Semiclassical analysis of the Schrödinger equation with a partially confining potential

Naoufel Ben Abdallah and Florian Méhats

MIP, Laboratoire CNRS (UMR 5640), Université Paul Sabatier,
118, route de Narbonne, 31062 Toulouse Cedex 04, France

`email: naoufel@mip.ups-tlse.fr ; mehats@mip.ups-tlse.fr`

Abstract

The semiclassical limit of a partially confined electron gas is performed. The length scale in the confined direction is of the order of magnitude of the electron de Broglie length whereas the non confined lengthscale is larger. A partial semiclassical limit of the Schrödinger equation (in the non confined direction) is performed and leads to the so-called subband model. The limiting behaviour is described by an infinite number of quasistatic Schrödinger equations for the confined direction and an infinite number of time-dependent Vlasov equations in the non confined direction.

Résumé

Nous appliquons la limite semiclassique au système formé par un gaz d'électrons partiellement confinés. Dans la direction du confinement, l'échelle spatiale caractéristique est de l'ordre de grandeur de la longueur de de Broglie des électrons, tandis que cette échelle est bien plus grande dans les directions non confinées. Une limite semiclassique partielle appliquée à l'équation de Schrödinger mène au modèle du transport par sous-bandes. Ce modèle limite est constitué d'un nombre infini d'équations de Schrödinger quasistatiques pour la direction du confinement couplé à un nombre infini d'équations de Vlasov non stationnaires pour les directions non confinées.

Keywords: Semiclassical limit, Wigner measures, subband transport, two dimensional electron gas.

Mots-clés : Limite semiclassique, mesures de Wigner, transport par sous-bandes, gaz d'électrons bidimensionnel.

1 Introduction

The operation of many electronic nanostructures, like quantum waveguides or transistors, relies on the formation of a bidimensional electron gas. Such a system is obtained by confining the electrons in one direction and allowing for improved transport properties in the two other directions. Other nanostructures, like quantum wires or nanotubes [13, 31], are confined in two directions while the transport is allowed in the remaining one.

In this paper, we are interested in situations where the length scale in the confined direction is of the order the de Broglie wavelength of electrons, while the non confined directions have a much bigger length scale. In other terms, electrons are in a quantum regime in the confined direction and exhibit classical behaviour in the non confined ones. This gives rise to the theory of subbands which is widely used in the semiconductor physics litterature [1, 2, 10, 34]. The aim of this paper is to derivate rigorously the subband model from a partial semiclassical limit of the Schrödinger equation.

The variables of the longitudinal directions are semiclassical and are denoted by $x \in \mathbb{R}^m$, where $m \in \mathbb{N}^*$. In the one-dimensional transversal direction z, the electrons have a quantum behaviour. The spatial domain is $\Omega \subset \mathbb{R}^{m+1}$. After an adequate rescaling we consider the linear (one particle) Schrödinger equation:

$$i\varepsilon\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x\psi^\varepsilon - \frac{1}{2}\partial_z^2\psi^\varepsilon + V^\varepsilon\psi^\varepsilon, \tag{1.1}$$

$$\psi^\varepsilon(0,x,z) = \psi_I^\varepsilon(x,z), \tag{1.2}$$

where ε denote the ratio between the length scales in the transversal and longitudinal directions. The external potential $V^\varepsilon = V^\varepsilon(t,x,z)$ is a data of the problem and is assumed to be regular enough with respect to t and x (see Assumption 2.1). Furthermore, the confinement in the transversal direction is modeled through a hard-wall potential assumption:

$$\psi^\varepsilon(t,x,z=0) = \psi^\varepsilon(t,x,z=1) = 0, \tag{1.3}$$

so that the spatial domain is the slab:

$$\Omega = \{(x,z) \in \mathbb{R}^m \times (0,1)\}.$$

In Section 6 we generalize the analysis to the more general case of a domain with a varying width. The general framework developed in this paper can also be applied to analyze a smoother confinement, modeled by (1.1)-(1.2) where $\Omega = \mathbb{R}^{m+1}$ and $V^\varepsilon \to +\infty$ for $|z| \to +\infty$.

Let us now introduce the subbands of the system. Thanks to the confinement (1.3), the transversal Hamiltonian has a discrete spectrum and admits a complete set of eigenfunctions. We denote by $\chi_p^\varepsilon(t,x,\cdot)$ and $\epsilon_p^\varepsilon(t,x)$ the eigenfunctions (chosen real-valued throughout this paper) and the eigenvalues of the operator $-\frac{1}{2}\partial_z^2 + V^\varepsilon$

acting on the z variable with homogeneous boundary conditions at $z = 0$ and $z = 1$. They depend parametrically on t and x and satisfy

$$\begin{cases} -\dfrac{1}{2}\,\partial_z^2 \chi_p^\varepsilon + V^\varepsilon \chi_p^\varepsilon = \epsilon_p^\varepsilon\, \chi_p^\varepsilon\,, \\ \chi_p^\varepsilon(t,x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p^\varepsilon\, \chi_q^\varepsilon\, dz = \delta_{pq}. \end{cases} \tag{1.4}$$

Since the confinement occurs in dimension 1, the eigenvalues are simple and do not cross. We denote by $\mathcal{H}_p^\varepsilon = \operatorname{span}(\chi_p^\varepsilon) \subset L^2(0,1)$ the p-th eigenspace and by Π_p^ε the orthogonal projector on $\mathcal{H}_p^\varepsilon$. Then the p-th *subband* is defined as the space $L^2(\mathbb{R}^m) \otimes \mathcal{H}_p^\varepsilon = L^2(\mathbb{R}^m, \mathcal{H}_p^\varepsilon)$.

To perform rigourously the partial semiclassical limit $\varepsilon \to 0$, we shall make use of the Wigner transforms. This powerful tool was introduced by Wigner in the early thirties [35] and has received an increasing interest from the Applied Mathematics community during the last decade, when many important results have been obtained. Namely, in the early 90's, Gérard [15], Lions and Paul [20] and Markowich and Mauser [21] have obtained convergence results for the semiclassical limit using Wigner function techniques. A review of these techniques as well as further results can be found in [16]. The first reference [15] concerns linear Schrödinger operators with rapidly oscillating potentials, the second [20] concerns self-consistent Schrödinger-Poisson systems and the last one [21] concern a slightly mollified version of the Schrödinger-Poisson system. Following the technique developed in [20], Wigner series has been introduced by Markowich, Mauser and Poupaud [22] in order to analyze the asymptotic behaviour in the presence of periodic potentials and a series of results concerning Schrödinger and Dirac equations, coupled to the Poisson equation with periodic and non periodic potentials, have been obtained by Béchouche, Gérard, Markowich, Mauser and Poupaud [3, 4, 5, 6, 16, 22, 23]. The Wigner function technique has also been used by Poupaud and Ringhofer [29] to deal with the effective mass approximation in crystals and by Markowich and Poupaud [24] to analyze some finite difference schemes. Boundary problems have been studied in [26] and collisions have been treated in [9, 11, 12, 27].

Throughout this paper the property of isolated subbands is fundamental. In the case of crossing energies, the picture is much more complicated; Fermanian-Kammerer and Gérard [14] have recently constructed double scale Wigner transform in order to analyze the so-called Landau-Zener effect.

Let us come back to our problem. In order to have an insight into this problem, we can derive *formally* the limit model by analogy with the Born-Oppenheimer theory [8] in molecular dynamics. For the mathematical analysis of the Born-Oppenheimer approximation, we refer to [17, 25, 32, 33] and to the references therein. At the limit $\varepsilon \to 0$, one expects that two physical effects occur: the adiabatic decoupling of the subbands and the semiclassical transport within each subband [32, 33]. The adiabatic decoupling states that at the leading order in ε the subbands are transported through decoupled equations. More precisely, denote by $H^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x - \frac{1}{2}\,\partial_z^2 + V^\varepsilon$

the Hamiltonian of the system, and by H^ε_{diag} the following Hamiltonian:

$$H^\varepsilon_{diag} = \sum_{p\geq 1} \Pi^\varepsilon_p\, H^\varepsilon\, \Pi^\varepsilon_p.$$

Then we expect that the dynamics generated by these two operators are asymptotically close: in an operator norm, we have for stationary and smooth enough potentials V,

$$\left\| e^{-iH^\varepsilon t/\varepsilon} - e^{-iH^\varepsilon_{diag} t/\varepsilon} \right\| \to 0 \qquad \text{as } \varepsilon \to 0.$$

In other words, the solution of (1.1) is close to the function

$$\sum_{p\geq 1} \psi^\varepsilon_p(t,x,z) = \sum_{p\geq 1} \phi^\varepsilon_p(t,x)\, \chi^\varepsilon_p(x,z),$$

where $\psi^\varepsilon_p(t,x,\cdot) \in \mathcal{H}^\varepsilon_p$ and solves

$$i\varepsilon\partial_t \psi^\varepsilon_p = \Pi^\varepsilon_p(-\frac{\varepsilon^2}{2}\Delta_x \psi^\varepsilon_p + \epsilon^\varepsilon_p\, \psi^\varepsilon_p).$$

The above equation can be written in terms of the ϕ^ε_p's under the following form:

$$i\varepsilon\partial_t \phi^\varepsilon_p = -\frac{\varepsilon^2}{2}\Delta_x \phi^\varepsilon_p + \left(\epsilon^\varepsilon_p + \frac{\varepsilon^2}{2}\int_0^1 |\nabla_x \chi^\varepsilon_p|^2 dz \right) \phi^\varepsilon_p.$$

In a second step, we apply formally the semiclassical limit to this equation. Let us form the Wigner transform of ϕ^ε_p:

$$f^\varepsilon_p(t,x,v) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\eta\cdot v}\, \phi^\varepsilon_p\left(t, x - \varepsilon\frac{\eta}{2}\right) \overline{\phi}^\varepsilon_p\left(t, x + \varepsilon\frac{\eta}{2}\right) d\eta.$$

From [20, 21], it is clear that if the χ^ε_p and ϵ^ε_p are smooth enough, this function f^ε_p converges as $\varepsilon \to 0$ to a bounded measure f_p which satisfies the Vlasov equation

$$\partial_t f_p + v\cdot\nabla_x f_p - \nabla_x \epsilon^0_p \cdot \nabla_v f_p = 0. \tag{1.5}$$

The transport on each subband is driven by $-\nabla_x \epsilon^0_p$, where ϵ^0_p is the eigen-energy of the subband: the classical equations of motion on the p-th subband are

$$\dot{X} = V\,, \qquad \dot{V} = -\nabla_x \epsilon^0_p\,.$$

Remark that at the limit, the electron system is described by the following "density matrix"

$$\sum_p f_p(t,x,v)\chi^0_p(x,z)\chi^0_p(x,z').$$

The aim of this paper is to justify these arguments for (given) time dependent potentials V^ε. The analysis of the partial semiclassical limit for (1.1) in the nonlinear case, where V^ε solves the Poisson equation, will be the object of a future work. In this nonlinear case, the limit model (1.4)–(1.5) coupled with the Poisson equation was analyzed in [7].

2 Assumptions and main results

For each $\varepsilon \in [0,1]$, the external potential V^ε is defined on $\mathbb{R}^+ \times \Omega$ and we have the following assumption:

Assumption 2.1 *For any $T > 0$, the function $(\varepsilon, t, x, z) \mapsto V^\varepsilon(t,x,z)$ is nonnegative and belongs to*

$$C^0([0,1], C_b^1([0,T] \times \mathbb{R}^m, L^\infty(0,1))).$$

In this assumption and in the sequel of the paper, C_b^1 denotes the space $C^1 \cap W^{1,\infty}$. Then it is clear that the time-dependent Hamiltonian $H = -\frac{\varepsilon^2}{2}\Delta_x - \frac{1}{2}\partial_z^2 + V^\varepsilon$ is nonnegative and self-adjoint on its domain $\mathcal{D}(H) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover the subbands are well-defined by (1.4) and the eigen-elements of $-\frac{1}{2}\partial_z^2 + V^\varepsilon$ have the same regularity as V^ε with respect to (ε, t, x): the function $(\varepsilon, t, x) \mapsto \epsilon_p^\varepsilon(t,x)$ belongs to $C^0([0,1], C_b^1([0,T]\times\mathbb{R}^m))$ and the function $(\varepsilon,t,x,z) \mapsto \chi_p^\varepsilon(t,x,z)$ belongs to $C^0([0,1], C_b^1([0,T] \times \mathbb{R}^m, H^2(0,1)))$, for any $T > 0$. This result comes from the perturbation theory of linear operators (see for instance [19, 30]).

Now we state the assumption on the initial wavefunction:

Assumption 2.2 *The Cauchy data ψ_I^ε belongs to $H_0^1(\Omega)$ and satisfies:*

$$\iint_\Omega \left(|\psi_I^\varepsilon|^2 + \varepsilon^2|\nabla_x\psi_I^\varepsilon|^2 + |\partial_z\psi_I^\varepsilon|^2\right) dx\,dz \le C,$$

independently of ε.

Under the two Assumptions 2.1 and 2.2, the Schrödinger equation (1.1)–(1.3) admits a unique weak solution

$$\psi^\varepsilon \in C^0(\mathbb{R}_+, H_0^1(\Omega)) \cap C^1(\mathbb{R}_+, H^{-1}(\Omega))$$

(see for instance [28, 30]). Moreover, since $\partial_t V^\varepsilon$ is bounded independently of ε in $L^\infty_{loc}(\mathbb{R}_+, L^\infty(\Omega))$, the energy estimate is propagated:

$$\forall t > 0 \qquad \iint_\Omega \left(|\psi^\varepsilon(t)|^2 + \varepsilon^2|\nabla_x\psi^\varepsilon(t)|^2 + |\partial_z\psi^\varepsilon(t)|^2\right) dx\,dz \le C(t), \tag{2.1}$$

where $C(t)$ is a continuous function of t, independent of ε (for details, see further the proof of Lemma A.1 in the Appendix). Let us define the particle charge and current densities by

$$n^\varepsilon(t,x,z) = |\psi^\varepsilon(t,x,z)|^2 \quad ; \quad j_x^\varepsilon(t,x,z) = \varepsilon\,\mathcal{I}m\,(\overline{\psi^\varepsilon}\nabla_x\psi^\varepsilon) \tag{2.2}$$

as well as the surface charge and surface current densities by

$$n_s^\varepsilon(t,x) = \int_0^1 n^\varepsilon(t,x,z)\,dz \quad ; \quad j_{x,s}^\varepsilon(t,x,z) = \int_0^1 \varepsilon\,\mathcal{I}m\,(\overline{\psi^\varepsilon}\nabla_x\psi^\varepsilon)\,dz. \tag{2.3}$$

The motivation of this paper is the study of the limit $\varepsilon \to 0$ of these macroscopic quantities. The result is naturally expressed by means of the partial Wigner transform in the x variable. For any function $\varphi \in \mathcal{S}'(\mathbb{R}^m, L^2(0,1))$, we define

$$W^\varepsilon(\varphi)(x,v,z,z') = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\eta\cdot v}\, \varphi\left(x - \varepsilon\frac{\eta}{2}, z\right) \overline{\varphi}\left(x + \varepsilon\frac{\eta}{2}, z'\right) d\eta. \tag{2.4}$$

With an obvious abuse of notation, we will denote the Wigner transform of ψ^ε by $W^\varepsilon(\psi^\varepsilon)(t,x,v,z,z')$.

