Fourth Order Linear Differential Equations with Imprimitive Group

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ABSTRACT

In this paper we study fourth order linear differential equations whose differential Galois groups are imprimitive. We derive optimal bounds for the degree of the minimal polynomial of the logarithmic derivative of a Liouvillian solution. This is the lowest possible order where imprimitive non monomial groups occur.

1. INTRODUCTION

Let k be a field and δ a derivation on k whose field of constants $\mathcal{C}$ is algebraically closed of characteristic 0 (e.g. $\overline{\mathbb{Q}}(x)$ with the usual derivation d/dx). For the derivation δ of k and $a \in k$ we write $\delta^m(a) = a^{(m)}$ and also $a^{(1)} = a'$, $a^{(2)} = a'', \ldots$. Any n-th order linear differential equation can be transformed to

$$L(y) = y^{(n)} + a_{n-2}\, y^{(n-2)} + \ldots + a_0\, y \;=\; 0,\, a_i \in k \qquad (1)$$

The differential Galois group $\mathcal{G}$ of $L(y)$ is a subgroup of $SL(n, \mathcal{C})$.

Definition 1.1. *Let V be a $\mathcal{C}$-vector space, let $Y_1, \ldots, Y_n$ be a basis for V and let $G \subseteq GL(V)$ be a linear group. Define a group action of $\sigma \in G$ on $I \in \mathcal{C}[Y_1, \ldots, Y_n]$ by $\sigma \cdot (I(Y_1, \ldots, Y_n)) = I(\sigma(Y_1), \ldots, \sigma(Y_n))$. If a homogeneous $I \in \mathcal{C}[Y_1, \ldots, Y_n]$ of degree m has the property that*

$$\forall \sigma \in G, \quad \sigma \cdot I(Y_1, \ldots, Y_n) = \psi_I(\sigma) \cdot (I(Y_1, \ldots, Y_n)), \quad \textit{with } \psi_I(\sigma) \in \mathcal{C}$$

then $\psi_I : G \to \mathcal{C}$ is a one dimensional character of G and I is a semi-invariant *of degree m of G with character ψ_I. If ψ_I is the trivial character $\mathbb{1}$ of G, then I is an* invariant *of degree m of G. If $j \in \mathbb{N}$ is minimal such as $(\psi_I)^j = \mathbb{1}$, then j is the* order *of the character ψ_I and of the semi-invariant I.*

From [12] Theorem 3 we get that the existence of a Liouvillian solution of the form $z = e^{\int u}$ where u is algebraic over k is equivalent to the existence of a semi-invariant that factors into linear forms. If $m = [k(u) : k]$ is minimal, then m is the minimal degree of a form that factors into linear forms. For each order n it is possible to derive a finite list of possible m that have to be considered in order to compute Liouvillian solutions of $L(y) = 0$. For second order equations the smallest possible list is given in [7] and for third order equations in [10].

Definition 1.2. Let $G \subset GL(V)$ be a linear group acting irreducibly on the vector space V of dimension n over $\mathcal{C}$. Then G is said to be *imprimitive* if there exist subspaces $V_1, \cdots, V_k$ with $k > 1$ such that $V = V_1 \oplus \cdots \oplus V_k$ and, for each $g \in G$, the mapping $V_i \to g(V_i)$ is a permutation of the set $\mathcal{S} = \{V_1, \ldots, V_k\}$. The set $\mathcal{S}$ is called a *system of imprimitivity* of G. If all the subspaces V_i are one dimensional, then G is called *monomial*. An irreducible group $G \subset GL(V)$ which is not imprimitive is called *primitive*.

For primitive groups of degree 4, the list of smallest possible m, $\{4, 5, 8, 10, 12, 16, 20, 24, 40, 48, 60, 72, 120\}$ is derived in [3].

A representation of a group G is imprimitive if the character of the representation is irreducible and induced by the character of a subgroup. If a given system of imprimitivity cannot be further decomposed, then the restriction of the stabilizer $\mathcal{G}_i$ of V_i in $\mathcal{G}$ to V_i, which we denote $H_i = \rho_i(\mathcal{G}_i)$, is a primitive group of degree $d = \dim(V_i)$. If $n = 4$ and $d = 2$ then $H_i/Z(H_i)$ is conjugated to either A_4, S_4, A_5 or $PSL(2, \mathbb{C})$ ([1, 7, 10]). In this case we say that the imprimitive representation is *of type* A_4, S_4, A_5 or $PSL(2, \mathbb{C})$.

