Note on Algebraic solutions of differential equations with known finite Galois group

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Abstract

Given a linear differential equation with known finite differential Galois group, we discuss methods to construct the minimal polynomial of a solution. We first outline a well known general method based on a basis transformation of the basis of formal solutions at a singularity. In the second part we show how to directly construct the minimal polynomial of an eigenvector of some monodromy matrix. The method is very efficient for any irreducible second and third order linear differential equation where there is always a one dimensional eigenspace.
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1 Introduction and notation

In the algorithms computing Liouvillian solutions of linear differential equations one always has to consider cases where the differential Galois group is a finite group [5, 7, 13, 15]. The determination of the finite group is done via necessary conditions [7, 14, 15], but the computation of the minimal polynomial remains a difficult task. The existing approaches either start from an algebraic logarithmic derivative $u = z'/z$ of a solution z and compute $e^{\int u}$ [13, 4] or use a basis transformation via invariants [3, 13]. In this paper we first describe the basis transformation approach used in [9] to compute accessory parameters which allows to compute the minimal polynomial of any solution at the cost of a Gröbner basis computation. We then focus on the computation of the minimal polynomial P of the eigenvector z of some monodromy matrix at a singular point. We obtain information on the degree and on the sparsity of P by using the knowledge of the group and of the ramification data given by the exponents. Using the series expansion of the solution z it is then possible to compute the coefficients of P using linear algebra. The proposed approach is a simple combination of existing methods to which we added the use of the ramification data. The efficiency depends on the specific group structure and on arithmetic properties of the exponents. Unlike the basis transformation approach, this approach needs very little group knowledge and no Gröbner basis computation. In the last two sections we give tables for the second and third order finite primitive (and some imprimitive) groups showing that the proposed method is in general efficient for second and third order equations.

In this paper, since most results use formal power series and monodromy considerations, we always consider the field $\mathbb{C}(x)$ with the derivation $\frac{d}{dx}$. We write $\frac{d^j}{dx^j}(a) = a^{(j)}$ for $j \in \mathbb{N}$ and $a^{(1)} = a'$, $a^{(2)} = a''$, ... Let

$$L(y) = y^{(n)} + b_{n-1}\, y^{(n-1)} + \ldots + b_1\, y' + b_0\, y \;=\; 0, \qquad b_i \in \mathbb{C}(x) \tag{1}$$

be a linear differential equation of order n over $\mathbb{C}(x)$ and denote K the Picard Vessiot extension associated to L (cf. [8]). We will always assume that the differential Galois group $\mathcal{G} \subseteq GL(n, \mathbb{C})$ of $L(y)$ is finite, i.e. that all solutions of $L(y) = 0$ are algebraic.

We will work with some given representation of the finite group $\mathcal{G} \subset GL(n, \mathbb{C})$, i.e. with some abstract basis $y_1, \ldots, y_n$ of the solution space. We note that the linear group $\mathcal{G}$ is uniquely determined by a generating system $I_1, \ldots, I_t \in \mathbb{C}[x_1, \ldots, x_n]$ of the ring $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{G}}$ of polynomial invariants of $\mathcal{G}$.

Definition 1.1 *Let $L(y)$ be a n-th order linear differential equation with Galois group $\mathcal{G} \in GL(n, \mathcal{C})$. To a basis $y_1, \ldots, y_n$ of the solution space of $L(y) = 0$ we associate the evaluation morphism*

$$\begin{array}{rcl} \Phi\colon \mathbb{C}[x_1, \ldots, x_n] & \rightarrow & K \\ x_i & \mapsto & y_i \end{array}$$

The morphism Φ maps polynomial invariants of $\mathcal{G}$ into rational functions.

Definition 1.2 *Let $L(y) = 0$ be an n-th order homogeneous linear differential equation and let $y_1, \dots, y_n$ be a fundamental system of solutions. The differential equation $L^{\circledS m}(y)$ whose solution space, denoted V_m, is spanned by all monomials of degree m in $y_1, \dots, y_n$ is called the m-*th symmetric power *of $L(y) = 0$.*

An algorithm to construct $L^{\circledS m}(y)$ is given in [11]. The image $\Phi(I_j)$ of a homogeneous invariant I_j of degree N is a rational solution of $L^{\circledS m}(y) = 0$ ([13] Lemma 1.6) and it is possible to bound the order M_j of the numerator and the orders $\alpha_{j,1}, \dots, \alpha_{j,t}$ of the poles at the singularities $c_1, \dots, c_t$ of the rational functions resulting from the evaluation of the invariant under Φ ([14], Lemma 3.1).

In order to identify the possible conjugacy classes of a monodromy matrix M_c at a singularity c, we introduce the following notation: Since $\mathcal{G}$ is finite, according to [14] Lemma 2.2 (3) all singular points c of $L(y)$ must be regular singular points and all exponents are rational. From [14] Lemma 2.4 we get that the monodromy matrix $M_{c,L}$ corresponding to a small loop γ around c containing no other singularities of $L(y)$ is conjugated to

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\alpha_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}\alpha_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & e^{2\pi\sqrt{-1}\alpha_n} \end{pmatrix} \tag{2}$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{Q}$ are the exponents of $L(y)$ at c. Therefore the eigenvalues of $M_{c,L}$ determine the exponents at c up to integers. We write $\alpha_i = \frac{a_i}{b_i} + n_i$ where $n_i \in \mathbb{Z}$ and $\frac{a_i}{b_i} \in]0, 1]$, then $\frac{a_i}{b_i}$ is the part of the exponents at c that is uniquely determined by $M_{c,L}$.