Theorem 2.3 *Suppose that Assumptions* 2.1 *and* 2.2 *are satisfied. Then, for a subsequence still indexed by* ε*, we have the following results at the semiclassical limit* $\varepsilon \to 0$*:*
(i) For any $p \in \mathbb{N}^*$*, the Wigner transform* $W^\varepsilon(\Pi_p^\varepsilon \psi_I^\varepsilon)$ *of the projection on the p-th subband of the initial data converges in* $\mathcal{S}'(\mathbb{R}^{2m}_{x,v}, L^2(0,1) \times L^2(0,1))$ *to*

$$f_{p,I}(t,x,v)\, \chi_p(0,x,z)\chi_p(0,x,z') \in \mathcal{M}_b(\mathbb{R}^{2m}_{x,v}, L^2(0,1) \times L^2(0,1)).$$

(ii) The Wigner transform $W^\varepsilon(\psi^\varepsilon)$ *of the solution of* (1.1)–(1.3) *converges in the* $L^\infty(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^{2m}_{x,v}, L^2(0,1) \times L^2(0,1)))$ *weak* $*$ *topology to*

$$\sum_{p\geq 1} f_p(t,x,v)\, \chi_p(t,x,z)\chi_p(t,x,z') \in C^0(\mathbb{R}_+, \mathcal{M}_b(\mathbb{R}^{2m}_{x,v}, L^2(0,1) \times L^2(0,1))),$$

where $f_p \geq 0$ *solves*

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0 \tag{2.5}$$

$$f_p(0,x,v) = f_{p,I}(x,v). \tag{2.6}$$

(iii) For every $T > 0$*, the charge density and the current density defined by* (2.2) *converge in the* $L^\infty((0,T), \mathcal{M}_b(\mathbb{R}^m_x, L^1(0,1) - weak))$ *weak* $*$ *topology to:*

$$n(t,x,z) = \sum_{p\geq 1} \left(\int_{\mathbb{R}^m} f_p(t,x,v)\, dv \right) |\chi_p(t,x,z)|^2,$$

and

$$j_x(t,x,z) = \sum_{p\geq 1} \left(\int_{\mathbb{R}^m} v f_p(t,x,v)\, dv \right) |\chi_p(t,x,z)|^2.$$

(iv) The surface charge and current densities defined by (2.3) *converge locally uniformly on* $\mathbb{R}_+$ *in the* $\mathcal{M}_b(\mathbb{R}^m_x)$ *weak* $*$ *topology to* $n_s = \int_0^1 n(\cdot,z)\, dz$ *and* $j_{x,s} = \int_0^1 j_x(\cdot,z)\, dz$*, which satisfy*

$$\partial_t n_s + \operatorname{div}_x j_{x,s} = 0.$$

In this theorem, as well as in the sequel of the paper, for $\varepsilon = 0$ we use the notations ϵ_p, χ_p and Π_p instead of ϵ_p^0, χ_p^0 and Π_p^0. Moreover, if E is a Banach space, $\mathcal{S}'(\mathbb{R}^m, E)$ denotes the space of E-valued tempered distributions and $\mathcal{M}_b(\mathbb{R}^m, E)$ is the space of E-valued bounded measures.

The analysis of this problem can be put in the general framework of semiclassical limits of a Schrödinger equation with $\mathcal{H}$-valued wavefunctions, where $\mathcal{H}$ is a Hilbert space. Section 3 is devoted to the presentation of the general framework and to the description of the required hypotheses. In Section 4, we define Wigner measures of $\mathcal{H}$-valued density matrices and we state their properties. We show that the known results on matrix-valued Wigner functions extend to our case. These properties, combined to the *a priori* bounds derived in Section 3, allow to pass to the limit $\varepsilon \to 0$. The approach that we adopted follows closely the one found in [16] and [4]. The convergence results are obtained in Section 5 and are given in Theorems 5.2. Theorem 2.3 is then shown to be a consequence of this theorem. In Section 6, we give an extension to this problem by treating the case of a slab with a varying width.

3 The general framework

Let $\mathcal{H}$ be a separable complex Hilbert space equipped with its inner product $(\cdot,\cdot)_{\mathcal{H}}$ and the associated norm $\|\cdot\|_{\mathcal{H}}$. We consider the following initial value problem:

$$i\varepsilon\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x\psi^\varepsilon + A^\varepsilon\psi^\varepsilon \qquad (3.1)$$

$$\psi^\varepsilon(0,x) = \psi^\varepsilon_I(x), \qquad (x\in\mathbb{R}^m) \qquad (3.2)$$

where the unknown $\psi^\varepsilon(t,x)$ is an $\mathcal{H}$-valued function on $\mathbb{R}_+\times\mathbb{R}^m$ and $A^\varepsilon(t,x)$ is an unbounded operator on $\mathcal{H}$. We first make some hypotheses on the operator A^ε and on ψ^ε.

Assumption 3.1 *For all $(\varepsilon,t,x)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}^m$, $A^\varepsilon(t,x)$ is a densely defined self-adjoint unbounded operator, whose domain $\mathcal{D}(A)$ is independent of (ε,t,x). Moreover we have*

$$\forall(\varepsilon,t,x),\quad \forall\varphi\in\mathcal{D}(A),\qquad (A^\varepsilon(t,x)\varphi,\varphi)_{\mathcal{H}}\geq\|\varphi\|^2_{\mathcal{H}}.$$

Remark that –up to a multiplication of ψ^ε by a phase factor $e^{iCt/\varepsilon}$ in (3.1)– the last assumption is equivalent to saying that A^ε is bounded from below with a lower bound independent of (ε,t,x).

Assumption 3.2 *For any given $(\varepsilon,t,x)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}^m$, the operator $A^\varepsilon(t,x)$ has a compact resolvent.*

Denote by

$$1\leq\epsilon^\varepsilon_1(t,x)<\epsilon^\varepsilon_2(t,x)<...<\epsilon^\varepsilon_p(t,x)<\epsilon^\varepsilon_{p+1}(t,x)<...,$$

the sequence of its eigenvalues counted with their multiplicities, by $\mathcal{H}^\varepsilon_p(t,x)$ the corresponding eigenspaces and by Π^ε_p the orthogonal projectors on these eigenspaces. For $\varepsilon=0$, we shall use the notation Π_p, ϵ_p instead of Π^0_p and ϵ^0_p. Recall that the Π^ε_p are self-adjoint. We define their derivatives $\partial_t\Pi^\varepsilon_p$ and $\nabla_x\Pi^\varepsilon_p$ by the following commutators:

$$\partial_t\Pi^\varepsilon_p=[\partial_t,\Pi^\varepsilon_p]\quad;\quad\nabla_x\Pi^\varepsilon_p=[\nabla_x,\Pi^\varepsilon_p].$$

Assumption 3.3 *For any $p \in \mathbb{N}^*$ and for any $T > 0$, the functions $(\varepsilon, t, x) \mapsto \epsilon_p^\varepsilon(t,x)$ and $(\varepsilon, t, x) \mapsto \Pi_p^\varepsilon(t,x)$, as well as their derivatives with respect to t and x are continuously bounded functions of $(\varepsilon, t, x) \in [0,1] \times [0,T] \times \mathbb{R}^m$, respectively valued in $\mathbb{R}$ and $\mathcal{L}(\mathcal{H})$.*

Let us now assume the existence of a solution to (3.1)–(3.2) in an adequate energy space. Denoting by X_A the completion of $\mathcal{D}(A)$ in $\mathcal{H}$ with respect to the following norm (dependent on t and x):

$$\varphi \mapsto (A^\varepsilon(t,x)\varphi(x), \varphi(x))_{\mathcal{H}}^{1/2},$$

the "energy space" is then

$$H_A^1 = H^1(\mathbb{R}^m, \mathcal{H}) \cap L^2(\mathbb{R}^m, X_A).$$

For $\varphi \in H_A^1$, we set

$$\mathcal{E}^\varepsilon(\varphi) = \int_{\mathbb{R}^m} \|\varphi(x)\|_{\mathcal{H}}^2 \, dx + \varepsilon^2 \int_{\mathbb{R}^m} \|\nabla\varphi(x)\|_{\mathcal{H}}^2 \, dx + \int_{\mathbb{R}^m} (A^\varepsilon(t,x)\varphi(x), \varphi(x))_{\mathcal{H}} \, dx.$$

Assumption 3.4 *For any $\varepsilon > 0$ the initial value problem* (3.1)–(3.2) *admits a weak solution*

$$\psi^\varepsilon \in C^0(\mathbb{R}_+, H_A^1) \cap C^1(\mathbb{R}_+, H^{-1}(\mathbb{R}^m, \mathcal{H})) \cap C^1(\mathbb{R}_+, L^2(\mathbb{R}^m, X_A')).$$

Moreover there exists a continuous function $C(t)$ independent of ε such that

$$\forall t \geq 0 \qquad \mathcal{E}^\varepsilon(\psi^\varepsilon(t, \cdot)) \leq C(t). \tag{3.3}$$

We shall not consider here the minimal hypotheses on A^ε which imply that this Assumption 3.4 is satisfied. Nevertheless, we remark that if the first part of the assumption is satisfied (the existence of ψ^ε), and if in addition we have

$$\forall \varphi \in \mathcal{D}(A) \qquad |(\partial_t A^\varepsilon(t,x)\varphi, \varphi)_{\mathcal{H}}| \leq C_1(t)(A^\varepsilon(t,x)\varphi, \varphi)_{\mathcal{H}}$$

where $C_1(t)$ is a locally bounded function of time, then the second part (3.3) of the assumption can be deduced.

Let us now derive the equation satisfied by the projection of ψ^ε on the eigenspaces of A^ε. We denote $\psi_{p,I}^\varepsilon = \Pi_p^\varepsilon \psi_I^\varepsilon$ and state a last assumption:

Assumption 3.5 *We assume that*

$$\lim_{p \to +\infty} \frac{1}{\inf\limits_{x \in \mathbb{R}^m} \epsilon_p^\varepsilon(t,x)} = 0$$

locally uniformly with respect to $(\varepsilon, t) \in [0,1] \times \mathbb{R}_+$.

Lemma 3.6 *Under Assumptions* 3.1–3.4, $\psi_p^\varepsilon = \Pi_p^\varepsilon \psi^\varepsilon$ *satisfies*

$$\psi_p^\varepsilon \in C^0(\mathbb{R}_+, H^1(\mathbb{R}^m, \mathcal{H})) \cap C^0(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{D}(A)) \cap C^1(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{H})) \tag{3.4}$$

and solves in the distribution sense

$$i\varepsilon\, \partial_t \psi_p^\varepsilon = -\frac{\varepsilon^2}{2}\, \Delta_x \psi_p^\varepsilon + \epsilon_p^\varepsilon\, \psi_p^\varepsilon + R_p^\varepsilon, \tag{3.5}$$

$$\psi_p^\varepsilon(0,x) = \psi_{p,I}^\varepsilon(x), \qquad (x \in \mathbb{R}^m) \tag{3.6}$$

where the remainder writes

$$R_p^\varepsilon = i\varepsilon\, \left(\partial_t \Pi_p^\varepsilon\right) \psi^\varepsilon + \frac{\varepsilon^2}{2}\, \left(\nabla_x \Pi_p^\varepsilon\right) \cdot \left(\nabla_x \psi^\varepsilon\right) + \frac{\varepsilon^2}{2}\, \mathrm{div}_x\, \left(\left(\nabla_x \Pi_p^\varepsilon\right) \psi^\varepsilon\right).$$

If in addition Assumption 3.5 *holds true then* ψ_p^ε *satisfies for any given* $T > 0$

$$\lim_{N\to+\infty} \sup_{\varepsilon\in(0,1]} \sup_{t\in[0,T]} \left\| \sum_{p\geq N} \psi_p^\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R}^m,\mathcal{H})} = 0 \tag{3.7}$$

Proof. To prove the first part of the lemma, we recall that by Assumption 3.3 the following commutators

$$\partial_t \Pi_p^\varepsilon = [\partial_t, \Pi_p^\varepsilon] \quad ; \quad \nabla_x \Pi_p^\varepsilon = [\nabla_x, \Pi_p^\varepsilon]$$

belong to $C^0(\mathbb{R}_+ \times \mathbb{R}^m, \mathcal{L}(\mathcal{H}))$. Moreover we remark that for any (t,x) the operator $\Pi_p^\varepsilon(t,x)$ maps $\mathcal{H}$ to $\mathcal{D}(A)$ and can be prolonged into a bounded operator from $\mathcal{D}(A)'$ to $\mathcal{H}$. Consequently

$$\psi_p^\varepsilon \in C^0(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{D}(A)),$$

$$\nabla_x \psi_p^\varepsilon = \Pi_p^\varepsilon(\nabla_x \psi^\varepsilon) + (\nabla_x \Pi_p^\varepsilon)\psi^\varepsilon \in C^0(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{H}))$$

and

$$\partial_t \psi_p^\varepsilon = \Pi_p^\varepsilon(\partial_t \psi^\varepsilon) + (\partial_t \Pi_p^\varepsilon)\psi^\varepsilon \in C^0(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{H})).$$

This gives (3.4). Equation (3.5) can be obtained by a straightforward calculation (one can check that all the terms in R_p^ε make sense).

Next, the proof of (3.7) is immediate. Indeed, Assumption 3.4 insures that we have for some constant C_T independent of ε

$$\int_{\mathbb{R}^m} \left(A^\varepsilon(t,x)\psi^\varepsilon(t,x)\,,\, \psi^\varepsilon(t,x)\right)_{\mathcal{H}} dx \leq C_T$$

for $\varepsilon \in (0,1]$ and $t \in [0,T]$. Now we remark the left-hand side of this inequality is nothing but

$$\sum_{p\geq 1} \int_{\mathbb{R}^m} \epsilon_p^\varepsilon(t,x) \|\psi_p^\varepsilon(t,x)\|_{\mathcal{H}}^2\, dx$$

which can be bounded from below by

$$\sum_{p\geq 1} \inf_{y\in\mathbb{R}^m} \{\epsilon_p^\varepsilon(t,y)\} \int_{\mathbb{R}^m} \|\psi_p^\varepsilon(t,x)\|_{\mathcal{H}}^2\, dx.$$

Therefore, we have

$$\left\|\sum_{p\geq N} \psi_p^\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R}^m,\mathcal{H})} \leq \frac{\displaystyle\sum_{p\geq N} \inf_{y\in\mathbb{R}^m} \{\epsilon_p^\varepsilon(t,y)\} \int_{\mathbb{R}^m} \|\psi_p^\varepsilon(t,x)\|_{\mathcal{H}}^2\, dx}{\displaystyle\inf_{y\in\mathbb{R}^m} \{\epsilon_N^\varepsilon(t,y)\}} \leq \frac{C_T}{\displaystyle\inf_{y\in\mathbb{R}^m} \{\epsilon_N^\varepsilon(t,y)\}}$$

and the lemma is proved thanks to Assumption 3.5. □

4 Wigner transform

4.1 Definitions

In order to perform the semiclassical limit of the Schrödinger equation (3.1), we need to construct density matrices on the Hilbert space $\mathcal{H}$. This is a generalisation of density matrices on $L^2(\mathbb{R}^m)$ defined for instance in [20]. For this purpose, we start with a little algebra (see [18] or [30] for details).