From [10] Theorem 3.2 and [9] Proposition 2.1 we get the following upper bounds for m where $z = e^{\int u}$ is a Liouvillian solution with $[k(u) : k] = m$

1. 4 if $\mathcal{G} \subset SL(4, \mathcal{C})$ monomial.
2. $8 = 2 \cdot 4$ if $\mathcal{G} \subset SL(4, \mathcal{C})$ is of type A_4.
3. $12 = 2 \cdot 6$ if $\mathcal{G} \subset SL(4, \mathcal{C})$ is of type S_4.

ISSAC'03, August 3–6, 2003, Philadelphia, Pennsylvania, USA.

4. $24 = 2 \cdot 12$ if $\mathcal{G} \subset SL(4, \mathcal{C})$ is of type A_5.

5. There are no Liouvillian solutions if $\mathcal{G} \subset SL(4, \mathcal{C})$ is of type $PSL(2, \mathbb{C})$.

In the following we will refer to the above bounds as the *standard* bounds. It is known that the standard bounds are not best possible for $n = 6$ ([2, 3]) and we will show that for $n = 4$ they are also not best possible. The above notion of *type* of an imprimitive group is ambiguous since we will see that a group can be of several distinct types (cf. example 2.2 and example 3.1). We thus define the type of a group as the group A_4, S_4 or A_5 of lowest possible order among the possible types. This will associate the smallest possible standard bound to a group.

In this paper we investigate if the bounds above are best possible for each type.

2. MONOMIAL GROUPS

The following explicit version of [9] Proposition 3.6 shows that it is possible to characterize the equation $L(y)$ whose Galois group $\mathcal{G} \subset SL(4, \mathcal{C})$ is monomial:

LEMMA 2.1. *Let $L(y) = 0$ be an irreducible linear differential equation of order n with $\mathcal{G} \subset GL(n, \mathcal{C})$.*

1. *If there exists a solution $z = e^{\int u}$ with $[k(u) : k] = m$, then $m \geq n$.*

2. *The differential Galois group $\mathcal{G}$ of $L(y)$ is monomial if and only if there exists a solution $z = e^{\int u}$ such that $[k(u) : k] = n$.*

PROOF. 1. Suppose that $L(y) = 0$ has a Liouvillian solution $z = e^{\int u}$ with $[k(u) : k] = m$ and such that m is minimal. Since $\mathcal{G}$ sends the logarithmic derivative $u = z'/z$ into another logarithmic derivative, the minimal polynomial $P \in k[U]$ of u must be of the form

$$P = \prod_{i=1}^{m} \left(U - \frac{z_i'}{z_i} \right)$$

where the z_i are solutions of $L(y) = 0$. Since $\mathcal{G}$ sends z_i'/z_i to z_j'/z_j, it sends z_i to a multiple of z_j. In particular the space spanned by $z_1, \ldots, z_m$ is $\mathcal{G}$-invariant. Since $L(y)$ is irreducible, the group $\mathcal{G} \subset GL(n, \mathcal{C})$ is an irreducible linear group, showing that $m \geq n$.

2. Keeping the above notation, suppose now that $m = n$. Then in the basis $z_1, \ldots, z_n$ the group $\mathcal{G}$ sends a solution z_i into a multiple of some z_j. Therefore in this basis the representation of $\mathcal{G}$ is monomial. Conversely, if $\mathcal{G}$ is monomial in some basis $y_1, \ldots, y_n$, then $\mathcal{G}$ permutes the y_i'/y_i which are therefore of degree at most n. Since $\mathcal{G} \subset GL(n, \mathcal{C})$ is irreducible, we get from the above that the orbit of $u = y_1'/y_1$ cannot be less than n, showing that $[k(u) : k] = n$.