Definition 1.3 *With the above notation we define $\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\}$ as the* ramification data *of $L(y)$ at the singularity c or, equivalently, of $M_{c,L} \in \mathcal{G}$.*

2 A general method to compute the minimal polynomial of a solution

Most approaches to the computation of algebraic solutions of a linear differential equation first compute an algebraic solution of the Riccati equation (cf. [4, 13]). In this section we review a more direct method outlined in [9] which is similar to our later approach. The basic idea is to relate the basis $s_1, \dots, s_n$ of formal series at some singular point c to the above "abstract" basis $y_1, \dots, y_n$, i.e. to find a basis transformation T that maps the basis $s_1, \dots, s_n$ into $y_1 = \Phi(x_1), \dots, y_n = \Phi(x_n)$. For that we proceed as follows:

1. Select the basis $y_1, \dots, y_n$ so that in the conjugation class of the monodromy matrix M_c at c there is a diagonal element (their may be several choices for the conjugacy class of M_c in $\mathcal{G}$).

2. Compute a $\mathbb{C}$-basis $I_1, \dots, I_s$ of the ring $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{G}}$ of polynomial invariants of $\mathcal{G} \subset GL(n, \mathbb{C})$.

3. For each invariant I_j of degree m_j in the above basis, bound the order M_j of the numerator and the orders $\alpha_{j,1}, \dots, \alpha_{j,t}$ of the poles at the finite singularities $c_1, \dots, c_t$ of the rational functions resulting from the evaluation of the invariant under Φ

4. Fix some element v in the $\mathbb{C}$-span of $x_1, \dots, x_n$. For example choose v such that its minimal polynomial is of small degree, i.e. such that stabilizer $\mathrm{Stab}_{\mathcal{G}}(v)$ is large.

5. Express the coefficients of the minimal polynomial of v over $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{G}}$ as
$$P = \prod_{\sigma \in \mathcal{T}} (Y - \sigma(v)),$$
where $\mathcal{T}$ is a left transversal of $\mathrm{Stab}_{\mathcal{G}}(v)$, in terms of $I_1, \dots, I_t$ (cf. [13], Section 4.3).

6. Consider a matrix T with unknown entries and substitute x_i by $T(s_i)$ into I_j in order to obtain a series expansion of $\Phi(I_j)$ at c. Since the coefficients of the series expansion of the solutions of $L^{\circledS m_j}(y) = 0$ at c satisfy a linear relation of finite length B_{m_j} (cf. [10]), it is sufficient to set the coefficients from $M+1$ to $M+1+B_{m_j}$ of the series
$$\Phi(I_j) \prod_{i=1}^{t} (x - c_t)^{-\alpha_{j,t}}$$
equal to zero in order to guaranty that all $\Phi(I_j)$ are non zero rational functions or equivalently that all I_j are invariants of $\mathcal{G}$. This gives a polynomial system for the entries of T.

7. Solve the system for the entries of T and substitute the values in the above expressions of $\Phi(I_j)$ in order to find I_j. This gives the minimal polynomial $\Phi(P)$ of the solution $\Phi(v)$.

The number of variables needed to represent T depends on the dimension of the eigenspaces of the monodromy matrix M_c at c.

Remark: If $L^{\circledS m_j}(y)$ is of maximal order (this can always be archived, cf. [14], Theorem 3.5 and its proof), then the differential Galois group of $L(y) = 0$ is a subgroup of the group $\mathcal{G}$ if and only if all invariants I_s of the basis of $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{G}}$ evaluate to non zero rational functions. Therefore, by also testing the maximal subgroups of $\mathcal{G}$, the above method can also be used to verify that a given finite linear group is the differential Galois group of a linear differential equation.

Example 2.1 *In [15] we prove that the group G_{54} generated by*

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 & 0 \\ 0 & -\omega^3 - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\omega^6 + \omega^3 + 1 = 0$, is the differential Galois group of the differential operator

$$\frac{d^3}{dx^3} + \frac{3(3x^2-1)}{x(x-1)(x+1)}\frac{d^2}{dx^2} + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2}\frac{d}{dx} + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3}.$$

The ramification data at $0, 1, -1, \infty$ is

$$\left\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right\}, \left\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right\}, \left\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right\}, \left\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right\}.$$

In order to diagonalize an element (in the conjugacy class of elements) with ramification data $\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\}$ into

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\frac{1}{3}} & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}\frac{5}{6}} & 0 \\ 0 & 0 & e^{2\pi\sqrt{-1}\frac{5}{6}} \end{pmatrix},$$

we perform the basis transformation

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\frac{1}{3}} & 0 & -1 \\ -e^{2\pi\sqrt{-1}\frac{1}{3}} & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In this new basis the invariants of G_{54} *are:*

$$\begin{aligned}
I_1 &= x_1^6+5x_1^2x_2^4+40x_1^2x_2x_3^3-\frac{2}{3}x_2^6+\frac{40}{3}x_2^3x_3^3+\frac{16}{3}x_3^6 \\
I_2 &= x_1^4x_2^2+\frac{2}{3}x_1^2x_2^4-\frac{8}{3}x_1^2x_2x_3^3+\frac{1}{9}x_2^6-\frac{8}{9}x_2^3x_3^3+\frac{16}{9}x_3^6 \\
I_3 &= x_1^4x_3^2-2x_1^2x_2^2x_3^2+x_2^4x_3^2 \\
I_4 &= x_1^4x_2x_3-\frac{2}{3}x_1^2x_2^3x_3-\frac{4}{3}x_1^2x_3^4-\frac{1}{3}x_2^5x_3+\frac{4}{3}x_2^2x_3^4 \\
I_5 &= x_1^9-6x_1^5x_2^4-48x_1^5x_2x_3^3+8x_1^3x_2^6-160x_1^3x_2^3x_3^3-64x_1^3x_3^6-3x_1x_2^8 \\
&\quad -48x_1x_2^5x_3^3-192x_1x_2^2x_3^6 \\
I_6 &= x_1^{13}x_2x_3-\frac{2}{3}x_1^{11}x_2^3x_3-\frac{4}{3}x_1^{11}x_3^4-\frac{19}{3}x_1^9x_2^5x_3-\frac{140}{3}x_1^9x_2^2x_3^4+12x_1^7x_2^7x_3 \\
&\quad -120x_1^7x_2^4x_3^4-\frac{19}{3}x_1^5x_2^9x_3+56x_1^5x_2^6x_3^4+\frac{256}{3}x_1^5x_3^{10}-\frac{2}{3}x_1^3x_2^{11}x_3+100x_1^3x_2^8x_3^4+ \\
&\quad \frac{512}{3}x_1^3x_2^2x_3^{10}+x_1x_2^{13}x_3+12x_1x_2^{10}x_3^4-256x_1x_2^4x_3^{10}
\end{aligned}$$