We first construct the *conjugate tensor product* of $\mathcal{H}$ with itself, denoted by $\mathcal{H}\overline{\otimes}\mathcal{H}$. For $\varphi_1 \in \mathcal{H}$ and $\varphi_2 \in \mathcal{H}$, the *simple tensor* $\varphi_1\overline{\otimes}\varphi_2$ denotes the bilinear form which acts on $\mathcal{H}\times\mathcal{H}$ as follows:

$$\forall(\psi_1,\psi_2)\in\mathcal{H}\times\mathcal{H} \qquad (\varphi_1\overline{\otimes}\varphi_2)(\psi_1,\psi_2) = (\varphi_1,\psi_1)_{\mathcal{H}}\,\overline{(\varphi_2,\psi_2)_{\mathcal{H}}} = (\varphi_1,\psi_1)_{\mathcal{H}}\,(\psi_2,\varphi_2)_{\mathcal{H}}.$$

Next, we define $\mathcal{E}$ as the set of linear combination of simple tensors. This vectorial space is equipped with the inner product $(\cdot,\cdot)_{\mathcal{H}\overline{\otimes}\mathcal{H}}$, defined on simple tensors by

$$(\varphi_1\overline{\otimes}\varphi_2\,,\,\psi_1\overline{\otimes}\psi_2)_{\mathcal{H}\overline{\otimes}\mathcal{H}} = (\varphi_1,\psi_1)_{\mathcal{H}}\,\overline{(\varphi_2,\psi_2)_{\mathcal{H}}}$$

and extended by linearity. Then the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$ is defined as the completion of the space $\mathcal{E}$ with respect to this inner product.

The tensor product $\mathcal{H}\overline{\otimes}\mathcal{H}$ can be identified with the ideal $\mathcal{J}_2 \subset \mathcal{L}(\mathcal{H})$ of the *Hilbert-Schmidt operators* on $\mathcal{H}$. Indeed, there exists an isometric isomorphism from $\mathcal{H}\overline{\otimes}\mathcal{H}$ onto $\mathcal{J}_2$ such that the action of simple tensors on elements of $\mathcal{H}$ is defined as follows:

$$\forall(\varphi_1,\varphi_2)\in\mathcal{H}\times\mathcal{H} \qquad \forall\psi\in\mathcal{H} \qquad (\varphi_1\overline{\otimes}\varphi_2)\psi = \varphi_1\,(\psi,\varphi_2)_{\mathcal{H}}.$$

Thanks to this identification, to describe an element of $\mathcal{H}\overline{\otimes}\mathcal{H}$, we can employ the operator language as well as the Hilbert framework. For instance, if $\mathbb{P}\in\mathcal{L}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$, we have

$$\mathbb{P}(\varphi_1\overline{\otimes}\varphi_2) = (\mathbb{P}\varphi_1)\overline{\otimes}\varphi_2 \quad ; \quad (\varphi_1\overline{\otimes}\varphi_2)\mathbb{P}^* = \varphi_1\overline{\otimes}(\mathbb{P}\varphi_2).$$

Let us introduce two other ideals of $\mathcal{L}(\mathcal{H})$: the subspace of *compact operators* on $\mathcal{H}$, denoted by $\mathrm{Com}(\mathcal{H})$, and the subspace of *trace class operators*, denoted by $\mathcal{J}_1$. We recall that

$$\mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathrm{Com}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}),$$
$$(\mathrm{Com}(\mathcal{H}))' = \mathcal{J}_1, \qquad \mathcal{J}_1' = \mathcal{L}(\mathcal{H}),$$

(see [30]). In particular, for any $\mathbb{L} \in \mathcal{L}(\mathcal{H})$, the application $\rho \mapsto \mathrm{tr}\,(\mathbb{L}\,\rho)$ is a linear form on $\mathcal{J}_1$; this defines the weak topology on $\mathcal{J}_1$. Remark also that

$$\forall (\varphi_1, \varphi_2) \in \mathcal{H} \times \mathcal{H} \qquad \mathrm{tr}\,(\varphi_1 \overline{\otimes} \varphi_2) = (\varphi_1, \varphi_2)_{\mathcal{H}}.$$

As in [20], we are interested by a particular class of operators: the *density matrices* on $\mathcal{H}$. These are the nonnegative trace class Hermitian operators on $\mathcal{H}$. Let ρ be such an operator. Generically, a density matrix takes the form

$$\rho = \sum_{p \in \mathbb{N}^*} \lambda_p \, e_p \overline{\otimes} e_p,$$

where $(e_p)_{p \in \mathbb{N}^*}$ is a Hilbertian basis of $\mathcal{H}$ and $(\lambda_p)_{p \in \mathbb{N}^*}$ is a sequence of real numbers satisfying

$$\forall p \in \mathbb{N}^* \quad \lambda_p \geq 0, \qquad\qquad \mathrm{tr}\,\rho = \sum_{p \in \mathbb{N}^*} \lambda_p < +\infty.$$

In analogy with [16, 20], we now introduce the tool which will enable to derive the semiclassical limit of (3.1): the Wigner transform. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two $\mathcal{H}$-valued distributions which belong to $\in \mathcal{S}'(\mathbb{R}^m_x, \mathcal{H})$. For $\varepsilon > 0$, the *Wigner transform* of φ_1 and φ_2 is the $\mathcal{H}$-valued distribution on $\mathbb{R}^m_x \times \mathbb{R}^m_v$ defined by

$$W^\varepsilon(\varphi_1, \varphi_2)(x, v) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\eta \cdot v} \, \varphi_1 \left(x - \varepsilon \frac{\eta}{2} \right) \overline{\otimes} \, \varphi_2 \left(x + \varepsilon \frac{\eta}{2} \right) \, d\eta.$$

This defines a continuous sesquilinear mapping from $\mathcal{S}'(\mathbb{R}^m_x, \mathcal{H}) \times \mathcal{S}'(\mathbb{R}^m_x, \mathcal{H})$ to $\mathcal{S}'(\mathbb{R}^m_x \times \mathbb{R}^m_v, \mathcal{H} \overline{\otimes} \mathcal{H})$.

Example 4.1 *To illustrate this, the standard example is* $\mathcal{H} = L^2(\mathbb{R}^D, \mathbb{C})$. *Then* $\mathcal{H} \overline{\otimes} \mathcal{H}$ *can be identified with* $L^2(\mathbb{R}^D \times \mathbb{R}^D)$ *thanks to* $(\varphi_1 \overline{\otimes} \varphi_2)(z, z') = \varphi_1(z) \, \overline{\varphi_2}(z')$. *Let* $\varphi_1(x, z)$, $\varphi_2(x, z)$ *be in* $\in \mathcal{S}'(\mathbb{R}^m_x, L^2(\mathbb{R}^D_z))$. *We have*

$$W^\varepsilon(\varphi_1, \varphi_2)(x, v, z, z') = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\eta \cdot v} \, \varphi_1 \left(x - \varepsilon \frac{\eta}{2}, z \right) \, \overline{\varphi}_2 \left(x + \varepsilon \frac{\eta}{2}, z' \right) \, d\eta. \quad (4.1)$$

4.2 General properties of the Wigner transform

In this section, we adapt some results on the Wigner transforms which can be found in [16] and [20]. We introduce the vector space of $\mathcal{H} \overline{\otimes} \mathcal{H}$-valued test functions:

$$\mathcal{A}_{\mathcal{H}} = \left\{ \varphi \in C^0_c(\mathbb{R}^m_x \times \mathbb{R}^m_v, \mathcal{H} \overline{\otimes} \mathcal{H}) \big/ (\mathcal{F}_v \varphi)(x, \eta) \in L^1(\mathbb{R}^m_\eta, C^0_c(\mathbb{R}^m_x, \mathcal{H} \overline{\otimes} \mathcal{H})) \right\}$$

(C_c^0 denotes the space of compactly supported functions). When equipped with the norm

$$\|\mathcal{F}_v\varphi\|_{L^1(\mathbb{R}^m_\eta,(C^0(\mathbb{R}^m_x,\mathcal{H}\overline{\otimes}\mathcal{H}))}\,,$$

$\mathcal{A}_\mathcal{H}$ is a separable Banach space. We will show that if (ψ^ε) is a bounded sequence in $L^2(\mathbb{R}^m,\mathcal{H})$, then (a subsequence of) the Wigner transform $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ converges to a nonnegative measure called Wigner measure.

Let us now define the equivalent of the macroscopic quantities introduced in Section 2. Let $\psi^\varepsilon\in H^1(\mathbb{R}^m_x,\mathcal{H})$. We associate to this wavefunction the density matrix $\psi^\varepsilon(x)\overline{\otimes}\psi^\varepsilon(x')$. Then we define a "density" and a "current" by

$$N^\varepsilon(x)=\psi^\varepsilon(x)\overline{\otimes}\psi^\varepsilon(x)\quad;\quad J^\varepsilon(x)=\frac{\varepsilon}{2i}\left[(\nabla_x\psi^\varepsilon(x))\overline{\otimes}\psi^\varepsilon(x)-\psi^\varepsilon(x)\overline{\otimes}\nabla_x\psi^\varepsilon(x)\right].$$

We have obviously $N^\varepsilon\in L^1(\mathbb{R}^m,\mathcal{J}_1)$ and $J^\varepsilon\in(L^1(\mathbb{R}^m,\mathcal{J}_1))^m$. In order to prove the convergence of these macroscopic quantities, we will need two properties. The first one says that the wavefunction ψ^ε is *ε-oscillatory* [16]:

$$\varepsilon^2\int_{\mathbb{R}^m}\|\nabla_x\psi^\varepsilon\|^2_\mathcal{H}\,dx\le C. \tag{4.2}$$

The second property is a compactness property:

There exists a sequence $(\mathbb{P}^\varepsilon_n)_{n\in\mathbb{N}}$ of functions in $C^0([0,1]_\varepsilon\times\mathbb{R}^m_x,\mathrm{Com}(\mathcal{H}))$ such that

$$\left\{\begin{array}{l}\lim_{n\to\infty}\mathbb{P}^0_n=\mathbb{I}\text{ in the }\mathcal{L}(\mathcal{H})\text{ weak }*\text{ topology, locally uniformly w.r.t. }x\\ \qquad\lim_{n\to\infty}\limsup_{\varepsilon\to0}\|(\mathbb{I}-\mathbb{P}^\varepsilon_n)\psi^\varepsilon\|_{L^2(\mathbb{R}^m,\mathcal{H})}=0.\end{array}\right. \tag{4.3}$$

Proposition 4.2 *Let ψ^ε be a bounded family of $L^2(\mathbb{R}^m_x,\mathcal{H})$. Then we have*

$$\|W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)\|_{\mathcal{A}'_\mathcal{H}}\le\|\psi^\varepsilon\|^2_{L^2(\mathbb{R}^m_x,\mathcal{H})}\,.$$

After extraction of a subsequence, $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ converges to W^0 in $\mathcal{A}'_\mathcal{H}$ weak $*$. The limit W^0 is a bounded measure with values in the space of nonnegative Hermitian $\mathcal{J}_1$ operators and is called the Wigner measure *associated to this subsequence of ψ^ε. Furthermore, if ψ^ε satisfies* (4.2) *and* (4.3) *then we have*

$$N^\varepsilon\rightharpoonup\int_{\mathbb{R}^m}W^0(\cdot,v)dv\quad\text{in }\mathcal{M}_b(\mathbb{R}^m,\mathcal{J}_1-\text{weak})\text{ weak }* \tag{4.4}$$

and

$$J^\varepsilon\rightharpoonup\int_{\mathbb{R}^m}v\,W^0(\cdot,v)dv\quad\text{in }\mathcal{M}_b(\mathbb{R}^m,\mathcal{J}_1-\text{weak})\text{ weak }*. \tag{4.5}$$

Proof. This proposition is mostly a generalization of the results on the matrix Wigner transform proved in [20] and [16]. The bound of $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ in $\mathcal{A}'_\mathcal{H}$ can be obtained by an immediate adaptation of [20, Proposition III.1]. In parallel the bounds of ψ^ε and $\varepsilon\nabla_x\psi^\varepsilon$ imply that N^ε and J^ε are bounded in $L^1(\mathbb{R}^m, \mathcal{J}_1)$. Hence after extraction of sequences we have

$$W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon) \rightharpoonup W^0 \quad \text{in } \mathcal{A}'_{\mathcal{H}} \text{ weak } *,$$

$$N^\varepsilon \rightharpoonup N^0 \quad \text{and} \quad J^\varepsilon \rightharpoonup J^0 \quad \text{in } \mathcal{M}_b(\mathbb{R}^m, \mathcal{J}_1) \text{ weak } *.$$

This means that

$$\operatorname{tr} \int_{\mathbb{R}^m} \mathbb{L}(x)\, N^\varepsilon(x)\, dx \to \operatorname{tr} \int_{\mathbb{R}^m} \mathbb{L}(x)\, N^0(x)\, dx$$

for any function $\mathbb{L}$ in $C_c^0(\mathbb{R}_x^m, \mathrm{Com}(\mathcal{H}))$ (and analogously for J^ε). However, we shall see in the proof of Theorem 2.3 at the end of Section 5, that we need to pass to the limit for operators $\mathbb{L}$ which are in $C_c^0(\mathbb{R}_x^m, \mathcal{L}(\mathcal{H}))$ which is the topology given in (4.4).

Let us first prove that W^0 belongs to $\mathcal{M}_b(\mathbb{R}_x^m, \mathcal{J}_1)$. To this aim, we proceed like in [20] and consider the *Husimi transform* of W^ε:

$$\widetilde{W^\varepsilon} = W^\varepsilon * G^\varepsilon,$$

where

$$G^\varepsilon = \frac{1}{(\pi\varepsilon)^m}\, e^{-(|x|^2+|v|^2)/\varepsilon}$$

and the convolution is taken with respect to $(x, v) \in \mathbb{R}^{2m}$. The Husimi transform has the following properties:

(i) $\widetilde{W^\varepsilon} \geq 0$ in the sense of operators on $\mathcal{H}$;

(ii) $\displaystyle\iint_{\mathbb{R}^{2m}} \operatorname{tr} \widetilde{W^\varepsilon}\, dx\, dv = \|\psi^\varepsilon\|^2_{L^2(\mathbb{R}^m, \mathcal{H})}$;

(iii) if $U \in \mathcal{A}_{\mathcal{H}}$ then $U * G^\varepsilon$ converges to U in $\mathcal{A}_{\mathcal{H}}$ as $\varepsilon \to 0$.

Properties (i) and (ii) imply that $\widetilde{W^\varepsilon}$ is bounded in $L^1(\mathbb{R}^{2m}, \mathcal{J}_1)$, hence a subsequence converges in the $\mathcal{M}_b(\mathbb{R}^{2m}_{x,v}; \mathcal{J}_1)$ weak $*$ topology and this measure is nonnegative (in the sense of operators on $\mathcal{H}$). Property (iii) implies that the subsequences of $\widetilde{W^\varepsilon}$ and W^ε have the same weak limit W^0.

