2) in which $\rho_p = \int f_p \, dv$. By definition, we set

$$\mathcal{S}^\varepsilon(V) = V^*.$$

The key point is to find T_ε such that for $t \in [T_1, T_1 + T_\varepsilon]$, we have $(\rho(t)) \in \mathcal{U}^+$, where $\mathcal{U}^+$ is defined in Theorem 3.2, which insures that V^* is uniquely defined.

To find such a T_ε, let us define for any $\tau > 0$ the bounded, closed, convex set

$$\mathcal{K}_\tau = \{V \in L^{3/2}((T_1, \tau + T_1) \times \Omega) \ : \quad 0 \leq \|V\|_{L^{3/2}} \leq 1\}. \tag{4.25}$$

We shall prove here that there exists $T_\varepsilon > 0$ such that $\mathcal{S}^\varepsilon(\mathcal{K}_{T_\varepsilon}) \subset \mathcal{K}_{T_\varepsilon}$ and such that for any $V \in \mathcal{K}_{T_\varepsilon}$ and $t \in (T_1, T_1 + T_\varepsilon)$, the ρ_p defined above satisfy $(\rho(t)) \in \mathcal{U}^+$.

We first deduce from Lemma 4.6 *(i)* and *(iii)* and from the hypotheses of this proposition, that $L_1(T_1) < M$ and $L_2(T_1) < \delta_M$, where L_1 and L_2 are defined in (4.17),(4.18). Besides, we deduce from (4.9), (4.10), that the sequence $(f_p(t))_{p\geq 1}$ satisfies independently of T_ε

$$\forall t \in [T_1, T_1 + T_\varepsilon], \quad L_2(t) < \delta_M, \quad L_1(t) < M + \sum_{p\geq 1} \iint v^2 f_p(t,x,v) \, dx \, dv.$$

Therefore, in virtue of Lemma 4.7, the only thing we need to check in order to define T_ε is that

$$\mathcal{S}^\varepsilon(\mathcal{K}_{T_\varepsilon}) \subset \mathcal{K}_{T_\varepsilon}, \quad \text{and} \quad \sum_{p\geq 1} \iint v^2 f_p(t,x,v) \, dx \, dv < 3M.$$

For this purpose we first notice that for any τ and for any $V \in \mathcal{K}_\tau$, the eigenvalues ϵ_p corresponding to the potential $R^\varepsilon[V] + V_{ext}$ satisfy

$$\|\nabla_x \epsilon_p\|_{L_t^{3/2}(W_x^{1,\infty})} \leq C_\varepsilon, \tag{4.26}$$

where C_ε only depends on ε and V_{ext} (and not on τ nor on p). Applying (4.11) yields

$$\begin{aligned}\sum_{p\geq 1}\iint v^2 f_p(t,x,v)\,dx\,dv &\leq \sum_{p\geq 1}\iint v^2 f_p(T_1,x,v)\,dx\,dv + M + \Phi^\varepsilon(t) + \Phi^\varepsilon(t)^4\\ &\leq 2M + \Phi^\varepsilon(t) + \Phi^\varepsilon(t)^4,\end{aligned}$$

where

$$\Phi^\varepsilon(t) \leq C M^{1/4} \sup_{p\in\mathbb{N}^*}\int_{T_1}^t \|\nabla_x \epsilon_p(s,\cdot)\|_{L^\infty_x(\omega)}\,ds \leq C_{\varepsilon,M}\,(t-T_1)^{1/3},$$

and where the generic constant $C_{\varepsilon,M}$ depends on M and ε. Defining T_ε^0 by

$$C_{\varepsilon,M}\,(T_\varepsilon^0)^{1/3} + (C_{\varepsilon,M}\,(T_\varepsilon^0)^{1/3})^4 = M,$$

we have

$$\forall t \in (T_1, T_1+T_\varepsilon^0) \quad \sum_{p\geq 1}\iint v^2 f_p(t,x,v)\,dx\,dv < 3M. \tag{4.27}$$