Using the approach in [13] Section 4.2 we get that $z = x_1 + \frac{2\omega^3+1}{3}x_2 + \frac{4\omega^3+2}{3}x_3$*, being an eigenvector of a one-reducible subgroup of minimal index, has a minimal polynomial of smallest possible degree. The minimal polynomial* P *of* z *over* $\mathbb{C}[x_1,\ldots,x_3]^{\mathcal{G}}$ *can be expressed (cf. [13] Section 4.2) in terms of invariants as follows:*

$$\begin{aligned}
&Y^{18}+(40I_4+40I_3+10I_2-2I_1)\,Y^{12}+ \\
&\left(-\frac{64}{3}I_4I_3+\frac{56}{3}I_4I_2+8I_4I_1+\frac{112}{3}I_3^2+56I_3I_2+8I_3I_1-\frac{29}{3}I_2^2+2I_2I_1+I_1^2\right)Y^6 \\
&+\left(-\frac{128}{9}I_4I_3^2-\frac{1280}{27}I_4I_3I_2-\frac{128}{9}I_4I_2^2+\frac{64}{27}I_3^3+\frac{320}{9}I_3^2I_2+\frac{320}{9}I_3I_2^2+\frac{64}{27}I_2^3\right)
\end{aligned}$$

We now want to link those abstract considerations to the given equation. We compute a basis of series solutions at 0 *and order the basis elements according to the ramification type* $\frac{1}{3},\frac{5}{6},\frac{5}{6}$ *of the above diagonal matrix:*

$$s_1 = x^{-2/3}(1+\frac{41}{81}x^2+\frac{874}{2187}x^4+\ldots)$$

$$
\begin{aligned}
s_2 &= x^{-1/6}(-1 - \frac{37}{54}x^2 - \frac{90797}{157464}x^4 + \ldots) \\
s_3 &= x^{-1/6}(x + \frac{137}{162}x^3 + \frac{13249}{17496}x^5 + \ldots)
\end{aligned}
$$

In this example a precision of $O(x^{30})$ was sufficient. Using the exponents we get that the values under Φ of invariants of degree 6, 9 and 15 are of the form

$$
\frac{\sum_{j=0}^{4} \alpha_j x^j}{x^4(x-1)^4(x+1)^4}, \frac{\sum_{j=0}^{6} \beta_j x^j}{x^6(x-1)^6(x+1)^6}, \frac{\sum_{j=0}^{10} \gamma_j x^j}{x^{10}(x-1)^{10}(x+1)^{10}}.
$$

Replacing x_1 by s_1, x_2 by $k_1 s_2 + k_2 s_3$ and x_3 by $k_3 s_2 + k_4 s_3$ in the invariants $I_1, \ldots, I_6$ and setting in the numerators the terms of the coefficients of the monomials whose degree is larger than the above degree bounds equal to zero we get a polynomial system for the k_i. Solving the resulting polynomial system we get:

$$
\begin{aligned}
0 &= k_1 + \frac{717897987691852588770249}{3573412790272}k_4^{15} - \frac{1397493}{3328}k_4^3 \\
0 &= k_2 + \frac{30594122838806286238301435385493 71}{29273397577908224}k_4^{21} - \frac{7541715912652143}{27262976}k_4^9 \\
0 &= k_3 + \frac{801192891401586246420164787498 9}{3659174697238528}k_4^{19} - \frac{19675924374843}{3407872}k_4^7 \\
0 &= k_4^{24} - \frac{10737418240}{4052555153018976267}k_4^{12} - \frac{1152921504606846976}{443426488243037769948249630619149892803}
\end{aligned}
$$

and from there the values of the invariants. Plugging into the equation above gives:

$$
\begin{aligned}
& Y^{18} + \frac{-2x^4 - 20(\xi\phi - \xi)x^3 - 4x^2 + 20(\xi\phi - \xi)x - 2}{x^4(x-1)^4(x+1)^4} Y^{12} \\
& + \left(\frac{27x^8 - 108(\xi\phi - \xi)x^7 + 28(4\phi - 3)x^6 - 108(\xi\phi - \xi)x^5 - 14(-39 + 16\phi)x^4}{27x^8(x-1)^8(x+1)^8} \right. \\
& \qquad \left. \frac{+108(\xi\phi - \xi)x^3 + 28(4\phi - 3)x^2 + 108(\xi\phi - \xi)x + 27}{27x^8(x-1)^8(x+1)^8} \right) Y^6 \\
& + \frac{512(6\xi\phi - 13\xi)}{729x^9(x-1)^9(x+1)^9}
\end{aligned}
$$

where ϕ is a root of $3X^2 - 6X - 1$ and ξ is a root of $3X^2 - \phi$. □

Note that once the above basis transformation is known, it is easy to compute the minimal polynomial of any solution. Therefore this approach gives more information than needed. Computational experiments indicate that the method becomes unpractical for differential equations of order ≥ 4 if large eigenspaces are involved.