Let us now prove the convergence of N^ε in the sense of (4.4). The proof of (4.5) can be done with the same argument and is skipped here. Consider a function $\mathbb{L}(x) \in C_c^0(\mathbb{R}^m, \mathcal{L}(\mathcal{H}))$ and let $\mathbb{P}_n^\varepsilon(x)$ be a sequence such as in Property (4.3). We are interested in the limit of $\operatorname{tr} \int_{\mathbb{R}^m} \mathbb{L}(x)\, N^\varepsilon(x)\, dx$. Since $N^\varepsilon \rightharpoonup N^0$ in $\mathcal{M}_b(\mathbb{R}_x^m, \mathcal{J}_1)$ weak $*$ and since, as $\varepsilon \to 0$, $\mathbb{P}_n^\varepsilon$ converges to $\mathbb{P}_n^0$ locally uniformly on $\mathbb{R}^m$ in the $\mathcal{L}(\mathcal{H})$ norm topology, we have for any $n \in \mathbb{N}$, as $\varepsilon \to 0$,

$$\operatorname{tr} \int_{\mathbb{R}^m} \mathbb{P}_n^\varepsilon(x)\, \mathbb{L}(x)\, N^\varepsilon(x)\, dx \to \operatorname{tr} \int_{\mathbb{R}^m} \mathbb{P}_n^0(x)\, \mathbb{L}(x)\, N^0(x)\, dx.$$

By (4.3), it is clear that the right-hand side converges to tr $\int_{\mathbb{R}^m} \mathbb{L}(x)\, N^0(x)\, dx$ as $n \to +\infty$ and that

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \operatorname{tr} \int_{\mathbb{R}^m} (\mathbb{I} - \mathbb{P}_n^\varepsilon(x))\, \mathbb{L}(x)\, \psi^\varepsilon(x) \overline{\otimes} \psi^\varepsilon(x)\, dx = 0.$$

Hence

$$\operatorname{tr} \int_{\mathbb{R}^m} \mathbb{L}(x)\, N^\varepsilon(x)\, dx \to \operatorname{tr} \int_{\mathbb{R}^m} \mathbb{L}(x)\, N^0(x)\, dx \qquad \text{as } \varepsilon \to 0$$

and the convergence of N^ε holds in the $\mathcal{M}_b(\mathbb{R}^m, \mathcal{J}_1\text{-weak})$ weak $*$ topology. Besides we know that

$$\int_{\mathbb{R}^m} \widetilde{W^\varepsilon}\, dx = (2\pi\varepsilon)^{-m} \left(\widehat{\psi^\varepsilon} \overline{\otimes} \widehat{\psi^\varepsilon}(\frac{v}{\varepsilon}) \right) *_v \frac{e^{-|v|^2/\varepsilon}}{(\pi\varepsilon)^{m/2}} \tag{4.6}$$

(in this expression $\widehat{\ }$ denotes the Fourier transform). Choose a function $\varphi \in C_c^\infty(\mathbb{R}^m)$, $0 \le \varphi \le 1$ with $\varphi(v) = 1$ in a neighbourhood of $v = 0$. From (4.2) and (4.6) we can deduce that

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \operatorname{tr} \iint_{\mathbb{R}^{2m}} (1 - \varphi(v/R)) \widetilde{W^\varepsilon}\, dx\, dv = 0.$$

Therefore, as $\varepsilon \to 0$ and for any n, we have

$$\operatorname{tr} \iint_{\mathbb{R}^{2m}} \mathbb{P}_n^\varepsilon(x) \mathbb{L}(x) \widetilde{W^\varepsilon}(x, v)\, dx\, dv \to \operatorname{tr} \iint_{\mathbb{R}^{2m}} \mathbb{P}_n^0(x) \mathbb{L}(x) W^0(x, v)\, dx\, dv.$$

By using the identity

$$\int_{\mathbb{R}^m} \widetilde{W^\varepsilon}\, dv = N^\varepsilon *_x \frac{e^{-|x|^2/\varepsilon}}{(\pi\varepsilon)^{m/2}}$$

and by letting $n \to \infty$, we conclude that $N^0 = \int W^0(\cdot, v)\, dv$. □

We now state three lemmas which will be useful further to pass to the limit in the Wigner equation. The first one is given without proof and can be obtained by a simple integration by parts:

Lemma 4.3 *Let φ, $\psi \in \mathcal{S}'(\mathbb{R}^m, \mathcal{H})$. We have*

$$\frac{\varepsilon}{2} \left(W^\varepsilon(\nabla_x \varphi, \psi) - W^\varepsilon(\varphi, \nabla_x \psi) \right) = i\, v\, W^\varepsilon(\varphi, \psi). \tag{4.7}$$

Lemma 4.4 *Let $T > 0$ and let $\varphi^\varepsilon(t, x), \psi^\varepsilon(t, x)$ be two families in $\mathcal{H}$ indexed by $(\varepsilon, t, x) \in (0, 1] \times \mathbb{R}^m \times (0, T)$. We assume that $\varphi^\varepsilon(t, \cdot)$, $\psi^\varepsilon(t, \cdot)$ are bounded in $L^2(\mathbb{R}^m, \mathcal{H})$ uniformly with respect to $t \in (0, T)$ and $\varepsilon \in (0, 1]$. Let $(\varepsilon, t, x) \mapsto \mathbb{P}^\varepsilon(t, x)$ and $(\varepsilon, t, x) \mapsto \mathbb{Q}^\varepsilon(t, x)$ be two continuous mappings from $[0, 1] \times [0, T] \times \mathbb{R}^m$ with values in $\mathcal{L}(\mathcal{H})$. Assume that*

$$\sup_{(\varepsilon,t,x)\in[0,1]\times[0,T]\times\mathbb{R}^m} \|\mathbb{P}^\varepsilon(t, x)\|_{\mathcal{L}(\mathcal{H})} + \sup_{(\varepsilon,t,x)\in[0,1]\times[0,T]\times\mathbb{R}^m} \|\mathbb{Q}^\varepsilon(t, x)\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Then we have

$$W^\varepsilon(\mathbb{P}^\varepsilon\,\varphi^\varepsilon,\mathbb{Q}^\varepsilon\psi^\varepsilon)-\mathbb{P}^\varepsilon\,W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)(\mathbb{Q}^\varepsilon)^*\rightharpoonup 0\quad \textit{as } \varepsilon\to 0$$

in $\mathcal{A}'_{\mathcal{H}}$ *weak* $*$*, uniformly with respect to* t *on* $(0,T)$*.*

Proof. This proof is adapted from [20]. To simplify and without loss of generality, we can assume that $\mathbb{Q}^\varepsilon=\mathbb{I}$. Consider a test-function $\Theta\in\mathcal{S}(\mathbb{R}^m_x\times\mathbb{R}^m_v,\mathcal{H}\overline{\otimes}\mathcal{H})$ such that $\mathcal{F}_v\Theta\in C^\infty_c(\mathbb{R}^m_x\times\mathbb{R}^m_\eta,\,\mathcal{H}\overline{\otimes}\mathcal{H})$. We can proceed by a density argument since the set of such test-functions is densely embedded in $\mathcal{A}_{\mathcal{H}}$ and, by Proposition 4.2, we have

$$\|W^\varepsilon(\mathbb{P}^\varepsilon\,\varphi^\varepsilon,\psi^\varepsilon)\|_{\mathcal{A}'_{\mathcal{H}}}\le C,\qquad \|\mathbb{P}^\varepsilon\,W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)\|_{\mathcal{A}'_{\mathcal{H}}}\le C.$$

For $t\in(0,T)$, let

$$\begin{aligned}R^\varepsilon(t)\ &=\langle W^\varepsilon(\mathbb{P}^\varepsilon\,\varphi^\varepsilon,\psi^\varepsilon)\,,\,\Theta\rangle_{\mathcal{A}'_{\mathcal{H}},\mathcal{A}_{\mathcal{H}}}-\langle \mathbb{P}^\varepsilon\,W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)\,,\,\Theta\rangle_{\mathcal{A}'_{\mathcal{H}},\mathcal{A}_{\mathcal{H}}}\\ &=(2\pi)^{-m}\iiint_{\mathbb{R}^{3m}}e^{i\eta\cdot v}\Big(\Big\{(\mathbb{P}^\varepsilon(x-\varepsilon\frac{\eta}{2})-\mathbb{P}^\varepsilon(x))\,\varphi^\varepsilon(x-\varepsilon\frac{\eta}{2})\Big\}\,\overline{\otimes}\psi^\varepsilon(x+\varepsilon\frac{\eta}{2}),\\ &\qquad\qquad ,\Theta(x,v)\Big)_{\mathcal{H}\overline{\otimes}\mathcal{H}}\,d\eta\,dx\,dv\\ &=\iint_{\mathbb{R}^{2m}}\Big(\Big\{(\mathbb{P}^\varepsilon(x-\varepsilon\frac{\eta}{2})-\mathbb{P}^\varepsilon(x))\,\varphi^\varepsilon(x-\varepsilon\frac{\eta}{2})\Big\}\,\overline{\otimes}\psi^\varepsilon(x+\varepsilon\frac{\eta}{2}),\\ &\qquad\qquad ,(\mathcal{F}_v\Theta)(x,\eta)\Big)_{\mathcal{H}\overline{\otimes}\mathcal{H}}\,dx\,d\eta\,.\end{aligned}$$

Hence

$$\begin{aligned}|R^\varepsilon(t)|\ &\le\iint_{\mathbb{R}^{2m}}\Big\|\mathbb{P}^\varepsilon(x-\varepsilon\frac{\eta}{2})-\mathbb{P}^\varepsilon(x)\Big\|_{\mathcal{L}(\mathcal{H})}\Big\|\varphi^\varepsilon(x-\varepsilon\frac{\eta}{2})\Big\|_{\mathcal{H}}\Big\|\psi^\varepsilon(x+\varepsilon\frac{\eta}{2})\Big\|_{\mathcal{H}}\times\\ &\qquad\times\|(\mathcal{F}_v\Theta)(x,\eta)\|_{\mathcal{H}\overline{\otimes}\mathcal{H}}\,dx\,d\eta\,.\end{aligned}\tag{4.8}$$

Let $\mathcal{K}\subset\mathbb{R}^{2m}$ be the compact support of $\mathcal{F}_v\Theta$. Then we have

$$|R^\varepsilon(t)|\ \le\sup_{(x,\eta)\in\mathcal{K}}\Big\|\mathbb{P}^\varepsilon(x-\varepsilon\frac{\eta}{2})-\mathbb{P}^\varepsilon(x)\Big\|_{\mathcal{L}(\mathcal{H})}\,\|\varphi^\varepsilon\|_{L^2(\mathbb{R}^m,\mathcal{H})}\,\|\psi^\varepsilon\|_{L^2(\mathbb{R}^m,\mathcal{H})}\,\|\Theta\|_{\mathcal{A}_{\mathcal{H}}}\,.$$

Since $\mathbb{P}^\varepsilon$ depends continuously on (ε,t,x), we deduce that $R^\varepsilon\to 0$ as $\varepsilon\to 0$ locally uniformly with respect to t. □

If the dependence of $\mathbb{P}^\varepsilon(t,x)$ in the variable x is more regular, one can get an expansion of $W^\varepsilon(\mathbb{P}^\varepsilon\varphi^\varepsilon,\psi^\varepsilon)$ with respect to ε. Once again, this result –given here without proof– is directly adapted from [16] or [20] where it is stated in the stationary case for scalar or matrix Wigner functions:

Lemma 4.5 *Let* φ^ε*,* ψ^ε *be as in Lemma* 4.4 *and let* $\mathbb{P}^\varepsilon(t,x)$ *be an* $\mathcal{L}(\mathcal{H})$*-valued function. We assume that* $\mathbb{P}^\varepsilon$ *and* $\nabla_x\mathbb{P}^\varepsilon$ *are continuous and bounded functions of the variables* $(\varepsilon,t,x)\in[0,1]\times[0,T]\times\mathbb{R}^m$ *to* $\mathcal{L}(\mathcal{H})$*. Then, we have the following Anszäts*

$$W^\varepsilon(\mathbb{P}^\varepsilon\,\varphi^\varepsilon,\psi^\varepsilon)=\mathbb{P}^\varepsilon\,W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)+\frac{i}{2}\,\varepsilon\,\nabla_x\mathbb{P}^\varepsilon\cdot\nabla_vW^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)+\varepsilon\,r^\varepsilon_1,\tag{4.9}$$

$$W^\varepsilon(\varphi^\varepsilon,\mathbb{P}^\varepsilon\,\psi^\varepsilon)=W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)(\mathbb{P}^\varepsilon)^*-\frac{i}{2}\,\varepsilon\,\nabla_v W^\varepsilon(\varphi^\varepsilon,\psi^\varepsilon)\cdot(\nabla_x\mathbb{P}^\varepsilon)^*+\varepsilon\,r_2^\varepsilon, \tag{4.10}$$

where $r_1^\varepsilon\rightharpoonup 0$ *and* $r_2^\varepsilon\rightharpoonup 0$ *in* $\mathcal{S}'(\mathbb{R}^{2m},\mathcal{H}\overline{\otimes}\mathcal{H})$ *weak* $*$ *uniformly with respect to* $t\in(0,T)$ *as* ε *goes to 0.*

5 The Limit $\varepsilon\to 0$

Let ψ^ε be the solution of the Schrödinger equation (3.1)-(3.2) given by Assumption 3.4. By (3.3), it is clear that for all $t\geq 0$, $\psi^\varepsilon(t,\cdot)$ is a bounded family in $L^2(\mathbb{R}^m,\mathcal{H})$, which satisfies (4.2). Moreover, by setting

$$\mathbb{P}_n^\varepsilon=\sum_{p\leq n}\Pi_p^\varepsilon,$$

and by applying (3.7) and Assumption 3.3, we deduce that the second condition (4.3) is satisfied locally uniformly in time. Consequently, Proposition 4.2 applies; in particular a subsequence of $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ converges to W^0 in $L^\infty(\mathbb{R}_+,\mathcal{A}'_{\mathcal{H}})$ weak $*$. Let us define the Wigner functions

$$W_{p,q}^\varepsilon=W^\varepsilon(\Pi_p^\varepsilon(\psi^\varepsilon),\Pi_q^\varepsilon(\psi^\varepsilon))=W^\varepsilon(\psi_p^\varepsilon,\psi_q^\varepsilon)$$

(for notational simplicity, we have dropped the (t,x) dependence of Π_p^ε and ψ_p^ε). For $p=q$, we will simply denote $W_p^\varepsilon=W_{p,p}^\varepsilon$. A simple algebra starting from (3.5) shows that $W_{p,q}^\varepsilon$ satisfies

$$\begin{aligned}\partial_t W_{p,q}^\varepsilon= & -\frac{i}{\varepsilon}\left(W^\varepsilon(\epsilon_p^\varepsilon\psi_p^\varepsilon,\psi_q^\varepsilon)-W^\varepsilon(\psi_p^\varepsilon,\epsilon_q^\varepsilon\psi_q^\varepsilon)\right)\\ & +\frac{i\varepsilon}{2}\left(W^\varepsilon(\Delta_x\psi_p^\varepsilon,\psi_q^\varepsilon)-W^\varepsilon(\psi_p^\varepsilon,\Delta_x\psi_q^\varepsilon)\right)\\ & +W^\varepsilon((\partial_t\Pi_p^\varepsilon)\psi^\varepsilon,\psi_q^\varepsilon)+W^\varepsilon(\psi_p^\varepsilon,(\partial_t\Pi_q^\varepsilon)\psi^\varepsilon)\\ & -\frac{i\varepsilon}{2}W^\varepsilon([\Delta_x,\Pi_p^\varepsilon]\psi^\varepsilon,\psi_q^\varepsilon)+\frac{i\varepsilon}{2}W^\varepsilon(\psi_p^\varepsilon,[\Delta_x,\Pi_q^\varepsilon]\psi^\varepsilon).\end{aligned} \tag{5.1}$$

Thanks to Lemma 4.3, we can simplify the second line of this equation:

$$\frac{i\varepsilon}{2}\left(W^\varepsilon(\Delta_x\psi_p^\varepsilon,\psi_q^\varepsilon)-W^\varepsilon(\psi_p^\varepsilon,\Delta_x\psi_q^\varepsilon)\right)=-v\cdot\nabla_x W_{p,q}^\varepsilon. \tag{5.2}$$