□

EXAMPLE 2.2. *The transitive group G of degree 16 of number 189 in the classification of [6] is of order 96. We will use the method presented in [15] to compute a linear differential equation with this group as differential Galois group and whose solution space is a subspace of the space of holomorphic one forms of some Galois cover with Galois group G. This group has two characters of degree 4 which are both unimodular and induced by the one dimensional character of a subgroup and thus monomial. The group can be generated by elements σ_1 and σ_2 belonging to the unique conjugacy class of size 12 whose elements are of order 4 and to a conjugacy class of elements of order 12 with the property that $\sigma_3 = (\sigma_1\sigma_2)^{-1}$ is of order 8. Consider a Galois cover of $\mathbf{P}^1$ with Galois group G ramified in $0, 1, \infty$ according to $\sigma_1, \sigma_2, \sigma_3$. The character of G on the space of holomorphic one forms (cf. [15], Theorem 4.2) contains both characters of degree 4 exactly once. From the character table (cf. [15] Section 5.2) we get that the smallest possible exponents at $0, 1, \infty$ are $\left\{-\frac{1}{2}, \frac{1}{2}, 0, 1\right\}$, $\left\{-\frac{1}{4}, \frac{3}{4}, -\frac{11}{12}, -\frac{7}{12}\right\}$ and $\left\{1+\frac{5}{8}, 1+\frac{1}{8}, 1+\frac{7}{8}, 1+\frac{3}{8}\right\}$. Since they add up to 6, there is no apparent singularity and the above must be the actual exponents (cf. [15] Section 6.3.1). Therefore an equation with those exponents at $0, 1, \infty$ must exist. According to ([5] Section 15.4) the exponents determine the equation up to 3 accessory parameters b_0, c_1 and c_0:*

$$\frac{d^4y}{dx^4} + \frac{12\,x-5}{x\,(x-1)}\,\frac{d^3y}{dx^3} + \frac{11043\,x^2 - 9359\,x + 1080}{288\,(x-1)^2\,x^2}\,\frac{d^2y}{dx^2} +$$

$$\frac{4806\,x^2 + (144\,b_0 - 4655)\,x - 144\,b_0}{144\,(x-1)^3\,x^2}\,\frac{dy}{dx} +$$

$$\left(\frac{57915\,x^3 + (12288\,c_1 - 59147)\,x^2}{12288 x^3\,(x-1)^4} \right.$$

$$\left. + \frac{(12288\,c_0 - 12288\,c_1)\,x - 12288\,c_0}{12288 x^3\,(x-1)^4} \right) y.$$

Since we have three tuples of exponents at some singularities where the exponents differ by integers, it is possible to compute the accessory parameters using the approach given in [15] Section 6.2 in order to get:

$$\frac{d^4y}{dx^4} + \frac{12\,x-5}{x\,(x-1)}\,\frac{d^3y}{dx^3} + \frac{11043\,x^2 - 9359\,x + 1080}{288\,(x-1)^2\,x^2}\,\frac{d^2y}{dx^2} +$$

$$\frac{19224\,x^2 - 25097\,x + 6477}{576\,(x-1)^3\,x^2}\,\frac{dy}{dx} +$$

$$\frac{47171 + 57915\,x^2 - 106318\,x}{12288\,x^2\,(x-1)^4}\,y.$$

By construction we must have $\mathcal{G} \cong G$. It is interesting to note that $\mathcal{G}$ can also be induced by a 2 dimensional character of a subgroup which is projectively equivalent to A_4. The group G is thus monomial and of type A_4. □

3. IMPRIMITIVE NON MONOMIAL GROUPS

The finite imprimitive non monomial subgroups of $Gl(4, \mathcal{C})$ are classified projectively, i.e. up to scalar multiples, by H.F. Blichfeldt in [1] and we will use the notation introduced there in the following. Blichfeldt gives 14 types of finite imprimitive groups. The groups belonging to the families 1° to 6° are of type A_4, those belonging to the families 7° to 11° are of type S_4 and those belonging to the families type 12° to 14° are of type A_5. However those families are not disjoint:

Example 3.1. *Taking* $\alpha = 1$ *and* $\beta = -1$ *in the family* 7° *we get the group* G *of order* 192 *generated by*