Example 2.2 *The following algebraic equation whose Galois group is of order* 36 *(the transitive group* $(8,15)$ *in the notation of [1]) has been constructed by G. Malle:*

$$Y^8 + 8\,Y^6 + \left(16\,x^2 - 16\,x + 4\right)Y^4 + \left(8\,x^4 - 16\,x^3 + 16\,x^2 - 8\,x\right)Y^2$$
$$+x^6 - 3\,x^5 + 6\,x^4 - 7\,x^3 + 3\,x^2$$

Using the naive method given in the introduction of Section 2 in [2], we obtain a fourth order linear differential equation $L_{32}(y)$ *having the roots of the above equation as solutions . The equation* $L_{32}(y)$ *has seven true singular points at* $0, 1, \infty$ *and at the roots of* $x^2 - x + 3$ *and* $x^2 - x - 1$*. The exponents at* ∞ *are* $\{1/4, -3/4, 3/4, -1/4\}$ *and* $\{0, 1, 1/2, 3/2\}$ *at the other true singularities. The above method would introduce* 8 *parameters in the matrix of the basis transformation since the eigenspaces of the monodromy matrices are all of dimension two.*

For second order equations this approach uses only one variable and therefore reduces to gcd computation, like the approach in [13] Section 5 if one uses the differential relations between semi-invariants given by L. Fuchs.

3 The minimal polynomial of an eigenvector of a Monodromy matrix

Instead of looking for a basis transformation, we directly compute the minimal polynomial P of an eigenvector z of some monodromy matrix at a singularity c. Once the degree of P is known, the series expansion of z at c can be used to compute the coefficients of P.

We will use the following result

Lemma 3.1 *([13], Corollary 1.4) If for an algebraic solution* z *of* $L(y) = 0$ *we have* $[k(z'/z) : k] = m$*, then the minimal polynomial* $P(\mathbf{Y}) = 0$ *of* z *over* k *is of the form*

$$\mathbf{Y}^{i\cdot m} + a_{m-1}\mathbf{Y}^{i\cdot(m-1)} + \ldots + a_1\mathbf{Y}^i + a_0 \quad (a_j \in k,\, m = [k(z'/z) : k]).$$

where $i \in \mathbb{N}$.

For each finite group $\mathcal{G}$ it is possible, starting with a representation of $\mathcal{G}$ in some basis $y_1, \cdots, y_n$, to compute the numbers m, i for a solution $z = \sum_{j=1}^{n} \alpha_j y_j$. For this we note that the length of the orbit of z'/z under $\mathcal{G}$ and the length of the orbit of the line spanned by z under $\mathcal{G}$ are both equal to m and that $i = |Orb_G(z)|/m$.

3.1 Matrices with a one dimensional eigenspace

In this section we will assume that there exists a monodromy matrix at some singularity having a one dimensional eigenspace. For irreducible second and third order linear differential equations such a solution always exists and we will focus on those cases in the last two sections.

Since conjugated elements have the same eigenvalues and eigenvectors, and since the values m, i are the same for all multiples of the solution space we get:

Lemma 3.2 *Let the differential Galois group $\mathcal{G}$ of $L(y)$ be finite and C be a conjugation class of $\mathcal{G}$. All elements of C have the same eigenvalues and for all eigenvectors of elements of C corresponding to a one dimensional eigenspace of a same eigenvalue, the numbers m and i above are identical.*

It is therefore sufficient to compute the values m and i for some representative of each conjugacy class of $\mathcal{G}$.

Example 3.3 *Consider again the group G_{54}. In the table below each column represents a ramification type of the groups. For each eigenspace of dimension one, we list the numbers m and i of the corresponding eigenvectors. Ramification types where m, i correspond for the eigenvalues of one dimensional eigenspaces have been merged to a single column and the scalar* *elements of the group are not present.*

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{6}$
	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{6}$
	1	1	$\frac{2}{3}$	$\frac{1}{3}$
i		6		
m		3		
i		6		
m		3		
i	3	6	3	3
m	9	3	9	9

For the ramification data $\{1/3, 2/3, 1\}$ we obtain that all eigenvectors z are such that $[k(z^6) : k] = 3$ and $[k(z) : k] = 3 \cdot 6$. However, in general, different classes with the same ramification data can lead to different values of m, i. Therefore in general those values are only known up to a finite set of possibilities (cf. section 5.2).

We denote χ the character of the representation of $\mathcal{G}$ on the solution space of $L(y) = 0$. In order to compute the minimal polynomial of an element in a one dimensional eigenspace, we now proceed as follows:

1. From the exponents, we identify a finite set of possible conjugacy classes for a given monodromy matrix and, for each possibility, deduce from a table like the one above the possible values of m, i of an eigenvector v in a one dimensional eigenspace of M_c for each possible class with this ramification data.

2. Denote P the minimal polynomial of v. Since the coefficients of the minimal polynomial P are symmetric functions of the roots, they are rational solutions of symmetric powers. This allows to bound the poles and the degree of the numerator of the coefficients using the exponents of the differential equation ([14], Lemma 3.1).

3. Since the eigenspace is one dimensional, we can associate, up to multiple, a formal solution at c to it. The multiple is irrelevant, since the value of m, i is valid for all multiples.

4. Knowing the exact degree of the minimal polynomial of $\Phi(v)^i$, bounds on the coefficients and the series expansion s^i of $\Phi(v)^i$ up to any precision allows to compute the minimal polynomial P_i of $\Phi(v)^i$ using linear algebra (cf. [10]).

5. Replacing Y by Y^i in P_i gives the minimal polynomial P of $\Phi(v)$.

This gives a direct method based only on the above table and exponents of the equation. As the examples will show, the approach depends on arithmetic conditions on the exponents since this will determine the size of the linear system to be solved.