Moreover, straightforward calculations enable to write the last line of (5.1) in a more handy way:

$$-\frac{i\varepsilon}{2}W^\varepsilon\left([\Delta_x,\Pi_p^\varepsilon]\psi^\varepsilon,\psi_q^\varepsilon\right)+\frac{i\varepsilon}{2}W^\varepsilon\left(\psi_p^\varepsilon,[\Delta_x,\Pi_q^\varepsilon]\psi^\varepsilon\right)=S^\varepsilon+\operatorname{div}_x T^\varepsilon, \tag{5.3}$$

with

$$\begin{aligned}
S^\varepsilon = \quad & \frac{i}{2}W^\varepsilon\left((\nabla_x\Pi_p^\varepsilon)\psi^\varepsilon, \Pi_q^\varepsilon(\varepsilon\nabla_x\psi^\varepsilon)\right) - \frac{i}{2}W^\varepsilon\left(\Pi_p(\varepsilon\nabla_x\psi^\varepsilon), (\nabla_x\Pi_q^\varepsilon)\psi^\varepsilon)\right) \\
& -\frac{i}{2}W^\varepsilon\left((\nabla_x\Pi_p^\varepsilon)\cdot(\varepsilon\nabla_x\psi^\varepsilon), \Pi_q^\varepsilon\psi^\varepsilon\right) + \frac{i}{2}W^\varepsilon\left(\Pi_p^\varepsilon\psi^\varepsilon, (\nabla_x\Pi_q^\varepsilon)\cdot(\varepsilon\nabla_x\psi^\varepsilon)\right), \\
T^\varepsilon = \quad & -\frac{i\varepsilon}{2}W^\varepsilon\left((\nabla_x\Pi_p^\varepsilon)\psi^\varepsilon, \Pi_q^\varepsilon\psi^\varepsilon\right) + \frac{i\varepsilon}{2}W^\varepsilon\left(\Pi_p^\varepsilon\psi^\varepsilon, (\nabla_x\Pi_q^\varepsilon)\psi^\varepsilon\right).
\end{aligned}$$

Lemma 5.1 *Let $p \neq q$. Then we have*

$$W_{p,q}^\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{2m}, \mathcal{H}\overline{\otimes}\mathcal{H}) \quad \text{as } \varepsilon \to 0.$$

Proof. Consider a test function $\theta \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^{2m}, \mathcal{H}\overline{\otimes}\mathcal{H})$. We choose $T > 0$ such that $\theta(t, \cdot) = 0$ for $t > T$. By using (5.1), (5.2) and (5.3) we get

$$\begin{aligned}
\iiint_{\mathbb{R}^{2m+1}} & \left(\left\{W^\varepsilon(\epsilon_p^\varepsilon\psi_p^\varepsilon, \psi_q^\varepsilon) - W^\varepsilon(\psi_p^\varepsilon, \epsilon_q^\varepsilon\psi_q^\varepsilon)\right\}, \theta\right)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \\
& = -i\varepsilon \iiint_{\mathbb{R}^{2m+1}} \left(W_{p,q}^\varepsilon, (\partial_t\theta + v\cdot\nabla_x\theta)\right)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \\
& \quad - i\varepsilon \iiint_{\mathbb{R}^{2m+1}} (R^\varepsilon + S^\varepsilon, \theta)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \\
& \quad + i\varepsilon \iiint_{\mathbb{R}^{2m+1}} (T^\varepsilon, \nabla_x\theta)_{(\mathcal{H}\overline{\otimes}\mathcal{H})^m} \, dx\, dv\, dt,
\end{aligned} \tag{5.4}$$

where S^ε and T^ε are given above and

$$R^\varepsilon = W^\varepsilon((\partial_t\Pi_p^\varepsilon)\psi^\varepsilon, \psi_q^\varepsilon) + W^\varepsilon(\psi_p^\varepsilon, (\partial_t\Pi_q^\varepsilon)\psi^\varepsilon).$$

By Assumption 3.3, ϵ_p^ε depends continuously on (ε, t, x) and is bounded on $[0,1] \times [0,T] \times \mathbb{R}^m$. Moreover, $\psi^\varepsilon(t,\cdot)$ is bounded in $L^2(\mathbb{R}^m, \mathcal{H})$ uniformly with respect to (ε, t) (Assumption 3.4). Consequently, Lemma 4.4 implies

$$\begin{aligned}
\iiint_{\mathbb{R}^{2m+1}} & \left(\left\{W^\varepsilon(\epsilon_p^\varepsilon\psi_p^\varepsilon, \psi_q^\varepsilon) - W^\varepsilon(\psi_p^\varepsilon, \epsilon_q^\varepsilon\psi_q^\varepsilon)\right\}, \theta\right)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \\
& - \iiint_{\mathbb{R}^{2m+1}} (\epsilon_p^\varepsilon - \epsilon_q^\varepsilon)\,(W_{p,q}^\varepsilon, \theta)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \quad \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{5.5}$$

Let us now treat the right-hand side of (5.4). Thanks to the uniform $L^2(\mathbb{R}^m, \mathcal{H})$ bound on ψ^ε and to Proposition 4.2, $W_{p,q}^\varepsilon$ is bounded in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$, uniformly with respect to ε. Therefore

$$-i\,\varepsilon \iiint_{\mathbb{R}^{1+2m}} \left(W_{p,q}^\varepsilon, (\partial_t\theta + v\cdot\nabla_x\theta)\right)_{\mathcal{H}\overline{\otimes}\mathcal{H}} \, dx\, dv\, dt \quad \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.6}$$

On the other hand, the regularity of Π_p^ε, $\partial_t\Pi_p^\varepsilon$ and $\nabla_x\Pi_p^\varepsilon$ in ε, t and x and their uniform boundedness, as well as the boundedness of ψ^ε and $\varepsilon\nabla_x\psi^\varepsilon$ in $L^\infty((0,T), L^2(\mathbb{R}^m, \mathcal{H}))$ imply:

R^ε and S^ε are bounded in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$,

$$T^\varepsilon \text{ is bounded in } L^\infty((0,T),\mathcal{A}'_{\mathcal{H}})^m.$$

Consequently, we have

$$\varepsilon \int_0^T \iint_{\mathbb{R}^{2m}} \left((R^\varepsilon + S^\varepsilon, \theta)_{\mathcal{H}\overline{\otimes}\mathcal{H}} - (T^\varepsilon, \nabla_x \theta)_{(\mathcal{H}\overline{\otimes}\mathcal{H})^m} \right) dt\, dx\, dv \quad \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.7}$$

Finally from (5.4), (5.5), (5.6) and (5.7) we deduce

$$(\epsilon_p^\varepsilon - \epsilon_q^\varepsilon) W_{p,q}^\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2m}, \mathcal{H}\overline{\otimes}\mathcal{H}).$$

By using the regularity of ϵ_p^ε and ϵ_q^ε with respect to $\varepsilon \in [0,1]$ and $(\epsilon_p - \epsilon_q)(p-q) > 0$, we get the result. □

Since the Cauchy data ψ_I^ε is bounded in $L^2(\mathbb{R}^m, \mathcal{H})$, by Proposition 4.2 and after extraction of a subsequence, we have

$$W^\varepsilon(\psi_I^\varepsilon, \psi_I^\varepsilon) \rightharpoonup W_I^0 \quad \text{as } \varepsilon \to 0$$

in $\mathcal{A}'_{\mathcal{H}}$ weak $*$. Let $W_{p,I}^0 = \Pi_p W_I^0 \Pi_p$. We recall the definitions of the macroscopic quantities:

$$N^\varepsilon = \psi^\varepsilon \overline{\otimes} \psi^\varepsilon \quad ; \quad J^\varepsilon = \frac{\varepsilon}{2i}\left[(\nabla_x \psi^\varepsilon)\overline{\otimes}\psi^\varepsilon - \psi^\varepsilon \overline{\otimes}(\nabla_x \psi^\varepsilon)\right].$$

The main result of this Section is the

Theorem 5.2 *Let ψ^ε, W_p^ε, $W_{p,I}^0$, N^ε, J^ε be defined as above and let (ε) denote a sequence tending to zero. Then, up to a diagonal extraction, we have the following convergence results for $\varepsilon \to 0$:*
(i) W_p^ε converges, for every $p \in \mathbb{N}^*$ and locally uniformly on $\mathbb{R}_+$ in the $\mathcal{A}'_{\mathcal{H}}$ weak $*$ topology to $W_p^0 \in C^0(\mathbb{R}_+, \mathcal{M}_b(\mathbb{R}^{2m}, \mathcal{J}_1))$, which is nonnegative (in the sense of operators) and solves

$$\begin{aligned} \partial_t W_p^0 + v\cdot\nabla_x W_p^0 - \nabla_x \epsilon_p \cdot \nabla_v W_p^0 = {} & (\partial_t \Pi_p) W_p^0 \Pi_p + \Pi_p W_p^0 (\partial_t \Pi_p) \\ & + v\cdot(\nabla_x \Pi_p) W_p^0 \Pi_p + v\cdot \Pi_p W_p^0 (\nabla_x \Pi_p) \end{aligned} \tag{5.8}$$

$$W_p^0(0,x,v) = W_{p,I}^0(x,v). \tag{5.9}$$

(ii) $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ converges in the $L^\infty(\mathbb{R}_+, \mathcal{A}'_{\mathcal{H}})$ weak $*$ topology to

$$W^0 = \sum_{p\in\mathbb{N}^*} W_p^0 \quad \in C^0(\mathbb{R}_+, \mathcal{M}_b(\mathbb{R}^{2m}, \mathcal{J}_1)) \tag{5.10}$$

and we have

$$\forall p \in \mathbb{N}^* \qquad W_p^0 = \Pi_p W^0 \Pi_p.$$

(iii) For all $T > 0$, N^ε *and* J^ε *converge in the* $L^\infty((0,T),\mathcal{M}_b(\mathbb{R}^m,\mathcal{J}_1 - weak))$ *weak* $*$ *topology respectively to*

$$N^0 = \sum_{p\in\mathbb{N}^*}\int_{\mathbb{R}^m} W_p^0 dv \quad and \quad J^0 = \sum_{p\in\mathbb{N}^*}\int_{\mathbb{R}^m} vW_p^0 dv, \tag{5.11}$$

which belong to $C^0([0,T],\mathcal{M}_b(\mathbb{R}^m,\mathcal{J}_1))$.
(iv) The surface density $n_s^\varepsilon = \operatorname{tr} N^\varepsilon$ *and the surface current density* $j_s^\varepsilon = \operatorname{tr} J^\varepsilon$ *converge locally uniformly on* $\mathbb{R}_+$ *in the* $\mathcal{M}_b(\mathbb{R}^m)$ *weak* $*$ *topology respectively to* $n_s^0 = \operatorname{tr} N^0$ *and* $j_s^0 = \operatorname{tr} J^0$*, which satisfy the equation:*

$$\partial_t n_s^0 + \operatorname{div}_x j_s^0 = 0. \tag{5.12}$$

Remark 5.3 *Taking into account the identity* $W_p^0 = \Pi_p W_p^0 \Pi_p$*, the Vlasov equation* (5.8) *can be rewritten*

$$\Pi_p\left[\partial_t W_p^0 + v\cdot\nabla_x W_p^0 - \nabla_x\epsilon_p\cdot\nabla_v W_p^0\right]\Pi_p = 0. \tag{5.13}$$

Moreover, by taking the trace of (5.8)*, we deduce that* $f_p^0 = \operatorname{tr} W_p^0$ *solves*

$$\partial_t f_p^0 + v\cdot\nabla_x f_p^0 - \nabla_x\epsilon_p\cdot\nabla_v f_p^0 = 0, \tag{5.14}$$

$$f_p^0(0,x,v) = \operatorname{tr} W_{p,I}^0(x,v). \tag{5.15}$$

Proof. We start with the second item of the theorem. As we remarked at the beginning of this section, by Assumption 3.4 and Proposition 4.2, $W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)$ converges, up to the extraction of a subsequence, towards a Wigner measure W^0 in $L^\infty((0,T),\mathcal{A}'_\mathcal{H})$ weak $*$. Moreover,

$$\Pi_p^\varepsilon W^\varepsilon(\psi^\varepsilon,\psi^\varepsilon)\Pi_q^\varepsilon \rightharpoonup \Pi_p W^0 \Pi_q \quad \text{in } L^\infty((0,T),\mathcal{A}'_\mathcal{H}) \text{ weak } *\,.$$

For that, it suffices to remark that if $\varphi \in L^1((0,T),\mathcal{A}_\mathcal{H})$ then $\Pi_p^\varepsilon\varphi$ converges strongly to $\Pi_p\varphi$ as $\varepsilon \to 0$. Then, applying Lemma 4.4 once again, we obtain the following convergence:

$$W_{p,q}^\varepsilon \rightharpoonup \Pi_p W^0 \Pi_q \quad \text{in } L^\infty((0,T),\mathcal{A}'_\mathcal{H}) \text{ weak } *\,,$$

and Lemma 5.1 yields:

$$\text{for } p \neq q \text{ we have } \Pi_p W^0 \Pi_q = 0. \tag{5.16}$$

It is important to remark that by construction of the Hilbert tensor product $\mathcal{H}\overline{\otimes}\mathcal{H}$ we have

$$W^0 = \sum_{p,q} \Pi_p W^0 \Pi_q \tag{5.17}$$

which leads finally to

$$W^0 = \sum_{p\in\mathbb{N}^*} \Pi_p W^0 \Pi_p\,.$$

Since $W_p^0 = \Pi_p W^0 \Pi_p$ is the limit of $W^\varepsilon(\psi_p^\varepsilon, \psi_p^\varepsilon)$, we also deduce from Proposition 4.2 that W_p^0 is nonnegative.

Consider now the right-hand side of the Wigner equation (5.1) and let us perform the limit $\varepsilon \to 0$ in the case $q = p$. In the sequel of the proof, T is a given positive real number. For the first line of (5.1), Lemma 4.5 with $\mathbb{P}^\varepsilon(t,x) = \epsilon_p^\varepsilon(t,x)\mathbb{I}$ yields

$$-\frac{i}{\varepsilon}\left(W^\varepsilon(\epsilon_p^\varepsilon\psi_p^\varepsilon, \psi_p^\varepsilon) - W^\varepsilon(\psi_p^\varepsilon, \epsilon_p^\varepsilon\psi_p^\varepsilon)\right) \rightharpoonup \nabla_x\epsilon_p \cdot \nabla_v W_p^0$$

in $L^\infty((0,T), \mathcal{S}'(\mathbb{R}^{2m}; \mathcal{H}\overline{\otimes}\mathcal{H}))$ weak $*$. By (5.2), the second line of (5.1) converges to $-v \cdot \nabla_x W_p^0$ in the same topology. The third line of (5.1) can be treated thanks to Lemma 4.4, by using the boundedness of ψ^ε in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^m, \mathcal{H}))$ and the continuity and boundedness of the operator $\partial_t\Pi_p^\varepsilon$. More precisely, we have

$$\begin{aligned} &W^\varepsilon((\partial_t\Pi_p^\varepsilon)\psi^\varepsilon, \psi_p^\varepsilon) + W^\varepsilon(\psi_p^\varepsilon, (\partial_t\Pi_p^\varepsilon)\psi^\varepsilon) = \\ &\qquad (\partial_t\Pi_p^\varepsilon)W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon)\Pi_p^\varepsilon + \Pi_p^\varepsilon W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon)\partial_t\Pi_p^\varepsilon \, + \, r_3^\varepsilon, \end{aligned}$$

where both sides are bounded in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$ and r_3^ε tends to 0 uniformly on $[0,T]$ in $\mathcal{A}'_\mathcal{H}$ weak $*$. Next, using the same argument as above, we have

$$(\partial_t\Pi_p^\varepsilon)W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon)\Pi_p^\varepsilon \rightharpoonup (\partial_t\Pi_p)W^0\Pi_p = (\partial_t\Pi_p)W_p^0\Pi_p$$

in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$ weak $*$, which implies the convergence of the third line of (5.1) towards

$$(\partial_t\Pi_p)W_p^0\Pi_p + \Pi_p W_p^0 \partial_t\Pi_p$$

in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$ weak $*$. In order to study the limit of the fourth line of (5.1), we first recall that it can be rewritten under the form $S^\varepsilon + \operatorname{div}_x T^\varepsilon$ (see (5.3)). Using the same arguments as above, S^ε and $T^\varepsilon/\varepsilon$ are bounded in $L^\infty((0,T)\mathcal{A}'_\mathcal{H})$ and we have

$$\begin{aligned} S^\varepsilon = \; & \frac{i}{2}(\nabla_x\Pi_p^\varepsilon) \cdot \left(W^\varepsilon(\psi^\varepsilon, \varepsilon\nabla_x\psi^\varepsilon) - W^\varepsilon(\varepsilon\nabla_x\psi^\varepsilon, \psi^\varepsilon)\right)\Pi_p^\varepsilon \\ & + \frac{i}{2}\Pi_p^\varepsilon\left(W^\varepsilon(\psi^\varepsilon, \varepsilon\nabla_x\psi^\varepsilon) - W^\varepsilon(\varepsilon\nabla_x\psi^\varepsilon, \psi^\varepsilon)\right) \cdot \nabla_x\Pi_p^\varepsilon + r_4^\varepsilon, \end{aligned}$$

where r_4^ε tends to 0 in $\mathcal{A}'_\mathcal{H}$ weak $*$ uniformly on $[0,T]$. This expression can be simplified thanks to Lemma 4.3:

$$S^\varepsilon = v \cdot (\nabla_x\Pi_p)W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon)\Pi_p + v \cdot \Pi_p W^\varepsilon(\psi^\varepsilon, \psi^\varepsilon)\nabla_x\Pi_p + r_4^\varepsilon.$$

We deduce as above that S^ε converges to

$$v \cdot (\nabla_x\Pi_p)W_p^0\Pi_p + v \cdot \Pi_p W_p^0 \nabla_x\Pi_p \tag{5.18}$$

in $L^\infty((0,T), \mathcal{A}'_\mathcal{H})$ weak $*$. Finally, since T^ε tends to 0, the fourth line of (5.1) converges to (5.18) in $L^\infty((0,T), \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{H}\overline{\otimes}\mathcal{H}))$ weak $*$.