Therefore Lemma 4.7 can be applied and, since the f_p's are nonnegative, we have $(\rho(t)) \in \mathcal{U}^+$ for any $t \in (T_1, T_1+T_\varepsilon^0)$. By Theorem 3.2, the regularized Schrödinger-Poisson system (3.1)-(3.2) with $\rho_p = \int f_p\,dv$ admits a unique solution $(V^*, (\epsilon_p^*, \chi_p^*)_{p\geq 1})$, which satisfies the uniform in ε estimate

$$\|V^*\|_{L^\infty((T_1,T_1+T_\varepsilon^0),W^{2,3/2}(\Omega))} < C.$$

This implies that if $\tau \leq T_\varepsilon^0$ the mapping $\mathcal{S}^\varepsilon$ is well-defined and satisfies

$$\|\mathcal{S}^\varepsilon(V)\|_{L^{3/2}((0,\tau)\times\Omega)} < C_0\,\tau^{2/3}, \tag{4.28}$$

where C_0 only depends on V_{ext}. Setting $T_\varepsilon = \min(T_\varepsilon^0, C_0^{-3/2})$, (4.28) shows that the set $\mathcal{K}_{T_\varepsilon}$ is stable by $\mathcal{S}^\varepsilon$.

In order to show that $\mathcal{S}^\varepsilon$ is compact and continuous, we notice that for any V in $\mathcal{K}_{T_\varepsilon}$, the corresponding sequence $(f_p)_{p\geq 1}$ defined in the mapping $\mathcal{S}^\varepsilon$ satisfies

$$\sum_{p\geq 1}\left(\|f_p\|_{L^\infty((0,T_\varepsilon)\times\omega\times\mathbb{R}^2)} + \|(v^2+p^2)f_p\|_{L^1((0,T_\varepsilon)\times\omega\times\mathbb{R}^2)}\right) \leq C.$$

Since by (4.26) the force fields in the Vlasov equations are bounded, the compactness of $\mathcal{S}^\varepsilon$ is then a consequence of Lemma 4.4 and its continuity is a consequence of Lemma 4.5. Hence Schauder's fixed-point theorem applies and provides the existence of a solution of (1.5)–(1.7) on (T_1, T_1+T_ε): the proof is complete.

□

4.4 Proof of Theorem 1.1

Theorem 1.1 is obtained by passing to the limit $\varepsilon \to 0$ in the regularized problem (1.5)-(1.7). Its proof is rather straightforward; we shall only sketch it and leave the details for the reader.

First remark that, thanks to Lemma 4.6, we have the estimate

$$\sum_{p\geq 1}\left(\|f_p^\varepsilon\|_{L^\infty_{t,x,v}} + \|(v^2+p^2)f_p^\varepsilon\|_{L^\infty_t L^1_{x,v}}\right) \leq C,$$

where C does not depend on ε. Hence by Lemma 4.3 and Theorem 3.1 we deduce

$$\|V^\varepsilon\|_{L^\infty((0,T),W^{2,3/2}(\Omega))} \leq C.$$

Thanks to standard results about the eigenvalue problem (2.6) (see for instance [34, 50]), the eigenvalues ϵ_p^ε inherit the regularity of $R^\varepsilon[V^\varepsilon] + V_{ext}$ with respect to x, thus we obtain

$$\|\nabla_x \epsilon_p^\varepsilon\|_{L^\infty((0,T),W^{1,3/2}(\omega))} \leq C_p$$

where the constant C_p depends on p but not on ε. Therefore the compactness Lemma 4.4 implies that, up to an extraction of subsequence, (ρ^ε) converges to (ρ) in $\ell^1(L^{3/2}((0,T)\times\omega))$.

Moreover, for any $t \in (0,T)$ and $\varepsilon \in (0,\varepsilon_M)$, we have $(\rho^\varepsilon(t)) \in \mathcal{U}^+$: the quasistatic Schrödinger-Poisson part of the problem is always solved in the framework of Theorem 3.2, *i.e.* its solution $(V^\varepsilon,(\epsilon_p^\varepsilon,\chi_p^\varepsilon)_{p\geq 1})$ is continuous with respect to (ρ^ε). Denote by $(V,(\epsilon_p,\chi_p)_{p\geq 1})$ the solution of the *unregularized* Schrödinger-Poisson system with the occupation factors ρ_p and by $(V^{0,\varepsilon},(\epsilon_p^{0,\varepsilon},\chi_p^{0,\varepsilon})_{p\geq 1})$ the solution of the *unregularized* Schrödinger-Poisson system with the occupation factors ρ_p^ε :