Example 3.4 *We consider again the equation of Example 3.3 whose differential Galois group is G_{54}. According to the above table, the solution corresponding to the exponent $1/3$ at 0 will have values $m = 9, i = 3$ and its minimal polynomial will be of the form $P = \sum_{j=0}^{9} b_j (Y^3)^j$ where the b_j are rational solutions of $L^{\circledS(3\cdot(9-j))}(y) = 0$. Note that all one dimensional eigenspaces correspond to $m = 9, i = 3$ and therefore it is not possible to compute the solution found in Example 2.1 using this approach. According to ([14], Lemma 3.1) the rational functions b_j must be of the form:*

$$b_j = \frac{\left(\sum_{s=0}^{2(9-j)} \gamma_{j,s} x^s\right)}{x^{2(9-j)}(x-1)^{2(9-j)}(x+1)^{2(9-j)}}.$$

Evaluating the polynomial P at s^3 where s is a power series at 0 of a solution with exponent $-2/3$, we obtain a linear system in the unknown $\gamma_{j,s}$. This gives us the following minimal polynomial

$$Y^{27} - \frac{9\left(1+x^2\right)^2}{16\,x^4\,(x-1)^4\,(x+1)^4}\,Y^{21} - \frac{21\left(1+x^2\right)\left(x^2+2\,x-1\right)\left(x^2-2\,x-1\right)}{64\,x^6\,(x-1)^6\,(x+1)^6}\,Y^{18}$$
$$-\frac{27\left(-52\,x^6+202\,x^4-52\,x^2+7+7\,x^8\right)}{2048\,x^8\,(x-1)^8\,(x+1)^8}\,Y^{15}$$
$$-\frac{63\left(1+x^2\right)^3\left(x^2+2\,x-1\right)\left(x^2-2\,x-1\right)}{4096\,x^{10}\,(x-1)^{10}\,(x+1)^{10}}\,Y^{12}$$
$$-\frac{3\left(5\,x^4+18\,x^2+5\right)\left(7\,x^4+6\,x^2+7\right)\left(1+x^2\right)^2}{65536\,x^{12}\,(x-1)^{12}\,(x+1)^{12}}\,Y^{9}$$
$$-\frac{27\left(1+x^2\right)\left(x^2+2\,x-1\right)\left(x^2-2\,x-1\right)\left(x^8+20\,x^6-26\,x^4+20\,x^2+1\right)}{262144\,x^{14}\,(x-1)^{14}\,(x+1)^{14}}\,Y^{6}$$

$$-\frac{9\left(1028\,x^{12}+88\,x^{2}-2712\,x^{6}+1028\,x^{4}+4970\,x^{8}-2712\,x^{10}+88\,x^{14}+7+7\,x^{16}\right)}{16777216\,x^{16}\left(x-1\right)^{16}\left(x+1\right)^{16}}Y^{3}$$

$$-\frac{\left(1+x^{2}\right)^{3}\left(x^{2}+2\,x-1\right)^{3}\left(x^{2}-2\,x-1\right)^{3}}{16777216\,x^{18}\left(x-1\right)^{18}\left(x+1\right)^{18}}$$

of a solution of $L(y)=0$.

Using the above bounds it is also possible to use the function *seriestoalgeq* of GFUN (cf. [10]) to compute the minimal polynomial, but this program uses a global bound on the size of the coefficients and therefore introduces too many unknowns. The example above reduces to a linear system with 96 variables. The size of the linear system depends on m and on the exponents of $L(y)$. The following example illustrates that the approach can be very efficient.

Example 3.5 *Consider the differential equation*

$$\frac{d^3y}{dx^3}+\frac{7x-3}{x(x-1)}\frac{d^2y}{dx^2}+\frac{2(138x^2-115x+7)}{27x^2(x-1)^2}\frac{dy}{dx}++\frac{10(162x^2-28x-7)}{729x^3(x-1)^2}y.$$

whose differential Galois group is the imprimitive G_{81} *defined in Example 4.9 of [15]. The ramification data of* $L(y)$ *at* 0 *is* $\{\frac{4}{9},\frac{7}{9},\frac{7}{9}\}$ *and from the table given in Section 4.2 we see that the solution corresponding to the exponent* $-5/9$ *at* 0 *will have values* $m=3, i=9$. *The minimal polynomial of this solution will be of the form* $P=\sum_{i=0}^{3}b_j(Y^9)^j$ *where the* b_j *are rational solutions of* $L^{\circledS(9\cdot(3-j))}(y)=0$. *According to ([14], Lemma 3.1) the rational functions* b_j *must be of the form:*

$$b_j=\frac{\left(\sum_{s=0}^{2(3-j)}\gamma_{0,s}x^s\right)}{x^{5(3-j)}(x-1)^{6(3-j)}}.$$

Evaluating the polynomial P *at* s^9 *where* s *is a power series at* 0 *of a solution with exponent* $-5/9$, *we obtain a linear system in the unknown* $\gamma_{j,s}$. *This gives us the following minimal polynomial*

$$\left(Y^9\right)^3-\frac{x^2-9x+9}{3^2x^5(x-1)^6}\left(Y^9\right)^2+\frac{1}{3^5x^6(x-1)^{12}}Y^9-\frac{1}{3^9x^9(x-1)^{18}}$$

of a solution of $L(y)=0$. *Here one only has to solve a linear system with* 15 *variables.*

Remark: In the above example there was a unique possibility for m, i. Therefore, instead of using series of the length given by the bounds, one can start the computation with series of arbitrary length and increase the precision until a solution is found.

The assumption that there exists a monodromy matrix with a one dimensional eigenspace is valid for irreducible second and third order equations but is no longer valid for linear differential equations of order ≥ 4. Example 2.2 shows that for the differential equation $L_{32}(y)$ at each true singularity the eigenspaces of a monodromy matrix are of dimension 2. In the next section we show how to extend the method presented in this section to the case where there is no one dimensional eigenspace

3.2 If no monodromy has a one dimensional eigenspace

If no one dimensional eigenspace exists at a singularity, it is still possible to use the previous approach. Denote χ the character of $\mathcal{G}$. In order to get a finite set of possibilities for m, i for an eigenvector z for the eigenvalue α of the monodromy matrix M_c at c we proceed as follows. Denote ϕ_α the one dimensional character of $< M_c >$ corresponding to the eigenvalue α.