From this analysis, we deduce two facts. Firstly, by passing to the limit in (5.1), that W_p^0 verifies (5.8) in the sense of distributions $\mathcal{S}'((0,T)\times\mathbb{R}^{2m},\mathcal{H}\overline{\otimes}\mathcal{H})$. Secondly, that $\partial_t W_p^\varepsilon$ is bounded in $L^\infty((0,T),\mathcal{S}'(\mathbb{R}^{2m},\mathcal{H}\overline{\otimes}\mathcal{H}))$. Hence W_p^ε is equicontinuous in t with values in $\mathcal{S}'$ and converges locally uniformly with respect to t. This proves the first item of the theorem and (5.9).

To infer the continuity in time of W^0 stated in (5.10) it suffices now to remark that Lemma 3.6 and Proposition 4.2 imply the uniform convergence of the series

$$\sum_{p\in\mathbb{N}^*} W_p^\varepsilon \quad \in C^0([0,T],\mathcal{A}'_{\mathcal{H}}).$$

The third part of the theorem is an immediate consequence of Proposition 4.2, of (5.10) and of the uniform convergence of the series. Indeed, we have already checked at the beginning of this Section that Properties (4.2) and (4.3) are satisfied.

Then the first part of Item *(iv)* is immediate, since by the $\mathcal{J}^1$-weak convergence of N^ε and J^ε we have

$$n_s^\varepsilon = \operatorname{tr} N^\varepsilon \rightharpoonup n_s^0 = \operatorname{tr} N^0 \quad ; \quad j_s^\varepsilon = \operatorname{tr} J^\varepsilon \rightharpoonup j_s^0 = \operatorname{tr} J^0.$$

To prove (5.12), it suffices to take (5.14), to integrate with respect to v on $\mathbb{R}^m$ then to sum on p. □

The following Corollary provides an explicit way to compute W_p^0 in the case of simple eigenvalues:

Corollary 5.4 *Assume that the eigenvalue ϵ_p is simple. Let $\chi_p(t,x)$ be the corresponding unitary eigenfunction. Then*

$$W_p^0(t,x,v) = f_p(t,x,v)\chi_p(t,x)\overline{\otimes}\chi_p(t,x) \tag{5.19}$$

where f_p solves the Vlasov equation

$$\partial_t f_p + v\cdot\nabla_x f_p - \nabla_x\epsilon_p\cdot\nabla_v f_p = 0 \tag{5.20}$$

with the initial condition

$$f_p(0,x,v) = \operatorname{tr}\,[W_{p,I}^0(x,v)].$$

Proof. The proof of this corollary is immediate. Defining f_p by $f_p = \operatorname{tr} W_p^0$, since $W_p^0 = \Pi_p W^0 \Pi_p$, we have directly (5.19). Then we apply Remark 5.3 to get (5.20). □

Application : Proof of Theorem 2.3.

Setting

$$\mathcal{H} = L^2(0,1) \quad ; \quad A^\varepsilon = -\frac{1}{2}\,\partial_z^2 + V^\varepsilon \quad ; \quad \mathcal{D}(A) = H^2(0,1),$$

we have seen in the beginning of Section 2 that Assumption 2.1 implies that Assumptions 3.1, 3.2 and 3.3 are satisfied. Moreover, the $L^\infty((0,1)\times(0,T)\times\Omega)$ bound of V^ε implies

$$\left|\epsilon_p^\varepsilon - \frac{\pi^2 p^2}{2}\right| \le C,$$

uniformly for $(\varepsilon,t,x) \in (0,1)\times(0,T)\times\mathbb{R}^m$, thus Assumption 3.5 is satisfied. Next, Assumption 2.2 implies that Assumption 3.4 is satisfied: the assumptions of the general framework are fulfilled and Theorem 5.2 applies, as well as its Corollary 5.4 (indeed, the eigenvalues ϵ_p are simple). Then it is immediate to deduce Item *(i)*, *(ii)* and *(iv)* of Theorem 2.3. It remains to prove Item *(iii)*. We will only prove the result concerning the charge density n^ε; this proof can be easily adapted for the current density j_x^ε. Recall that

$$N^\varepsilon(t,x,z,z') = \psi^\varepsilon(t,x,z)\overline{\psi}^\varepsilon(t,x,z') \quad ; \quad n^\varepsilon(t,x,z) = |\psi^\varepsilon(t,x,z)|^2.$$

By (5.11) and (5.19) we have

$$N^0(t,x,v,z,z') = \sum_{p\in\mathbb{N}^*}\left(\int_{\mathbb{R}^m} f_p(t,x,v)\,dv\right)\chi_p(t,x,z')\chi_p(t,x,z).$$

Consider a function $\theta \in L^1((0,T), C_c^0(\mathbb{R}^m, L^\infty(0,1)))$. Pointwise in t and x, the multiplication by this function that we shall denote by Θ, is a bounded operator on $L^2(0,1)$ (but not compact). Hence by Theorem 5.2, *(iii)*, we have as $\varepsilon \to 0$

$$\int_0^T\int_{\mathbb{R}^m} \operatorname{tr}(\Theta N^\varepsilon)\,dx\,dt \to \int_0^T\int_{\mathbb{R}^m} \operatorname{tr}(\Theta N^0)\,dx\,dt.$$

Remark that

$$\operatorname{tr}(\Theta(t,x,\cdot)N^\varepsilon(t,x,\cdot,\cdot)) = \int_0^1 \theta(t,x,z)\,n^\varepsilon(t,x,z)\,dz$$

and

$$\operatorname{tr}(\Theta(t,x,\cdot)N^0(t,x,\cdot,\cdot)) = \int_{\mathbb{R}^m} f_p(t,x,v)\,\theta(t,x,z)|\chi_p|^2(t,x,z)dv\,dz.$$

By an identification, we deduce that n^ε converges to $(\int_{\mathbb{R}^m} f_p(\cdot,v)\,dv)|\chi_p|^2$ in the $L^\infty((0,T),\mathcal{M}_b(\mathbb{R}^m, L^1(0,1)-\text{weak}))$ weak $*$ topology.

6 Extension: slab with a varying width

In this section we analyze the limit of the Schrödinger equation (1.1)-(1.2) in the case when the domain is a slab of varying width:

$$\Omega = \{(x,z) : \quad x \in \mathbb{R}^m, \quad l_1(x) \le z \le l_2(x)\},$$

where l_1 and l_2 are two C^2 functions such that

$$\|l_i\|_{W^{2,\infty}(\mathbb{R}^m)} \le C_1 \quad (i=1,2) \quad ; \quad \frac{1}{C_1} \le l_2(x) - l_1(x) \le C_1 \quad (x \in \mathbb{R}^m), \tag{6.1}$$

$C_1 > 0$ being a constant. The hard-wall boundary condition (1.3) becomes

$$\psi^\varepsilon(t,x,l_1(x)) = \psi^\varepsilon(t,x,l_2(x)) = 0. \tag{6.2}$$

The general results developed in Section 3–5 cannot be applied directly here since the Hilbert space $L^2(l_1(x), l_2(x))$ depends on the x variable. This is the reason why we shall perform a change of variable $(t,x,z) \to (\widehat{t}, \widehat{x}, \widehat{z})$ to transform the domain Ω into the reference domain $\widehat{\Omega} = \{(x,z) \in \mathbb{R}^m \times (0,1)\}$. As a consequence, the Hamiltonian $-\frac{\varepsilon^2}{2}\Delta_x - \frac{1}{2}\partial_z^2 + V^\varepsilon$ will be transformed into an operator which does not take the form of (3.1). Nevertheless, we shall see that the additional terms can be treated with only little modification and we will prove the equivalent of Theorem 2.3, under the following stronger assumptions:

Assumption 6.1 *The function $(\varepsilon,t,x,z) \mapsto V^\varepsilon(t,x,z)$ is nonnegative and belongs to*

$$C^0([0,1], C^1(\mathbb{R}_+ \times \mathbb{R}^m, L^\infty(0,1))) \cap C^0([0,1], W^{1,\infty}_{loc}(\mathbb{R}_+, W^{1,\infty}(\Omega))).$$

Assumption 6.2 *For every integer n, q verifying $0 \le p+q \le 2$, the Cauchy data ψ_I^ε satisfies*

$$\int_\Omega |(\varepsilon\nabla_x)^p\, (\partial_z)^q\, \psi_I^\varepsilon|^2\, dx\, dz \le C, \tag{6.3}$$

with a constant C independent of ε.

Let us now write the main result of this Section. The subbands of the system are still defined by

$$\begin{cases} -\dfrac{1}{2}\,\partial_z^2 \chi_p^\varepsilon + V^\varepsilon \chi_p^\varepsilon = \epsilon_p^\varepsilon\, \chi_p^\varepsilon\,, \\[2ex] \chi_p^\varepsilon(t,x,\cdot) \in H_0^1(l_1(x), l_2(x)), \qquad \displaystyle\int_{l_1(x)}^{l_2(x)} \chi_p^\varepsilon\, \chi_q^\varepsilon\, dz = \delta_{pq}. \end{cases} \tag{6.4}$$

Since the domain Ω is not invariant with respect to x, the partial Wigner transform defined in (2.4) cannot be applied to ψ^ε. Instead, we consider the coefficients of the wavefunction on the subbands by writing

$$\psi^\varepsilon(t,x,z) = \sum_{p \ge 1} \phi_p^\varepsilon(t,x) \chi_p(t,x,z)$$

and we apply to these coefficients the standard Wigner transform in dimension m:

$$f_p^\varepsilon(t,x,v) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\eta\cdot v}\, \phi_p^\varepsilon\left(t, x - \varepsilon\frac{\eta}{2}\right) \overline{\phi}_p^\varepsilon\left(t, x + \varepsilon\frac{\eta}{2}\right) d\eta.$$

Denoting $f_{p,I}^\varepsilon = f_p^\varepsilon(0,x,v)$, we have the

Theorem 6.3 *If Assumptions* 6.1 *and* 6.2 *are satisfied and if* (6.1) *is fulfilled, then, for a subsequence still indexed by ε, we have the following convergence as $\varepsilon \to 0$:*
(i) For any $p \in \mathbb{N}^*$, $f_{p,I}^\varepsilon$ converges in $\mathcal{S}'(\mathbb{R}^{2m}_{x,v})$ to $f_{p,I} \in \mathcal{M}_b(\mathbb{R}^{2m}_{x,v})$.
(ii) For any $p \in \mathbb{N}^*$, f_p^ε converges locally uniformly in the $\mathcal{S}'(\mathbb{R}^{2m}_{x,v})$ weak $*$ topology to $f_p \geq 0$ which solves

$$\partial_t f_p + v\cdot\nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0 \tag{6.5}$$

$$f_p(0,x,v) = f_{p,I}(x,v). \tag{6.6}$$

(iii) For every $T > 0$, the charge density n^ε and the current density j_x^ε defined by (2.2) *converge in the $L^\infty((0,T),\mathcal{M}_b(\mathbb{R}^m_x))$ weak $*$ topology to:*

$$n(t,x,z) = \sum_{p\geq 1}\left(\int_{\mathbb{R}^m} f_p(t,x,v)\,dv\right)|\chi_p(t,x,z)|^2,$$

and

$$j_x(t,x,z) = \sum_{p\geq 1}\left(\int_{\mathbb{R}^m} v f_p(t,x,v)\,dv\right)|\chi_p(t,x,z)|^2.$$

(iv) The surface charge and current densities defined by

$$n_s^\varepsilon(t,x) = \int_{l_1(x)}^{l_2(x)} n^\varepsilon(t,x,z)\,dz \quad ; \quad j_{x,s}^\varepsilon(t,x,z) = \int_{l_1(x)}^{l_2(x)} \varepsilon\,\mathcal{I}m\,(\overline{\psi^\varepsilon}\nabla_x\psi^\varepsilon)\,dz.$$

converge locally uniformly on $\mathbb{R}_+$ in the $\mathcal{M}_b(\mathbb{R}^m_x)$ weak $*$ topology to $n_s = \int_{l_1}^{l_2} n(\cdot,z)\,dz$ and $j_{x,s} = \int_{l_1}^{l_2} j_x(\cdot,z)\,dz$, which satisfy

$$\partial_t n_s + \operatorname{div}_x j_{x,s} = 0.$$

Proof. As announced above, we perform a change of variable, setting

$$\left\{\begin{array}{lcll} \widehat{t} & = & t & (t\in\mathbb{R}) \\ \widehat{x} & = & x & (x\in\mathbb{R}^m) \\ \widehat{z} & = & \dfrac{z-l_1(x)}{l_2(x)-l_1(x)} & (z\in(l_1(x),l_2(x))). \end{array}\right. \tag{6.7}$$