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1-\xi^2}{2} & \frac{1-\xi^2}{2} & 0 & 0 \\ \frac{-1-\xi^2}{2} & \frac{1+\xi^2}{2} & 0 & 0 \\ 0 & 0 & \frac{1-\xi^2}{2} & \frac{1-\xi^2}{2} \\ 0 & 0 & \frac{-1-\xi^2}{2} & \frac{1+\xi^2}{2} \end{pmatrix}, \begin{pmatrix} -\xi^3 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix}$$

where $\xi^4 + 1 = 0$. *The group* G *has* 7 *subgroups of order* 96. *One of them, denoted* K_1 *is generated by:*

$$\begin{pmatrix} 0 & 0 & \frac{-\xi-\xi^3}{2} & \frac{\xi-\xi^3}{2} \\ 0 & 0 & \frac{\xi-\xi^3}{2} & \frac{-\xi-\xi^3}{2} \\ \frac{\xi+\xi^3}{2} & \frac{-xi+\xi^3}{2} & 0 & 0 \\ \frac{-\xi+\xi^3}{2} & \frac{\xi+\xi^3}{2} & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & \frac{\xi+\xi^3}{2} & \frac{\xi+\xi^3}{2} \\ 0 & 0 & \frac{\xi+\xi^3}{2} & \frac{-\xi-\xi^3}{2} \\ \frac{-\xi-\xi^3}{2} & \frac{-\xi-\xi^3}{2} & 0 & 0 \\ \frac{-\xi-\xi^3}{2} & \frac{\xi+\xi^3}{2} & 0 & 0 \end{pmatrix}.$$

Another one, denoted K_2 *is generated by*

$$\begin{pmatrix} 0 & 0 & \frac{1+\xi^2}{2} & \frac{1+\xi^2}{2} \\ 0 & 0 & \frac{1-\xi^2}{2} & \frac{-1+\xi^2}{2} \\ \frac{1+\xi^2}{2} & \frac{1+\xi^2}{2} & 0 & 0 \\ \frac{1-\xi^2}{2} & \frac{-1+\xi^2}{2} & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & \frac{1+\xi^2}{2} & \frac{1+\xi^2}{2} \\ 0 & 0 & \frac{-1+\xi^2}{2} & \frac{1-\xi^2}{2} \\ \frac{1+\xi^2}{2} & \frac{1+\xi^2}{2} & 0 & 0 \\ \frac{-1+\xi^2}{2} & \frac{1-\xi^2}{2} & 0 & 0 \end{pmatrix}.$$

The representation of each subgroup K_i *is a sum of two irreducible representations of dimension* 2 *and we denote one of those characters* χ_i. *Since* $(\chi_i)^G$ *is the character of* G *we get a system of imprimitivity for each group* K_i. *Denoting* H_i *the kernel of* χ_i *and* $Z(H_i)$ *the center of* H_i *we have* $|H_1/Z(H_1)| = 24$ *and* $|H_2/Z(H_2)| = 12$, *which proves that* G *is of type* A_4 *and* S_4. *It is however natural to consider this group as a group of type* A_4 *(cf. Section 1).* $\square$

Theorem 3.2. *Let* $L(y) = 0$ *be an irreducible fourth order linear differential equation whose differential Galois group is an imprimitive non monomial subgroup of* $SL(4, \mathcal{C})$. *If* $\mathcal{G}$ *is respectively of type* A_4, S_4 *or* A_5, *then there exists a solution* $z = e^{\int u}$ *where* $[k(u) : k]$ *is minimal and equal to resp.* 8, 12 *or* 24, *with the exception of the group of family number* 13° *with* $n = 1$ *of type* A_5 *where there exists a solution* $z = e^{\int u}$ *with* $[k(u) : k] = 20$.

Proof. We note that there exists a Liouvillian solution of $L(y) = 0$ if and only if the non monomial imprimitive group $\mathcal{G}$ is of type A_4, S_4 or A_5. The result is obtained by computation using the classification of the finite imprimitive non monomial groups given in [1]. Since we show that there is only one finite imprimitive non monomial group for which the classical bound is not best possible, the classical bound will always be best possible for the infinite imprimitive non monomial groups which, by construction, contain finite imprimitive non monomial groups of arbitrary large order (cf. [4], Theorem 2.2.3).

We illustrate the approach on the first family of subgroups G_1^n of [1] whose elements are of type A_4. An imprimitive group in this family is generated by

$$R_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$S_4(n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad S_3(n) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & -ib & ib \end{pmatrix}$$

where

$$a^n = 1 \quad \text{and} \quad b^{3n} = 1.$$

Note that the classification given in [1] is a *projective* classification, i.e. up to scalar multiples. From [10] Lemma 3.1 we get that the minimal index $[k(u) : k]$ is the index of a 1-reducible subgroup of minimal index, which is invariant up to projective equivalence [14]. Therefore it is possible to work with the matrices given in [1], even if they are not unimodular.