1. We are looking for $H_1 = \text{stab}_{\mathcal{G}}(z'/z)$. Consider all subgroups $H_1 \subset \mathcal{G}$ which are maximal with the property that
   (a) H_1 contains $< M_c >$
   (b) $\chi_{|H_1}$, the restriction of χ to H_1, has a summand ψ of degree one.
   (c) $\psi_{<M_c>} = \phi_\alpha$.
2. We are looking for $H_2 = \text{stab}_{\mathcal{G}}(z)$. For each group H_1 consider all subgroups $H_2 \subset \mathcal{G}$ which are maximal with the properties that
   (a) H_1 contains H_2 and H_2/H_1 is cyclic (cf. [13], Corollary 1.4).
   (b) $\psi_{|H_2}$, the restriction of ψ to H_2, is $\mathbb{1}_2$, the trivial character of H_2.
   (c) $\phi_{\alpha|H_2} = \mathbb{1}_2$.
3. The possible values for m are all possible indices $[\mathcal{G} : H_1]$ and the possible values for i are all possible values of $[H_2 : H_1]$.

Example 3.6 *Consider the differential equation $L_{32}(y) = 0$ defined in example 2.2. In order to have $H_1 = stab_{\mathcal{G}}(z'/z)$ of maximal order, we select the point ∞ where the monodromy matrix is of order 4. Translating ∞ to 0 gives* *us an equation* $\tilde{L}_{32}(y) = 0$ *whose exponents at* 0 *are* $\{1/4, -3/4, 3/4, -1/4\}$ *and* $\{0, 1, 1/2, 3/2\}$ *at* 1, ∞ *and at the roots of* $3x^2 - x + 1$ *and* $x^2 + x - 1$. *The group* G_{32} *has* 3 *cyclic subgroups of order* 4 *containing an element of the form*

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\frac{1}{4}} & 0 & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}\frac{1}{4}} & 0 & 0 \\ 0 & 0 & e^{2\pi\sqrt{-1}\frac{3}{4}} & 0 \\ 0 & 0 & 0 & e^{2\pi\sqrt{-1}\frac{3}{4}} \end{pmatrix}$$

having all the unique subgroup $< \pm 1 >$. *The groups of order* 4 *are contained in groups* H_1 *of order* 8 *having* 4 *eigenspaces of dimension one. From this we get that* H_2 *is always trivial and that the possibilities for* m, i *are* $8, 4$ *or* $4, 8$. *Therefore the minimal polynomial of an eigenvector of* $\tilde{M}_0$ *is of the form*

$$\tilde{P} = Y^{8\cdot 4} + \sum_{j=1}^{7} b_j Y^{j\cdot 4}$$

Using [14], Lemma 3.1 we get that the coefficients b_j *must be of the form*

$$b_j = \frac{\sum_{s=0}^{3\cdot j} \gamma_{j,s} x^s}{x^{3\cdot j}}$$

evaluating the polynomial at a power series for the exponent $-3/4$ *we obtain a linear system for the unknown* $\gamma_{j,s}$ *from which we get* $\tilde{P}$ *as*

$$Y^{32} - \frac{2(x^2 - x + 1)(x^4 - 6x^3 + 7x^2 - 2x + 1)}{x^6} Y^{24}$$
$$+ \left(\frac{3x^{12} - 54x^{11} + 193x^{10} - 338x^9 + 320x^8 - 182x^7 + 57x^6 + 10x^5}{x^{12}} \right.$$
$$\left. \frac{-4x^4 - 10x^3 + 13x^2 - 6x + 1}{x^{12}} \right) Y^{16}$$
$$- \frac{2(x^2 - x + 1)(x^4 - 6x^3 + 7x^2 - 2x + 1)(x^2 + x - 1)^4}{x^{14}} Y^8 + \frac{(x^2 + x - 1)^8}{x^{16}}$$

and, applying the transformation $x \mapsto \frac{1}{x}$, *the minimal polynomial* P *of a solution of* $L(y) = 0$.

The above approach may introduce many possibilities for m, i, especially if the monodromy matrices at the singularities all have small order.

4 The exceptional finite groups of degree 3

In this section we compute the number i, m for all exceptional finite groups of degree 3 that appear in the algorithm proposed in [15]. We refer to [15] and [13] for the definition of the groups. Together with [15] this gives a complete and efficient algorithm for the computation of Liouvillian solutions of third order linear differential equations. Note that from the tables we get that there will always be a solution for which the numbers m, i are uniquely defined. Comparing the result to the minimal possible values of m given in [13] we see that the m given by this approach are close to the minimal bound, with the exception of one conjugacy class for the group H_{216}. Note that for the imprimitive subgroups of $SL(n, \mathbb{C})$ the best possible m is n.

4.1 The group G_{162}

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{1}{3}$ $\frac{5}{6}$ $\frac{5}{6}$	$\frac{2}{3}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{8}{9}$ $\frac{5}{9}$ $\frac{5}{9}$	$\frac{5}{9}$ $\frac{2}{9}$ $\frac{2}{9}$	$\frac{7}{9}$ $\frac{1}{9}$ $\frac{1}{9}$	$\frac{4}{9}$ $\frac{7}{9}$ $\frac{7}{9}$	$\frac{1}{9}$ $\frac{4}{9}$ $\frac{4}{9}$	$\frac{2}{9}$ $\frac{8}{9}$ $\frac{8}{9}$	$\frac{2}{9}$ $\frac{13}{18}$ $\frac{1}{18}$	$\frac{5}{9}$ $\frac{1}{18}$ $\frac{7}{18}$	$\frac{4}{9}$ $\frac{17}{18}$ $\frac{11}{18}$	$\frac{7}{9}$ $\frac{17}{18}$ $\frac{5}{18}$	$\frac{8}{9}$ $\frac{13}{18}$ $\frac{7}{18}$	$\frac{1}{9}$ $\frac{11}{18}$ $\frac{5}{18}$
i		9	9	18	18	18	18	18	18	9	9	9	9	9	9
m		9	9	3	3	3	3	3	3	9	9	9	9	9	9
i										18	18	18	18	18	18
m										9	9	9	3	3	9
i	9									18	18	18	18	18	18
m	9									3	3	3	9	9	3