$$\begin{cases} -\dfrac{1}{2}\,\partial^2_{zz}\chi_p^{0,\varepsilon} + (V^{0,\varepsilon}+V_{ext})\chi_p^{0,\varepsilon} = \epsilon_p^{0,\varepsilon}\chi_p^{0,\varepsilon}\,, \\ -\Delta V^{0,\varepsilon} = \displaystyle\sum_{p\geq 1}\rho_p^\varepsilon\,\left|\chi_p^{0,\varepsilon}\right|^2. \end{cases}$$

By Item *(i)* of Theorem 3.2, $V^\varepsilon - V^{0,\varepsilon}$ converges to 0 in $L^{3/2}((0,T),W^{2,3/2}(\Omega))$ as $\varepsilon \to 0$. Moreover, by Item *(ii)* of the same theorem (applied to the *unregularized* S-P system), we have

$$\|V^{0,\varepsilon} - V\|_{L^{3/2}((0,T),W^{2,3/2}(\Omega))} \leq C\|\rho^\varepsilon - \rho\|_{\ell^1(L^{3/2}((0,T)\times\omega))}\,,$$

which implies that V^ε converges to V in $L^{3/2}((0,T),W^{2,3/2}(\Omega))$. Since V^ε and V belong to $W^{1,3/2}_\omega$, by Lemma 2.6 the regularization $R^\varepsilon[V^\varepsilon]$ converges to V in $L^{3/2}((0,T),W^{1,3/2}(\Omega))$. Consequently, the eigenvalues $\epsilon_p^\varepsilon := \epsilon_p[R^\varepsilon[V^\varepsilon]+V_{ext}]$ converge to $\epsilon_p[V+V_{ext}]$ in $L^{3/2}((0,T),W^{1,3/2}(\omega))$ as $\varepsilon \to 0$, and by Lemma 4.5 one can pass to the limit in the Vlasov equations and obtain a weak solution of the unregularized Vlasov-Schrödinger-Poisson system which satisfies (4.4) and (4.5).

The occupation numbers $(\rho_p)_{p\geq 1}$ are in $\ell^1(L^\infty((0,T),L^2(\omega)))$. A similar argument to the one developed in [19] shows that $(\rho_p)_{p\geq 1} \in \ell^1(C([0,T],L^2(\omega)))$, which yields $V \in C([0,T],H^2(\Omega))$ in view of Theorem 3.1.

5 The stationary situation

This section is devoted to the study of the stationary solutions of (1.1)–(1.3). The analysis relies on a fixed point procedure different from the one used in the time-dependent case. In particular it is independent of Section 3 and no smallness assumption will be necessary. Let us first write the system studied here. We seek $V(x,z)$, $(\epsilon_p(x), \chi_p(x,z), f_p(x,v))_{p\geq 1}$ solving

$$\begin{cases} v\cdot\nabla_x f_p - \nabla_x\epsilon_p\cdot\nabla_v f_p = 0, \\ \gamma^- f_p = g_p \quad \text{on } (0,T)\times\Sigma_-\,, \end{cases} \tag{5.1}$$

$$\begin{cases} -\dfrac{1}{2}\,\partial^2_{zz}\chi_p + (V+V_{ext})\chi_p = \epsilon_p\chi_p\,, \\ \chi_p(x,\cdot)\in H^1_0(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{5.2}$$

$$\begin{cases} -\Delta V = n = \displaystyle\sum_{p\geq 1}\left(\int_{\mathbb{R}^2} f_p\,dv\right)|\chi_p|^2\,, \\ V = 0 \quad \text{on } \partial\omega\times(0,1), \qquad \partial_z V = 0 \quad \text{on } \omega\times\{0\}\,\cup\,\omega\times\{1\}. \end{cases} \tag{5.3}$$

We make the following assumptions on the data of the stationary problem:

(**HS-1**) The external potential V_{ext} is nonnegative and lies in $C^2(\overline{\Omega})$.