We denote

$$d(\widehat{x}) = l_2(\widehat{x}) - l_1(\widehat{x}) \quad ; \quad Z(\widehat{x},\widehat{z}) = l_1(\widehat{x}) + d(\widehat{x})\widehat{z},$$

so that $z = Z(\widehat{x}, \widehat{z})$. Then, denoting

$$\widehat{\psi^\varepsilon}(\widehat{t}, \widehat{x}, \widehat{z}) = \psi^\varepsilon(t, x, z), \qquad \widehat{n}^\varepsilon(\widehat{t}, \widehat{x}, \widehat{z}) = n^\varepsilon(t, x, z), \qquad \widehat{V}^\varepsilon(\widehat{t}, \widehat{x}, \widehat{z}) = V^\varepsilon(t, x, z),$$

a straightforward but lengthy computation leads to the following Schrödinger equation in $(\widehat{t}, \widehat{x}, \widehat{z}) \in \mathbb{R}_+ \times \mathbb{R}^m \times (0,1)$:

$$i\varepsilon\partial_{\widehat{t}}\widehat{\psi^\varepsilon} = -\frac{\varepsilon^2}{2}\Delta_{\widehat{x}}\widehat{\psi^\varepsilon} + A^\varepsilon\widehat{\psi^\varepsilon} + \varepsilon^2 \mathrm{div}_{\,\widehat{x}}(R^1\widehat{\psi^\varepsilon}) + \varepsilon^2 R^2\widehat{\psi^\varepsilon} \tag{6.8}$$

$$\psi^\varepsilon(\widehat{t}, \widehat{x}, 0) = \psi^\varepsilon(\widehat{t}, \widehat{x}, 1) = 0, \tag{6.9}$$

where A^ε, R^1, R^2 are the linear operators indexed by x and acting on the variable $z \in (0,1)$ defined by

$$A^\varepsilon(t,x)\psi = -\frac{1}{2}\frac{\partial}{\partial z}\left(\alpha^\varepsilon(x,z)\frac{\partial\psi}{\partial z}\right) + \widehat{V}^\varepsilon(t,x,z)\psi, \tag{6.10}$$

$$R^1(x)\psi = \frac{1}{d(x)}\frac{\partial\psi}{\partial z}\nabla_x Z(x,z), \tag{6.11}$$

$$R^2(x)\psi = \beta(x,z)\frac{\partial\psi}{\partial z} \tag{6.12}$$

and

$$\alpha^\varepsilon(x,z) = \frac{1}{d^2(x)}\left[1 + \varepsilon^2|\nabla_x Z(x,z)|^2\right], \tag{6.13}$$

$$\beta(x,z) = -\frac{1}{2}\left[\mathrm{div}\left(\frac{\nabla_x Z(x,z)}{d(x)}\right) - \frac{\partial}{\partial z}\left(\frac{|\nabla_x Z(x,z)|^2}{2d^2(x)}\right)\right]. \tag{6.14}$$

Then, showing that n^ε converges towards

$$\sum_p \left\{\int_{\mathbb{R}^m} f_p(t,x,v)\,dv\right\}|\chi_p(t,x,z)|^2$$

is equivalent to showing that $\widehat{n}^\varepsilon$ converges to

$$\widehat{n}(\widehat{t}, \widehat{x}, \widehat{z}) = \sum_p \left\{\int_{\mathbb{R}^m} f_p(\widehat{t}, \widehat{x}, \widehat{v})\,d\widehat{v}\right\}|\chi_p(\widehat{t}, \widehat{x}, Z(\widehat{x}, \widehat{z}))|^2.$$

Let $(\widehat{\epsilon}_p, \widehat{\chi}_p)$ be the eigenvalues and the unitary eigenfunctions of the operator A^0 given by Formula (6.10) with $\varepsilon = 0$. Then it is readily seen that

$$\widehat{\epsilon}_p(\widehat{t}, \widehat{x}) = \epsilon_p(t,x) \quad ; \quad \widehat{\chi}_p(\widehat{t}, \widehat{x}, \widehat{z}) = \sqrt{d(\widehat{x})}\,\chi_p\left(\widehat{t}, \widehat{x}, Z(\widehat{x}, \widehat{z})\right)$$

which implies that the limit of $\widehat{n}^\varepsilon$ is

$$\widehat{n}(\widehat{t}, \widehat{x}, \widehat{z}) = \sum_p \left\{\int_{\mathbb{R}^m} \frac{f_p(\widehat{t}, \widehat{x}, \widehat{v})}{d(\widehat{x})}\,d\widehat{v}\right\}|\widehat{\chi}_p(\widehat{t}, \widehat{x}, \widehat{z})|^2.$$

Denoting by $\widehat{f}_p^0(\widehat{t},\widehat{x},\widehat{v}) = f_p(\widehat{t},\widehat{x},\widehat{v})/d(\widehat{x})$, we can write

$$\widehat{n}(\widehat{t},\widehat{x},\widehat{z}) = \sum_p \left\{ \int_{\mathbb{R}^m} \widehat{f}_p^0(\widehat{t},\widehat{x},\widehat{v})\, d\widehat{v} \right\} |\widehat{\chi}_p(\widehat{t},\widehat{x},\widehat{z})|^2.$$

Consequently showing that f_p solves (6.5) is equivalent to showing that $\widehat{f}_p^0$ solves

$$\partial_t \widehat{f}_p^0 + \widehat{v}\cdot\nabla_{\widehat{x}}\widehat{f}_p^0 - \nabla_{\widehat{x}}\widehat{\epsilon}_p \cdot \nabla_{\widehat{v}}\widehat{f}_p^0 = -\widehat{v}\cdot\frac{\nabla_{\widehat{x}}d}{d}\widehat{f}_p^0. \tag{6.15}$$

From this analysis, it appears that we only have to prove that the additional terms induced by R^1 and R^2 lead to the second-hand side $-\widehat{v}\cdot\frac{\nabla_{\widehat{x}}d}{d}\widehat{f}_p^0$ at the limit $\varepsilon \to 0$. Then Theorem 2.3 will be deduced from Theorem 5.2 and Corollary 5.4, where the Vlasov equation (5.20) is replaced by (6.15).

Since Assumption 2.2 is weaker than Assumption 6.2, the estimate (2.1) holds true. Besides, due to (6.1), the change of variable $(t,x,z) \mapsto (\widehat{t},\widehat{x},\widehat{z})$ is a C^2-diffeomorphism and belongs to $W^{2,\infty}$. Next we deduce:

$$\iint_{\widehat{\Omega}} \left(|\widehat{\psi^\varepsilon}(\widehat{t})|^2 + \varepsilon^2 |\nabla_{\widehat{x}}\widehat{\psi^\varepsilon}(\widehat{t})|^2 + |\partial_{\widehat{z}}\widehat{\psi^\varepsilon}(\widehat{t})|^2 \right) d\widehat{x}\, d\widehat{z} \le C(\widehat{t}).$$

From now on and for notational simplicity, we drop the $\widehat{\ }$'s and denote the new variables as the original ones. Denoting $\mathcal{H} = L^2(0,1)$, the wavefunction $\psi^\varepsilon(t,x)$ is an $\mathcal{H}$-valued function on $\mathbb{R}_+ \times \mathbb{R}^m$, solving

$$i\varepsilon\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x\psi^\varepsilon + A^\varepsilon\psi^\varepsilon + \varepsilon^2 \mathrm{div}_x(R^1\psi^\varepsilon) + \varepsilon^2 R^2\psi^\varepsilon \tag{6.16}$$

$$\psi^\varepsilon(x,0) = \psi_I^\varepsilon(x), \quad x\in\mathbb{R}^m. \tag{6.17}$$

The operators A^ε and R^2 are unbounded operators on $\mathcal{H}$ while R^1 is an unbounded operator from $\mathcal{H}$ on $\mathcal{H}\times\mathcal{H}$. The domain of A^ε is $D(A^\varepsilon) = H^2(0,1)\cap H_0^1(0,1)$ and Assumptions 3.1–3.5 are all satisfied. Moreover the operators R^1 and R^2 satisfy the uniform in ε estimate

$$\int_{\mathbb{R}^m} \|R^i\psi^\varepsilon(t,x)\|_{\mathcal{H}}^2\, dx \le \int_{\mathbb{R}^m} (A^\varepsilon\psi^\varepsilon,\psi^\varepsilon)_{\mathcal{H}}\, dx \le C(t). \tag{6.18}$$

The projection of the Schrödinger equation on the p-th subband yields the following equation: if ψ^ε is a solution of (6.16) then $\psi_p^\varepsilon = \Pi_p^\varepsilon\psi^\varepsilon$ is solution of

$$\begin{aligned} i\varepsilon\,\partial_t\psi_p^\varepsilon \;&=\; -\frac{\varepsilon^2}{2}\,\Delta_x\psi_p^\varepsilon + \epsilon_p^\varepsilon\psi_p^\varepsilon + i\varepsilon\,\left(\partial_t\Pi_p^\varepsilon\right)\psi^\varepsilon + \frac{\varepsilon^2}{2}\,\left[\Delta_x,\Pi_p^\varepsilon\right]\psi^\varepsilon \\ &\quad +\Pi_p^\varepsilon(\varepsilon^2\mathrm{div}_x\,(R^1\psi^\varepsilon) + \varepsilon^2 R^2\psi^\varepsilon), \end{aligned} \tag{6.19}$$

which leads to the following identity satisfied by the Wigner transform $W^\varepsilon_{p,q} = W^\varepsilon(\psi^\varepsilon_p, \psi^\varepsilon_q)$:

$$\begin{aligned}
\partial_t W^\varepsilon_{p,q} &= -\frac{i}{\varepsilon}\left(W^\varepsilon(\epsilon^\varepsilon_p \psi^\varepsilon_p, \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \epsilon^\varepsilon_q \psi^\varepsilon_q)\right) \\
&+\frac{i\varepsilon}{2}\left(W^\varepsilon(\Delta_x \psi^\varepsilon_p, \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \Delta_x \psi^\varepsilon_q)\right) + W^\varepsilon((\partial_t \Pi^\varepsilon_p)\psi^\varepsilon, \psi^\varepsilon_q) + W^\varepsilon(\psi^\varepsilon_p, (\partial_t \Pi^\varepsilon_q)\psi^\varepsilon) \\
&-\frac{i\varepsilon}{2} W^\varepsilon([\Delta_x, \Pi^\varepsilon_p]\psi^\varepsilon, \psi^\varepsilon_q) + \frac{i\varepsilon}{2} W^\varepsilon(\psi^\varepsilon_p, [\Delta_x, \Pi^\varepsilon_q]\psi^\varepsilon) \\
&-i\left(W^\varepsilon(\Pi^\varepsilon_p(\varepsilon \mathrm{div}_x\,(R^1\psi^\varepsilon) + \varepsilon R^2\psi^\varepsilon), \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \Pi^\varepsilon_q(\varepsilon \mathrm{div}_x\,(R^1\psi^\varepsilon) + \varepsilon R^2\psi^\varepsilon))\right).
\end{aligned} \tag{6.20}$$

Since by (6.18) $R^2\psi^\varepsilon$ is bounded in $L^2(\mathbb{R}^m, \mathcal{H})$, we have

$$\lim_{\varepsilon\to 0} W^\varepsilon(\Pi^\varepsilon_p(\varepsilon R^2\psi^\varepsilon), \psi^\varepsilon_q) = \lim_{\varepsilon\to 0} W^\varepsilon(\psi^\varepsilon_p, \Pi^\varepsilon_q(\varepsilon R^2\psi^\varepsilon)) = 0.$$

Therefore, the only additional term which has to be analyzed is

$$B^\varepsilon_{p,q} = -i\left(W^\varepsilon(\Pi^\varepsilon_p(\varepsilon \mathrm{div}_x\,(R^1\psi^\varepsilon)), \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \Pi^\varepsilon_q(\varepsilon \mathrm{div}_x\,(R^1\psi^\varepsilon)))\right).$$

This term can be rewritten thanks to (4.7):

$$\begin{aligned}
B^\varepsilon_{p,q} &= -i\left(W^\varepsilon(\varepsilon \mathrm{div}_x\,(\Pi^\varepsilon_p R^1\psi^\varepsilon), \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \varepsilon \mathrm{div}_x\,(\Pi^\varepsilon_q R^1\psi^\varepsilon))\right) \\
&\quad +i\left(W^\varepsilon(\varepsilon(\nabla_x \Pi^\varepsilon_p)\cdot(R^1\psi^\varepsilon), \psi^\varepsilon_q) - W^\varepsilon(\psi^\varepsilon_p, \varepsilon(\nabla_x \Pi^\varepsilon_q)\cdot(R^1\psi^\varepsilon))\right)
\end{aligned}$$

By (6.18) the terms appearing in the second line of the above identity tend to zero as ε tends to zero in $L^\infty_{loc}(\mathbb{R}_+, \mathcal{A}'_{\mathcal{H}})$, while the first line can be rewritten

$$\begin{aligned}
C^\varepsilon_{p,q} &= -i\left(W^\varepsilon(\Pi^\varepsilon_p R^1\psi^\varepsilon, \varepsilon\nabla_x \psi^\varepsilon_q) - W^\varepsilon(\varepsilon\nabla_x \psi^\varepsilon_p, \Pi^\varepsilon_q R^1\psi^\varepsilon)\right) \\
&\quad +2v\cdot W^\varepsilon(\Pi^\varepsilon_p R^1\psi^\varepsilon, \psi^\varepsilon_q) + 2v\cdot W^\varepsilon(\psi^\varepsilon_p, \Pi^\varepsilon_q R^1\psi^\varepsilon).
\end{aligned} \tag{6.21}$$

Since for every fixed p, q, $C^\varepsilon_{p,q}$ is bounded in $L^\infty_{loc}(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{H}\overline{\otimes}\mathcal{H}))$, it is readily seen from the proof of Lemma 5.1 that

$$\lim_{\varepsilon\to 0} W^\varepsilon_{p,q} = 0 \quad ; \text{ if } \ p \neq q.$$

For $p = q$, we note that the final Vlasov equation now reads

$$\Pi_p\left(\partial_t W^0_p + v\cdot\nabla_x W^0_p - \nabla_x \epsilon_p \cdot \nabla_v W^0_p\right)\Pi_p = \lim_{\varepsilon\to 0} C^\varepsilon_{p,p}. \tag{6.22}$$

Now it remains to determine the limit of $C^\varepsilon_{p,p}$. For that we shall make use of the following identity:

$$\Pi^\varepsilon_p R^1 \Pi^\varepsilon_p = -\frac{\nabla_x d}{2d}\,\Pi^\varepsilon_p \tag{6.23}$$

which can be proven easily from the following computation

$$\Pi^\varepsilon_p R^1 \Pi^\varepsilon_p\,\phi = \left(\int \chi^\varepsilon_p\, R^1 \chi^\varepsilon_p\, dz\right)\left(\int \chi^\varepsilon_p \phi\, dz\right)\chi^\varepsilon_p$$

and from

$$\int \chi_p^\varepsilon \, R^1 \chi_p^\varepsilon \, dz = \int \frac{\nabla_x Z}{d} \chi_p^\varepsilon \frac{\partial \chi_p^\varepsilon}{\partial z} \, dz = \frac{1}{2} \int \frac{\nabla_x Z}{d} \frac{\partial}{\partial z} |\chi_p^\varepsilon|^2 \, dz = -\frac{\nabla_x d}{2d},$$

where we have taken into account the fact that χ_p is real-valued.