4.2 The group G_{81}

ramification type	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{7}{9}$ $\frac{7}{9}$ $\frac{4}{9}$	$\frac{1}{9}$ $\frac{1}{9}$ $\frac{7}{9}$	$\frac{5}{9}$ $\frac{5}{9}$ $\frac{8}{9}$	$\frac{4}{9}$ $\frac{4}{9}$ $\frac{1}{9}$	$\frac{2}{9}$ $\frac{2}{9}$ $\frac{5}{9}$	$\frac{8}{9}$ $\frac{8}{9}$ $\frac{2}{9}$
i	3	9						
m	9	3						
i	3	9						
m	9	3						
i	3	9	9	9	9	9	9	9
m	9	3	3	3	3	3	3	3

4.3 The group G_{54}

The corresponding table is given in Example 3.3.

4.4 The group G_{27}

For any eigenvector of a non scalar monodromy matrix we have $m = 3$ and $i = 3$. In this case an eigenvector for a non scalar monodromy matrix always corresponds to an m which is minimal.

4.5 The group $H_{216}^{SL_3}$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{1}{3}$ $\frac{2}{3}$ 1	$\frac{1}{3}$ $\frac{2}{3}$ 1	$\frac{3}{4}$ $\frac{1}{4}$ 1	$\frac{2}{3}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{1}{3}$ $\frac{5}{6}$ $\frac{5}{6}$
i		18	3	12	9	9
m		12	72	54	9	9
i		18	3	12		
m		12	72	54		
i	9	18	3	9		
m	9	12	72	9		

ramification type	$\frac{7}{9}$ $\frac{1}{9}$ $\frac{1}{9}$	$\frac{5}{9}$ $\frac{2}{9}$ $\frac{2}{9}$	$\frac{2}{9}$ $\frac{8}{9}$ $\frac{8}{9}$	$\frac{4}{9}$ $\frac{7}{9}$ $\frac{7}{9}$	$\frac{8}{9}$ $\frac{5}{9}$ $\frac{5}{9}$	$\frac{1}{9}$ $\frac{4}{9}$ $\frac{4}{9}$	$\frac{2}{3}$ $\frac{5}{12}$ $\frac{11}{12}$	$\frac{1}{3}$ $\frac{7}{12}$ $\frac{1}{12}$	$\frac{1}{9}$ $\frac{11}{18}$ $\frac{5}{18}$	$\frac{2}{9}$ $\frac{1}{18}$ $\frac{13}{18}$	$\frac{5}{9}$ $\frac{1}{18}$ $\frac{7}{18}$	$\frac{8}{9}$ $\frac{7}{18}$ $\frac{13}{18}$	$\frac{7}{9}$ $\frac{5}{18}$ $\frac{17}{18}$	$\frac{4}{9}$ $\frac{11}{18}$ $\frac{17}{18}$
i	18	18	18	18	18	18	9	9	9	9	9	9	9	9
m	12	12	12	12	12	12	9	9	9	9	9	9	9	9
i							12	12	18	18	18	18	18	18
m							54	54	36	12	36	36	36	12
i							12	12	18	18	18	18	18	18
m							54	54	12	36	12	12	12	36

4.6 The group $H_{72}^{SL_3}$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{3}{4}$ $\frac{1}{4}$ 1	$\frac{5}{6}$ $\frac{5}{6}$ $\frac{1}{3}$	$\frac{1}{6}$ $\frac{1}{6}$ $\frac{2}{3}$	$\frac{2}{3}$ $\frac{11}{12}$ $\frac{5}{12}$	$\frac{1}{3}$ $\frac{7}{12}$ $\frac{1}{12}$
i		6	12			3	3
m		12	18			9	9
i		6	12			12	12
m		12	18			18	18
i	3	6	3	3	3	12	12
m	9	12	9	9	9	18	18

4.7 The group $F_{36}^{SL_3}$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{1}{4}$ $\frac{3}{4}$ 1	$\frac{1}{6}$ $\frac{1}{6}$ $\frac{2}{3}$	$\frac{5}{6}$ $\frac{5}{6}$ $\frac{1}{3}$	$\frac{7}{12}$ $\frac{1}{3}$ $\frac{1}{12}$	$\frac{5}{12}$ $\frac{2}{3}$ $\frac{11}{12}$
i		6	12			12	12
m		6	9			9	9
i		6	12			3	3
m		6	9			9	9
i	3	6	3	3	3	12	12
m	9	6	9	9	9	9	9

4.8 The group G_{168}

Ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ 1 $\frac{1}{3}$	$\frac{3}{4}$ $\frac{1}{4}$ 1	$\frac{4}{7}$ $\frac{2}{7}$ $\frac{1}{7}$	$\frac{5}{7}$ $\frac{6}{7}$ $\frac{3}{7}$
i		3	4	7	7
m		56	42	24	24
i		2	4	7	7
m		28	42	24	24
i	2	3	2	7	7
m	21	56	21	24	24