(**HS-2**) There exists a sequence of nonincreasing functions $(G_p)_{p\geq 1}$ such that

$$0 \leq g_p(x,v) \leq G_p\left(\frac{v^2}{2} + \frac{\pi^2 p^2}{2} + \max_{\partial\omega\times[0,1]}(V_{ext})\right),$$

$$\text{with} \quad \sum_{p\geq 1} p\int_a^{+\infty} G_p\left(e + \frac{\pi^2 p^2}{2}\right) de \; < +\infty \quad \forall a\in\mathbb{R}.$$

Theorem 5.1 *Under Hypotheses* (**HS-1**)–(**HS-2**)*, there exists a weak solution* $(V, (\epsilon_p, \chi_p, f_p)_{p\in\mathbb{N}^*})$ *of* (5.1)–(5.3) *such that* $V\in W^{2,s}(\Omega)$*,* $\epsilon_p\in W^{2,s}(\omega)$*,* $\chi_p\in W^{2,s}(\Omega)$ *for all* $s<+\infty$ *and the total charge density* n *belongs to* $L^\infty(\Omega)$*. Moreover this solution satisfies the following pointwise estimate*

$$0 \leq f_p(x,v) \leq G_p\left(\frac{v^2}{2} + \epsilon_p(x)\right), \quad (x,v)\in\omega\times\mathbb{R}^2. \tag{5.4}$$

Proof. The analysis relies on the application of Schauder fixed point theorem for a regularization of the problem and uses the supersolution technique developed by Poupaud in [48]. More precisely, let $\varepsilon\in(0,1)$ and $\lambda\in(0,1)$ be two positive regularization parameters that we shall let tend to zero. Consider the mapping $\mathcal{S}_{\lambda,\varepsilon}$

defined on $W^{1,4}_\omega$ in the following way : for a given potential $V \in W^{1,4}_\omega$, let $(\epsilon_p, \chi_p)_{p\geq 1}$ be the eigenvalues and eigenfunctions defined by

$$\left\{ \begin{array}{l} -\dfrac{1}{2}\,\partial^2_{zz}\chi_p(x;z) + (R^\varepsilon[V] + V_{ext})\chi_p(x;z) = \epsilon_p(x)\chi_p(x;z), \\ \chi_p(x,\cdot) \in H^1_0(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{array} \right. \tag{5.5}$$

where the regularization operator R^ε is defined in (1.4). Then, compute f_p by solving

$$\left\{ \begin{array}{l} \lambda f_p + v\cdot\nabla_x f_p - \nabla_x\epsilon_p\cdot\nabla_v f_p = 0, \\ \gamma^- f_p = g_p \quad \text{on } \Sigma_-\,. \end{array} \right. \tag{5.6}$$

Finally, define $V^* = \mathcal{S}_{\lambda,\varepsilon}(V)$ as the unique solution of

$$\left\{ \begin{array}{l} -\Delta V^* = R^\varepsilon\left[\displaystyle\sum_{p\geq 1}\left(\int_{\mathbb{R}^2} f_p\,dv\right)|\chi_p|^2\right], \\ V^* = 0 \quad \text{on } \partial\omega\times(0,1), \qquad \partial_z V^* = 0 \quad \text{on } \omega\times\{0\}\,\cup\,\omega\times\{1\}. \end{array} \right. \tag{5.7}$$

The mapping $\mathcal{S}_{\lambda,\varepsilon}$ is clearly uniquely defined. Indeed, the only thing to be checked is the existence and uniqueness of solutions of (5.6). This is a consequence of the fact that ϵ_p is in $C^2(\overline{\omega})$ (the ϵ_p's have the same regularity as the potential) and of the positivity of the damping coefficient λ. For this point, we refer to [48].