Since the existence of a solution $z = e^{\int u}$ with $[k(u) : k] = m$ minimal is equivalent to the existence of a semi-invariant of degree m that factors into linear forms, our goal is to show that for no value of the parameters there exists a semi-invariant that factors into linear forms of degree less than 8, which is the standard bound for the type A_4. Since the groups in [1] are irreducible, there will be no such semi-invariant of degree < 4 (cf. proof of Lemma 2.1).

In order to construct the semi-invariants of the group G_1 we first consider the finite subgroup $\tilde{G}_1$ generated by the non parametrized matrices R_1, S_1, S_2.

For each linear character ψ of $\tilde{G}_1$, we compute the dimensions of the spaces $V_{\psi,d}$ of semi-invariants of $\tilde{G}_1$ of degree less than 8 with character ψ. We use the Molien series (Theorem 2.2.1 of [13]) that we generalize to non necessarily trivial characters. For a group G and a character ψ of G we get :

$$\frac{1}{|G|} \sum_{u \in G} \frac{1}{\psi(u)} \frac{1}{det(I - z\,u)} = \sum_{d=0}^{\infty} dim(V_{\psi,d})\, z^d.$$

We start with the invariants of $\tilde{G}_1$, which are the semi-invariants corresponding to the trivial character of $\tilde{G}_1$. The Molien series of $\tilde{G}_1$ is

$$1 + 2u^2 + 7u^4 + 12u^6 + O(u^8)$$

showing that the space of invariants of degree $2, 4, 6$ is respectively $2, 7, 12$. Using the Reynolds operator, we compute invariants of degree 4 until we get 7 linearly independent ones and we get :

$$\begin{aligned}\tilde{B} \;=\; & [x_1^4+x_2^4+x_3^4+x_4^4,\\ & x_1^2x_2^2+x_3^2x_4^2,\\ & x_1^3x_3+x_1x_3^3+x_2^3x_4+x_2x_4^3,\\ & x_1^2x_2x_4+x_1x_2^2x_3+x_1x_3x_4^2+x_2x_3^2x_4,\\ & x_1^2x_3^2+x_2^2x_4^2,\\ & x_1^2x_4^2+x_2^2x_3^2,\\ & x_1x_2x_3x_4].\end{aligned}$$

Invariants of degree 4 of G_1 must be linear combinations of the above invariants of $\tilde{G}_1$ and thus be of the form $f = \sum_{I\in\tilde{B}} \alpha_I\, I$, but invariants of $\tilde{G}_1$ could become semi-invariants of G_1. However semi-invariants corresponding to different characters of $\tilde{G}_1$ will also correspond to different characters of G_1. Therefore the space of semi-invariants of G_1 must be subspaces of the spaces of semi-invariants of $\tilde{G}_1$ which are also semi-invariants of G_1, i.e. which are also semi-invariants of the parametrized matrices $S_4(n), S_3(n)$. We thus set

$$S_4(n).f - \lambda_1 f = 0 \;\; (1)$$

and

$$S_3(n).f - \lambda_2 f = 0 \;\; (2)$$

Under the condition that $a \neq 0$ and $b \neq 0$ we get a Gröbner basis whose elements involving $\lambda_i, \alpha_j, a, b$ are