4.9 The group $G_{168} \times C_3$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ 1 $\frac{1}{3}$	$\frac{2}{3}$ 1 $\frac{1}{3}$	$\frac{2}{3}$ 1 $\frac{1}{3}$	$\frac{1}{4}$ $\frac{3}{4}$ 1	$\frac{2}{3}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{1}{3}$ $\frac{5}{6}$ $\frac{5}{6}$	$\frac{3}{7}$ $\frac{5}{7}$ $\frac{6}{7}$	$\frac{2}{7}$ $\frac{1}{7}$ $\frac{4}{7}$	$\frac{1}{12}$ $\frac{7}{12}$ $\frac{1}{3}$	$\frac{2}{3}$ $\frac{11}{12}$ $\frac{5}{12}$	$\frac{1}{21}$ $\frac{4}{21}$ $\frac{16}{21}$	$\frac{17}{21}$ $\frac{20}{21}$ $\frac{5}{21}$	$\frac{11}{21}$ $\frac{8}{21}$ $\frac{2}{21}$	$\frac{10}{21}$ $\frac{19}{21}$ $\frac{13}{21}$
i		6	3	3	12	6	6	21	21	12	6	21	21	21	21
m		28	56	56	42	21	21	24	24	42	21	24	24	24	24
i		3	6	3	12			21	21	12	12	21	21	21	21
m		56	28	56	42			24	24	42	42	24	24	24	24
i	6	3	3	6	6			21	21	6	12	21	21	21	21
m	21	56	56	28	21			24	24	21	42	24	24	24	24

4.10 The group $A_5 \times C_3$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{2}{3}$ $\frac{1}{3}$ 1	$\frac{3}{5}$ 1 $\frac{2}{5}$	$\frac{1}{5}$ $\frac{4}{5}$ 1	$\frac{1}{3}$ $\frac{5}{6}$ $\frac{5}{6}$	$\frac{2}{3}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{8}{15}$ $\frac{2}{15}$ $\frac{1}{3}$	$\frac{2}{3}$ $\frac{4}{15}$ $\frac{1}{15}$	$\frac{1}{3}$ $\frac{11}{15}$ $\frac{14}{15}$	$\frac{2}{3}$ $\frac{13}{15}$ $\frac{7}{15}$
i		6	3	3	15	15	6	6	15	6	6	6
m		10	20	20	12	12	15	15	12	6	6	6
i		3	3	6	6	15			15	15	15	15
m		20	20	10	6	12			12	12	12	12
i	6	3	6	3	15	6			6	15	15	15
m	15	20	10	20	12	6			6	12	12	12

4.11 The group A_5

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{2}{3}$ 1 $\frac{1}{3}$	$\frac{4}{5}$ 1 $\frac{1}{5}$	$\frac{3}{5}$ 1 $\frac{2}{5}$
i		3	5	5
m		20	12	12
i		2	2	2
m		10	6	6
i	2	3	5	5
m	15	20	12	12

4.12 The Valentiner group $A_6^{SL_3}$

ramification type	$\frac{1}{2}$ $\frac{1}{2}$ 1	1 $\frac{1}{3}$ $\frac{2}{3}$	1 $\frac{1}{3}$ $\frac{2}{3}$	$\frac{1}{4}$ 1 $\frac{3}{4}$	$\frac{4}{5}$ 1 $\frac{1}{5}$	1 $\frac{2}{5}$ $\frac{3}{5}$	$\frac{2}{3}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{1}{3}$ $\frac{5}{6}$ $\frac{5}{6}$	$\frac{11}{12}$ $\frac{2}{3}$ $\frac{5}{12}$	$\frac{7}{12}$ $\frac{1}{3}$ $\frac{1}{12}$	$\frac{1}{15}$ $\frac{4}{15}$ $\frac{2}{3}$	$\frac{13}{15}$ $\frac{7}{15}$ $\frac{2}{3}$	$\frac{11}{15}$ $\frac{14}{15}$ $\frac{1}{3}$	$\frac{8}{15}$ $\frac{1}{3}$ $\frac{2}{15}$
i		6	6	12	15	6	6	6	12	12	15	15	15	15
m		60	60	90	72	36	45	45	90	90	72	72	72	72
i		6	6	6	6	15			6	6	15	15	15	6
m		60	60	45	36	72			45	45	72	72	72	36
i	6	6	6	12	15	15			12	12	6	6	6	15
m	45	60	60	90	72	72			90	90	36	36	36	72

From [13] Theorem 3.3 we get that the minimal m for this group is 36, while from the above table we can find an eigenvector at some singularity with $m \in \{36, 45, 60\}$.

5 The exceptional finite groups of degree 2

In this section we compute the number i, m for all exceptional finite groups of degree 2 that appear in the algorithm proposed in [16]. We refer to [16] and [7] for the definition of the groups.

5.1 The tetrahedral group

ramification type	$\frac{1}{3}$ $\frac{2}{3}$	$\frac{3}{4}$ $\frac{1}{4}$	$\frac{1}{6}$ $\frac{5}{6}$
i	6	4	6
m	4	6	4
i	6	4	6
m	4	6	4

5.2 The octahedral group

ramification type	$\frac{1}{3}$ $\frac{2}{3}$	$\frac{1}{4}$ $\frac{3}{4}$	$\frac{1}{4}$ $\frac{3}{4}$	$\frac{5}{6}$ $\frac{1}{6}$	$\frac{1}{8}$ $\frac{7}{8}$	$\frac{3}{8}$ $\frac{5}{8}$
i	6	8	4	6	8	8
m	8	6	12	8	6	6
i	6	8	4	6	8	8
m	8	6	12	8	6	6

5.3 The icosahedral group

ramification type	$\frac{2}{3}$ $\frac{1}{3}$	$\frac{3}{4}$ $\frac{1}{4}$	$\frac{4}{5}$ $\frac{1}{5}$	$\frac{2}{5}$ $\frac{3}{5}$	$\frac{5}{6}$ $\frac{1}{6}$	$\frac{1}{10}$ $\frac{9}{10}$	$\frac{3}{10}$ $\frac{7}{10}$
i	6	4	10	10	6	10	10
m	20	30	12	12	20	12	12
i	6	4	10	10	6	10	10
m	20	30	12	12	20	12	12

5.4 The Quaternion group

For any eigenvector of a non scalar monodromy matrix we have $m = 2$ and $i = 4$.

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