Consider (6.21) with $p = q$. We immediately get

$$\begin{aligned} C_{p,p}^\varepsilon &= -i \left(W^\varepsilon(\Pi_p^\varepsilon R^1 \psi^\varepsilon, \varepsilon \Pi_p^\varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon) - W^\varepsilon(\varepsilon \Pi_p^\varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon, \Pi_p^\varepsilon R^1 \psi^\varepsilon) \right) \\ &\quad + 2v \cdot W^\varepsilon(\Pi_p^\varepsilon R^1 \psi^\varepsilon, \psi_p^\varepsilon) + 2v \cdot W^\varepsilon(\psi_p^\varepsilon, \Pi_p^\varepsilon R^1 \psi^\varepsilon) + o(1). \end{aligned}$$

Let us analyze the first term of the right-hand side of this identity. Applying Lemma 4.4 with $\mathbb{P}^\varepsilon = \Pi_p^\varepsilon R^1$ (this can be done although $\Pi_p^\varepsilon R^1$ is an unbounded operator, since ψ^ε lies uniformly in the domain of this operator), we have

$$\begin{aligned} -i W^\varepsilon(\Pi_p^\varepsilon R^1 \psi^\varepsilon, \varepsilon \Pi_p^\varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon) &= -i \Pi_p^\varepsilon R^1 \, W^\varepsilon(\psi^\varepsilon, \varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon) \Pi_p^\varepsilon + o(1) \\ &= -i \Pi_p^\varepsilon R^1 \, W^\varepsilon(\varepsilon \nabla_x \psi^\varepsilon, \psi_p^\varepsilon) \Pi_p^\varepsilon \\ &\quad - 2v \cdot \Pi_p^\varepsilon R^1 W^\varepsilon(\psi^\varepsilon, \psi_p^\varepsilon) \Pi_p^\varepsilon + o(1). \end{aligned}$$

Similarly for the other terms we have

$$i W^\varepsilon(\varepsilon \Pi_p^\varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon, \Pi_p^\varepsilon R^1 \psi^\varepsilon) = i \Pi_p^\varepsilon \, W^\varepsilon(\varepsilon \nabla_x \psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^* + o(1)$$

and

$$\begin{aligned} & 2v \cdot W^\varepsilon(\Pi_p^\varepsilon R^1 \psi^\varepsilon, \psi_p^\varepsilon) + 2v \cdot W^\varepsilon(\psi_p^\varepsilon, \Pi_p^\varepsilon R^1 \psi^\varepsilon) \\ &\quad = 2v \cdot \Pi_p^\varepsilon R^1 W^\varepsilon(\psi^\varepsilon, \psi_p^\varepsilon) \Pi_p^\varepsilon + 2v \cdot \Pi_p^\varepsilon W^\varepsilon(\psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^* + o(1). \end{aligned}$$

Hence, summing up these identities, it follows

$$C_{p,p}^\varepsilon = 2v \cdot \Pi_p^\varepsilon W^\varepsilon(\psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^* + r_p^\varepsilon + o(1),$$

where

$$r_p^\varepsilon = -i \Pi_p^\varepsilon R^1 \, W^\varepsilon(\varepsilon \nabla_x \psi^\varepsilon, \psi_p^\varepsilon) \Pi_p^\varepsilon + i \Pi_p^\varepsilon \, W^\varepsilon(\varepsilon \nabla_x \psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^*.$$

Since $W^\varepsilon(\psi_p^\varepsilon, \psi^\varepsilon) = \Pi_p^\varepsilon W_{p,p}^\varepsilon \Pi_p^\varepsilon + o(1)$, we deduce from (6.23) that

$$C_{p,p}^\varepsilon = -v \cdot \frac{\nabla_x d}{d} W_{p,p}^\varepsilon + r_p^\varepsilon + o(1).$$

The only thing left to show now is that for all $p \in \mathbb{N}^*$ we have $r_p^\varepsilon = o(1)$. This is given by Lemma A.3 in the Appendix. □

Remark. Since R^1 is vector-valued, the above computations need some clarifications. Terms like $W^\varepsilon(\Pi_p^\varepsilon R^1 \psi^\varepsilon, \varepsilon \Pi_p^\varepsilon \nabla_x \Pi_p^\varepsilon \psi^\varepsilon)$ have to be understood in the following way:

$$\text{let} \quad x = (x_1, \dots, x_m), \quad v = (v_1, \dots, v_m), \quad R^1 = (R_1^1, \dots, R_m^1),$$

$$\text{then}\quad W^\varepsilon(\Pi_p^\varepsilon R^1\psi^\varepsilon, \varepsilon\Pi_p^\varepsilon\nabla_x\Pi_p^\varepsilon\psi^\varepsilon) = \sum_{j=1}^m W^\varepsilon(\Pi_p^\varepsilon R_j^1\psi^\varepsilon, \varepsilon\Pi_p^\varepsilon\partial_{x_j}\Pi_p^\varepsilon\psi^\varepsilon).$$

The algebra in the proof of Lemma A.3 is valid with R^1, ∇_x, v replaced by $R_j^1, \partial_{x_j}, v_j$ with $j = 1, \ldots, m$. The result is then obtained by summing up on j.

Appendix

The aim of this Appendix is to prove a technical result that we use in Section 6 in the proof of Theorem 6.3. We recall that

$$\Omega = \{(x,z) : x \in \mathbb{R}^m,\, l_1(x) \le z \le l_2(x)\}, \qquad \widehat{\Omega} = \{(x,z) : x \in \mathbb{R}^m,\, 0 \le z \le 1\}.$$

Lemma A.1 *Under Assumptions* 6.1 *and* 6.2*, for any integers n, p, q such that $0 \le 2n + p + q \le 2$, the solution of* (1.1)-(1.2)-(6.2) *satisfies*

$$\int_\Omega |(\varepsilon\partial_t)^n\,(\varepsilon\nabla_x)^p\,(\partial_z)^q\,\psi^\varepsilon|^2\,dx\,dz \le C(t), \tag{A.1}$$

where $C(t)$ is a continuous function on $\mathbb{R}_+$ independent of ε.

Proof. The cases where $n = 0$ and $0 \le p + q \le 1$ are consequences of the usual mass and energy conservations for the Schrödinger equation. We have indeed

$$\frac{d}{dt}\iint_\Omega |\psi^\varepsilon|^2\,dxdz = 0,$$

$$\frac{d}{dt}\iint_\Omega \left[\varepsilon^2|\nabla_x\psi^\varepsilon|^2 + |\partial_z\psi^\varepsilon|^2 + 2V^\varepsilon|\psi^\varepsilon|^2\right]\,dxdz = 2\iint_\Omega \partial_t V^\varepsilon|\psi^\varepsilon|^2\,dxdz,$$

which gives the result, under Assumption 2.1 (which holds true as since it is weaker than Assumption 6.1). Let us now differentiate (1.1) with respect to x:

$$i\varepsilon\partial_t(\varepsilon\nabla_x\psi^\varepsilon) = -\frac{\varepsilon^2}{2}\Delta_x(\varepsilon\nabla_x\psi^\varepsilon) - \partial_z^2(\varepsilon\nabla_x\psi^\varepsilon) + V^\varepsilon(\varepsilon\nabla_x\psi^\varepsilon) + \varepsilon\nabla_x V^\varepsilon\psi^\varepsilon.$$

Hence, multiplying this equation by $\varepsilon\partial_t\nabla_x\overline{\psi}^\varepsilon$, integrating on Ω and taking the real part, we obtain

$$\begin{aligned}
&\frac{d}{dt}\iint_\Omega \left[\frac{\varepsilon^4}{2}|\Delta_x\psi^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_z\nabla_x\psi^\varepsilon|^2 + V^\varepsilon\varepsilon^2|\nabla_x\psi^\varepsilon|^2\right]\,dxdz \\
&\quad = -2\varepsilon\frac{d}{dt}\mathcal{R}e\iint_\Omega \overline{\psi}^\varepsilon\nabla_x V^\varepsilon\cdot\varepsilon\nabla_x\psi^\varepsilon\,dxdz + 2\varepsilon\mathcal{R}e\iint_\Omega \overline{\psi}^\varepsilon(\partial_t\nabla_x V^\varepsilon)\cdot\varepsilon\nabla_x\psi^\varepsilon\,dxdz \\
&\qquad + 2\mathcal{R}e\iint_\Omega \varepsilon\partial_t\overline{\psi}^\varepsilon\nabla_x V^\varepsilon\cdot\varepsilon\nabla_x\psi^\varepsilon\,dxdz + \iint_\Omega \partial_t V^\varepsilon|\varepsilon\nabla_x\psi^\varepsilon|^2\,dxdz.
\end{aligned}$$

We can do the same with $\partial_z\psi^\varepsilon$ instead of $\varepsilon\nabla_x\psi^\varepsilon$. Then by integrating on $[0,T]$, by using (A.1) with $n = 0$ and $0 \le p + q \le 1$ and by Assumption 6.1, we get

$$\iint_\Omega \left[\varepsilon^4|\Delta_x\psi^\varepsilon|^2 + \varepsilon^2|\partial_z\nabla_x\psi^\varepsilon|^2 + |\partial_z^2\psi^\varepsilon|^2\right]\,dxdz \le C + C\int_0^t\iint_\Omega |\varepsilon\partial_t\psi^\varepsilon|^2\,dxdzds.$$

We can conclude by using Equation (1.1) and a Gronwall estimate. □

We consider now the solution of (6.16)-(6.17), where the operators A^ε, R^1 and R^2 are defined in (6.10)–(6.12).

Lemma A.2 *If Assumptions* 6.1 *and* 6.2, *as well as* (6.1), *are satisfied, then the solution of* (6.16)-(6.17)-(1.3) *denoted by* $\widehat{\psi^\varepsilon}(t,x,z)$ *on* $\mathbb{R}_+ \times \widehat{\Omega}$ *satisfies*

$$\int_{\widehat{\Omega}} |(\varepsilon\partial_t)^n\,(\varepsilon\nabla_x)^k\,(\partial_z)^l\,\widehat{\psi}^\varepsilon|^2\,dx\,dz \leq C(t), \tag{A.2}$$

for any integers n, k, l such that $0 \leq 2n+k+l \leq 2$. *Moreover, defining* $\widehat{\phi}^\varepsilon_p$ *for any* $p \in \mathbb{N}^*$ *by* $\phi^\varepsilon_p = \widehat{\Pi}^\varepsilon_p \widehat{\phi}^\varepsilon$, *where* $\widehat{\phi}^\varepsilon = \varepsilon\nabla_x\widehat{\psi^\varepsilon}$ *and* Π^ε_p *is the p-th spectral projector associated to A^ε, then we have for $p \neq q$*

$$\lim_{\varepsilon\to 0} W^\varepsilon(\widehat{\phi}^\varepsilon_p, \widehat{\psi}^\varepsilon_q) = 0,$$

where W^ε is defined by (4.1).

Proof. Estimate (A.2) is a straightforward consequence of (A.1). For that, it suffices to recall that the system (6.16)-(6.17)-(1.3) can be deduced from (1.1)-(1.2)-(6.2) by the change of variable (6.7), which is regular enough thanks to (6.1). Let us now prove the second part of the lemma. In order to simplify the notations, we skip the " $\widehat{\ }$ ". Multiplying (6.16) by ε and differentiating with respect to x leads to the following identity

$$i\varepsilon\partial_t\phi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_x\phi^\varepsilon + A^\varepsilon\phi^\varepsilon + \varepsilon^2\mathrm{div}_x\left(R^1\phi^\varepsilon\right) + \varepsilon^2R^2\phi^\varepsilon + \varepsilon S^\varepsilon \tag{A.3}$$

with homogeneous Dirichlet boundary conditions. The term S^ε contains expressions of the form $(\varepsilon\nabla_x)^k(\partial_z)^l\psi^\varepsilon$ with $0 \leq k+l \leq 2$, which are bounded in $L^\infty((0,T), L^2(\widehat{\Omega}))$ thanks to (A.2). Let us now write an equation satisfied by $W^\varepsilon(\phi^\varepsilon_p, \psi^\varepsilon_q)$ analogous to (6.20). We have

$$\begin{aligned}
\partial_t W^\varepsilon(\phi^\varepsilon_p,\psi^\varepsilon_q) = &-\frac{i}{\varepsilon}\left(W^\varepsilon(\epsilon^\varepsilon_p\phi^\varepsilon_p,\psi^\varepsilon_q) - W^\varepsilon(\phi^\varepsilon_p,\epsilon^\varepsilon_q\psi^\varepsilon_q)\right)\\
&+\frac{i\varepsilon}{2}\left(W^\varepsilon(\Delta_x\phi^\varepsilon_p,\psi^\varepsilon_q) - W^\varepsilon(\phi^\varepsilon_p,\Delta_x\psi^\varepsilon_q)\right) + W^\varepsilon((\partial_t\Pi^\varepsilon_p)\phi^\varepsilon,\psi^\varepsilon_q) + W^\varepsilon(\phi^\varepsilon_p,(\partial_t\Pi^\varepsilon_q)\psi^\varepsilon)\\
&-\frac{i\varepsilon}{2}W^\varepsilon([\Delta_x,\Pi^\varepsilon_p]\phi^\varepsilon,\psi^\varepsilon_q) + \frac{i\varepsilon}{2}W^\varepsilon(\phi^\varepsilon_p,[\Delta_x,\Pi^\varepsilon_q]\psi^\varepsilon)\\
&-i\left(W^\varepsilon(\Pi^\varepsilon_p(\varepsilon\mathrm{div}_x\left(R^1\phi^\varepsilon\right)+\varepsilon R^2\phi^\varepsilon),\psi^\varepsilon_q) - W^\varepsilon(\phi^\varepsilon_p,\Pi^\varepsilon_q(\varepsilon\mathrm{div}_x\left(R^1\psi^\varepsilon\right)+\varepsilon R^2\psi^\varepsilon))\right)\\
&-iW^\varepsilon(\Pi^\varepsilon_pS^\varepsilon,\psi^\varepsilon_q).
\end{aligned} \tag{A.4}$$

Taking advantage of the estimates (A.2), a similar argument to that developed in the proof of Lemma 5.1 shows that

$$(\epsilon_p - \epsilon_q)\lim_{\varepsilon\to 0} W^\varepsilon(\phi^\varepsilon_p,\psi^\varepsilon_q) = 0$$

which ends the proof of Lemma A.2. □

Lemma A.3 *Under the assumption of Theorem* 2.3, *for any* $p \in \mathbb{N}^*$, *the term* r_p^ε *defined in the proof of this Theorem by*

$$r_p^\varepsilon = -i\Pi_p^\varepsilon R^1\, W^\varepsilon(\varepsilon\nabla_x\psi^\varepsilon, \psi_p^\varepsilon)\Pi_p^\varepsilon + i\Pi_p^\varepsilon\, W^\varepsilon(\varepsilon\nabla_x\psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^*.$$

converges to 0 as $\varepsilon \to 0$ *in* $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{2m}, L^2((0,1) \times L^2(0,1)))$.

Proof. Denote $\phi^\varepsilon = \varepsilon\nabla_x\psi^\varepsilon$ and $\phi_q^\varepsilon = \Pi_q^\varepsilon\phi^\varepsilon$. By developing $\varepsilon\nabla_x\psi^\varepsilon$ in the first term of r_p^ε, we obtain

$$r_p^\varepsilon = \tilde{r}_p^\varepsilon + s_p^\varepsilon.$$

with

$$\tilde{r}_p^\varepsilon = -i\Pi_p^\varepsilon R^1\, W^\varepsilon(\phi_p^\varepsilon, \psi_p^\varepsilon)\Pi_p^\varepsilon + i\Pi_p^\varepsilon\, W^\varepsilon(\varepsilon\nabla_x\psi_p^\varepsilon, \psi^\varepsilon)(\Pi_p^\varepsilon R^1)^*$$

and

$$s_p^\varepsilon = -i\Pi_p^\varepsilon R^1 \sum_{q\neq p} W^\varepsilon(\phi_q^\varepsilon, \psi_p^\varepsilon)\Pi_p^\varepsilon.$$

Then we remark that

$$\tilde{r_p^\varepsilon} = -i(\Pi_p^\varepsilon R^1\Pi_p^\varepsilon)\, W^\varepsilon(\phi_p^\varepsilon, \psi_p^\varepsilon)\Pi_p^\varepsilon + i\Pi_p^\varepsilon\, W^\varepsilon(\phi_p^\varepsilon, \psi_p^\varepsilon)(\Pi_p^\varepsilon R^1\Pi_p^\varepsilon)^* + o(1),$$

whose limit is equal to zero in view of the identity (6.23). It remains to prove that $s_p^\varepsilon \rightharpoonup 0$. This is a consequence of Lemma A.2. Indeed the second part of this Lemma implies that each term of this term converges to 0 separately. Then the first part of the Lemma says that ϕ^ε is bounded in $L^\infty_{loc}(\mathbb{R}_+, L^2(\widehat{\Omega}))$, thus for any $T > 0$

$$\lim_{N\to+\infty} \sup_{\varepsilon\in(0,1]} \left\| \sum_{q\geq N} \phi_q^\varepsilon \right\|_{L^\infty((0,T),L^2(\widehat{\Omega}))} = 0,$$

which enables to conclude thanks to Proposition 4.2. □

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