$$\begin{gathered}\alpha_1\,(\lambda_1-1)\,,\alpha_2\,(\lambda_1-1)\,,\alpha_3\,(\lambda_1-a)\,,\alpha_4\,(\lambda_1-a)\,,\\ \alpha_5\left(\lambda_1-a^2\right),\alpha_6\left(\lambda_1-a^2\right),\alpha_7\left(\lambda_1-a^2\right),\\ \lambda_2\alpha_1-2\,\alpha_1+\alpha_2, \lambda_2\alpha_2-12\,\alpha_1-2\,\alpha_2, \lambda_2\alpha_3-2\,\alpha_3 b+2\,\alpha_4 b,\\ \lambda_2\alpha_4-6\,\alpha_3 b-2\,\alpha_4 b, \lambda_2\alpha_5-2\,\alpha_5 b^2+2\,\alpha_6 b^2+\alpha_7 b^2,\\ \lambda_2\alpha_6-2\,\alpha_5 b^2+2\,\alpha_6 b^2-\alpha_7 b^2, \lambda_2\alpha_7-8\,\alpha_5 b^2-8\,\alpha_6 b^2,\\ \alpha_1\left(a^4-1\right),\alpha_2\left(a^4-1\right),\alpha_1\left(b^4-1\right),\alpha_2\left(b^4-1\right),\\ \alpha_1\alpha_7\left(a^2-1\right),\alpha_2\alpha_7\left(a^2-1\right),\alpha_1\alpha_7\left(b^2-1\right),\alpha_2\alpha_7\left(b^2-1\right),\\ \alpha_1\alpha_4\,(a-1)\,,\alpha_2\alpha_4\,(a-1)\,,\alpha_3\alpha_7\,(a-1)\,,\alpha_4\alpha_7\,(a-1)\,,\\ \alpha_3\left(a^2-1\right),\alpha_4\left(a^2-1\right),\alpha_1\alpha_4\,(b-1)\,,\alpha_2\alpha_4\,(b-1)\,,\\ \alpha_3\alpha_7\,(b-1)\,,\alpha_4\alpha_7\,(b-1)\,,\alpha_3\left(b^2-1\right),\alpha_4\left(b^2-1\right),\\ {\alpha_1}^2+1/12\,{\alpha_2}^2, \alpha_1\alpha_3+1/6\,\alpha_2\alpha_4, \alpha_2\alpha_3-2\,\alpha_1\alpha_4, {\alpha_3}^2+1/3\,{\alpha_4}^2,\\ \alpha_1\alpha_5+1/8\,\alpha_2\alpha_7, \alpha_2\alpha_5-3/2\,\alpha_1\alpha_7, \alpha_3\alpha_5+1/4\,\alpha_4\alpha_7,\\ \alpha_4\alpha_5-3/4\,\alpha_3\alpha_7, {\alpha_5}^2+1/4\,\alpha_6\alpha_7+1/8\,{\alpha_7}^2,\\ 1/4\,\alpha_1\,(4\,\alpha_6-\alpha_7)\,,1/4\,\alpha_2\,(4\,\alpha_6-\alpha_7)\,,1/4\,\alpha_3\,(4\,\alpha_6-\alpha_7)\,,\\ 1/4\,\alpha_4\,(4\,\alpha_6-\alpha_7)\,,1/4\,\alpha_5\,(4\,\alpha_6-\alpha_7)\,,\\ 1/8\;(2\,\alpha_6+\alpha_7)\,(4\,\alpha_6-\alpha_7)\,.\end{gathered}$$

From this we get the following cases:

1. There are no conditions on a and b if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. In this case the most generic semi-invariant of G_1 "coming from" an invariant of $\tilde{G}_1$ is of the form

   $$f = \alpha_5\,(x_1^2\,x_3^2+x_2^2\,x_4^2)+\alpha_6\,(x_1^2\,x_4^2+x_2^2\,x_3^2)+\alpha_7\,x_1\,x_2\,x_3\,x_4$$

   where

   $$\alpha_5\,(\alpha_6 - 1/4\,\alpha_7) = \alpha_5^2 + 1/4\,\alpha_6\,\alpha_7 + 1/8\,\alpha_7^2 = \alpha_6^2 + 1/4\,\alpha_6\,\alpha_7 - 1/8\,\alpha_7^2 = 0.$$

   The possible dependence between α_i and λ_i shows that not all semi-invariants will always correspond to the same character of G_1, but ignoring these possibilities clearly gives us the "worst possible" case containing all possible splittings in spaces of invariants for distinct characters of G_1. Note that "worst possible" case here means that we may look for a semi-invariant that factors into linear form in a space of semi-invariants that is too large. If there exists a semi-invariant that factors into linear forms in this space, then $resultant(f, x_1 + c_2\,x_2 + c_3\,x_3 + c_4\,x_4, x_1) = 0$ (cf. [8]). Computing a Gröbner basis we get that in this case there is no solution (i.e. no values of c_i, α_j with the α_j not all zero).

2. Either $a^4 = b^4 = 1$ or $a^2 = b^2 = 1$. We now multiply each of the above matrices by the inverse of their determinant which gives us the unimodular groups H_{576}, H_{384}, H_{288}, H_{192} and H_{96} of respective order 576, 384, 288, 192 and 96. In the last case corresponding to $a = b = 1$, as noted by Blichfeldt, the group is reducible. In the other cases we use character theory to compute that there are no 1-reducible subgroups of index less than 8 (cf. [10] proof of Theorem 3.2). Consider for example the group H_{192}. We consider all faithful imprimitive unimodular characters χ of H_{192}. In order to find the minimal index of a 1-reducible group, we consider all subgroups K of H_{192} of index < 8 and decompose the restriction χ_K of the character χ to K. If the standard bound is not best possible, then we must find a group K such that χ_K has a summand of degree 1. It turned out that none of the above groups has a 1-reducible group of index < 8.