For any $A > 0$, we now define

$$B^A_+ = \{V \in W^{1,4}_\omega \ :\ \|V\|_{W^{1,4}} \leq A \quad \text{and} \quad 0 \leq V(x,z) \ \text{ a.e.}\}.$$

Proceeding by contradiction as for (4.21), one can show the existence of a constant $C(\varepsilon) > 0$ such that $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$ and

$$\|R^\varepsilon[V] - V\|_{L^\infty_x L^1_z} \leq C(\varepsilon)\|V\|_{W^{1,4}(\Omega)}.$$

Combining this inequality and (2.10), since V vanishes on $\partial\omega\times(0,1)$, one can find $\varepsilon_{A,\lambda} > 0$ (depending only on A and λ) such that for $\varepsilon\in(0,\varepsilon_{A,\lambda})$ and $V\in B^A_+$ we have

$$\forall x\in\partial\omega \quad \forall p\in\mathbb{N}^* \qquad |\epsilon_p[R^\varepsilon[V] + V_{ext}] - \epsilon_p[V_{ext}]|(x) < \lambda.$$

Thus from (2.8) we deduce that the ϵ_p's defined by (5.5) satisfy

$$\forall x\in\partial\omega \quad \forall p\in\mathbb{N}^* \qquad \epsilon_p(x) < \frac{\pi^2 p^2}{2} + \max_{\partial\omega\times[0,1]}(V_{ext}) + \lambda.$$

Thanks to Assumption (**HS-2**), this is enough to conclude that $G_p(\frac{v^2}{2} + \epsilon_p(x) - \lambda)$ is a supersolution of (5.6). This implies that

$$0 \leq f_p(x,v) \leq G_p\left(\frac{v^2}{2} + \epsilon_p(x) - \lambda\right). \tag{5.8}$$

This supersolution estimate shows that the corresponding $\rho_p = \int f_p \, dv$ satisfy

$$0 \leq \rho_p(x) \leq 2\pi \int_0^{+\infty} G_p\left(e + \epsilon_p(x) - \lambda\right) de.$$

Since $V + V_{ext} \geq 0$ and $\lambda \in (0,1)$, it follows from Hypothesis **(HS-2)** and from (2.8) that

$$0 \leq \rho_p(x) \leq M_p = 2\pi \int_0^{+\infty} G_p\left(e + \frac{\pi^2 p^2}{2} - 1\right) de.$$

Besides, applying (2.7), (2.9) and a Gagliardo-Nirenberg inequality yields, for any $x \in \omega$

$$\|\chi_p(x,\cdot)\|^2_{L^\infty_z} \leq C \|\chi_p(x,\cdot)\|_{L^2_z} \left\| \frac{d}{dz}\chi_p(x,\cdot)\right\|_{L^2_z} \leq C\,p + C\|V\|^{1/2}_{L^\infty} + C\|V_{ext}\|^{1/2}_{L^\infty},$$

thus, thanks to the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$,

$$0 \leq -\Delta V^* \leq C \sum_{p\geq 1} p M_p + C A^{1/2} \sum_{p \geq 1} M_p \,. \tag{5.9}$$

The first inequality implies the positivity of V^*, while the second one yields, by elliptic regularity and Sobolev embeddings,

$$\|V^*\|_{W^{1,4}(\Omega)} \leq C(1 + A^{1/2}).$$

Let A_0 be such that $A_0 \geq C(1 + A_0^{1/2})$ and $\varepsilon \in (0, \varepsilon_{A_0,\lambda})$. Then it is clear that the mapping $\mathcal{S}_{\lambda,\varepsilon}$ satisfies the following property:

$$\mathcal{S}_{\lambda,\varepsilon}(B^{A_0}_+) \subset B^{A_0}_+.$$

Moreover, coming back to (5.9) and using again elliptic regularity, we also have

$$\forall q \in [1,+\infty), \qquad \mathcal{S}_{\lambda,\varepsilon}(B^{A_0}_+) \subset \{V \in W^{2,q}(\Omega) \,:\quad \|V\|_{W^{2,q}(\Omega)} \leq C_q\},$$

where C_q is independent of ε and λ.

This shows the existence of a convex bounded set of $W^{1,4}_\omega$ which is let invariant by $\mathcal{S}_{\lambda,\varepsilon}$ and also shows the compactness of $\mathcal{S}_{\lambda,\varepsilon}$. The continuity of $\mathcal{S}_{\lambda,\varepsilon}$ is easily obtained (the details are left to the reader). Hence, $\mathcal{S}_{\lambda,\varepsilon}$ admits a fixed point $V_{\lambda,\varepsilon}$. Then we pass successively to the limit $\varepsilon \to 0$, $\lambda \to 0$. This can be done without difficulty since $V_{\lambda,\varepsilon}$ is bounded uniformly in $W^{2,q}$ for all $q < +\infty$. This shows the existence of solutions of the unmodified stationary Vlasov-Schrödinger-Poisson problem (5.1)–(5.3). Passing to the limit in (5.8) shows that the solution satisfies the supersolution estimate (5.4). To complete the proof of Theorem 5.1, it remains to recall that the regularity of the eigenvalues ϵ_p and eigenfunctions χ_p can be deduced from the regularity of the potential V (see for instance [50]). $\square$

6 Volume and boundary collisions

In this section, we describe two extensions which lead to similar existence results as those obtained in Sections 2–5.

Volume collisions.

In the model analyzed in this paper, the only couplings between the subbands which are taken into account are collective interactions through the selfconsistent potential. Moreover, no transition can occur from one subband to another one during the evolution of the system. The transport in the x direction is ballistic and described by the Vlasov equation.

In order to take into account collisions in this model, the Vlasov equation can be replaced by Boltzmann type equations

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = Q(f)_p$$

where $f = (f_n)_{n \in \mathbb{N}^*}$ and Q is a matrix collision operator which models intersubband transition (extra-diagonal terms) and intrasubband transitions (diagonal terms).

For the linear, diagonal, elastic collision operator, the existence Theorems 1.1 and 5.1 apply without any change. Namely, we take Q under the form

$$Q(f)_p = Q_p(f_p) = \int \alpha_p(t,x,v,v')\,(f(v') - f(v))\,\delta(|v|^2 - |v'|^2)\,dv'$$

and assume that α_p is nonnegative, symmetric (with respect to v and v') and satisfies

$$\int \alpha_p(t,x,v,v')\,\delta(|v|^2 - |v'|^2)\,dv' \in L^\infty.$$

The formula $\int h(v')\,\delta(|v'|^2 - a)\,dv'$ is defined for almost every a like in [9] thanks to the coarea formula by

$$\int h(v')\,\delta(|v'|^2 - a)\,dv' = \frac{1}{2}\int_{\mathbb{S}^1} h(w\sqrt{a})\,d\sigma(w),$$

where $d\sigma$ is the surface measure on the unit sphere $\mathbb{S}^1$. One can show, with the above collision operator that the results of Sections 2–5 hold true. The reason for this is that Q_p conserves any function of the energy, that it conserves the L^∞ norm and that any isotropic (in velocity) function belongs to the kernel of Q_p. Moreover, since there is no intersubband transition term, the initial data can still be arbitrary large on the first subband.

In [11], a Drift-Diffusion-Schrödinger-Poisson system was studied, obtained thanks to a diffusion limit of such a Boltzmann-Schrödinger-Poisson system.

Boundary collisions

The boundary operator which has been chosen in the paper was a purely absorbing boundary operator. More sophisticated mechanisms at the boundary can be described by an elastic-diffusive operator similar as in [9, 40]:

$$\gamma^- f_p = \beta g_p + (1-\beta)\mathcal{R}_p(\gamma^+ f_p),$$

where $\beta \in [0,1]$ and

$$\mathcal{R}_p(f)(t,x,v) = \int_{v'\cdot\nu(x)>0} \sigma_p(t,x,v',v)\,\delta(|v|^2 - |v'|^2) f(t,x,v')\,|v'\cdot\nu(x)|\,dv'.$$

If the cross section σ_p is a nonnegative function satisfying mass conservation and reciprocity:

$$\int_{v\cdot\nu(x)<0} \sigma_p(t,x,v',v)\delta(|v|^2 - |v'|^2)\,|v\cdot\nu(x)|\,dv = 1 \qquad \text{for } (t,x,v') \in (0,T)\times\Sigma_+$$

$$\sigma(t,x,v',v) = \sigma(t,x,-v,-v'),$$

then the analysis of the time-dependent case can be done as in Section 4. In the stationary situation, the analysis of Section 5 can be reproduced when the accomodation coefficient β is different from zero. If $\beta = 0$, the system is isolated and the total mass, as well as the profile in the energy of the solution, has to be prescribed. We do not develop this case here and refer to [21, 22, 28].

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