The above computations for the invariants of $\tilde{G}_1$ correspond to the trivial character of $\tilde{G}_1$. Next we have to consider all other linear characters of $\tilde{G}_1$, there are 7 in this case. For each linear character of $\tilde{G}_1$ we compute the corresponding Molien series and using a Reynolds operator for this character we now compute all semi-invariants of degree ≤ 8. We again consider those semi-invariants that are also semi-invariants of G_1 and do the above computations. Having done this, we can conclude that for this family of groups of type A_4 the standard bound is best possible.

The standard bound was always best possible, except for one group in the family number 13° of the classification given in [1] for the parameter $n = 1$ where we found 20 instead of 24. This group is presented in the example below. □

EXAMPLE 3.3. *The unimodular group G corresponding to $n = 1$ in the family 13 is of order 240 and generated by*

$$\sigma_1 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^5 & 0 & 0 \\ 0 & 0 & \omega^7 & 0 \\ 0 & 0 & 0 & \omega^{11} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \frac{-1-\omega^2}{3} & \frac{1}{4} & 0 & \frac{-1}{4} \\ \frac{-5}{3} & \frac{-2+\omega^2}{3} & 1 & 0 \\ 0 & \frac{-1}{4} & \frac{-1-\omega^2}{3} & \frac{-5}{12} \\ 1 & 0 & 1 & \frac{-2+\omega^2}{3} \end{pmatrix}$$

where $\omega^4 - \omega^2 + 1 = 0$. This group has an abelian subgroup of index 20 which must be 1-reducible. The construction of an equation for the group G via the method of [15] is possible. In the following we give all necessary ingredients for the construction, showing also why the construction is computationally difficult. Selecting the monodromy matrices at the singularities $0, 1, \infty$ to be $\sigma_1, \sigma_2, (\sigma_1\sigma_2)^{-1}$ we see that the unimodular imprimitive character of G appears twice in the G-module of the holomorphic one forms of the corresponding covering of $P^1(\mathbb{C})$. Therefore there exists a differential equation whose exponents at $0, 1, \infty$ are up to integers $\{-7/12, -1/12, -11/12, -5/12\}$, $\{-2/3, -1/3, 1/3, 2/3\}$, $\{1 + 1/8, 1 + 3/8, 1 + 5/8, 1 + 7/8\}$. Since the exponents add up to 4, we have to either:

1. *add an integer 2 at one exponent,*
2. *add two integers at two different exponents so that the exponents remain all distinct,*
3. *add one apparent singularity with exponents $0, 1, 2, 3, 5$ and one integer at one of the exponents at $0, 1, \infty$ so that the exponents remain all distinct,*
4. *add two apparent singularities with exponents $0, 1, 2, 3, 5$,*
5. *add one apparent singularity with exponents $0, 1, 2, 3, 6$.*

For each of the above 69 possibilities we have to compute the remaining accessory parameters using the invariants of G. There are 3 linearly independent invariants of lowest degree 8 of G.

We consider the case when we add the integer 2 to the exponent $-\frac{5}{12}$. From ([11], Lemma 3.1) we get that the values of the invariants of degree 8, which are rational solutions of the 8-th symmetric power, must be of the form

$$\frac{\sum_{i=0}^{3} \gamma_i x^i}{x^7 (x-1)^5}$$

Following the method of [15], we get polynomial equations for the accessory parameters.

We could not complete the computations of all cases yet, but the equation is guarantied to exist. $\square$

4. FINAL REMARKS

The method used in this paper relies on the classification of Blichfeldt via parametrized matrices. Such a simple classification is not available for representations of higher degree. In the proof of Theorem 3.2 the general case could always be treated using a "worst case" approach where we considered all space of semi-invariants of $\tilde{G}_i$ which where also semi-invariants of G_i together. Another notable fact was that for each family we always had to consider separately only a finite set of finite groups. It is not clear that the same approach would work for higher degree.

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