

Liouvillian solutions of third order differential equations

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Abstract

The Kovacic algorithm and its improvements give explicit formulae for the Liouvillian solutions of second order linear differential equations. Algorithms for third order differential equations also exist, but the tools they use are more sophisticated and the computations more involved. In this paper we refine parts of the algorithm to find Liouvillian solutions of third order equations. We show that, except for 4 finite groups and a reduction to the second order case, it is possible to give a formula in the imprimitive case. We also give necessary conditions and several simplifications for the computation of the minimal polynomial for the remaining finite set of finite groups (or any known finite group) by extracting ramification information from the character table. Several examples have been constructed, illustrating the possibilities and limitations.

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1. Introduction and notation

Classically the algorithm to compute Liouvillian solutions breaks into four cases. For third order equations $L(y) = 0$ over a differential field k with differential Galois group \mathcal{G} we get:

1. \mathcal{G} is a reducible linear group and there exists a decomposition $L(y) = L_1(L_2(y))$ into two operators of lower order.
2. \mathcal{G} is an imprimitive linear group and there exists a solution $z = e^{\int u}$ of $L(y) = 0$ with $[k(u) : k] = 3$.
3. \mathcal{G} belongs to a list of eight finite primitive linear groups. In particular all Liouvillian solutions are algebraic.
4. \mathcal{G} is an infinite primitive linear group and $L(y) = 0$ has no Liouvillian solution.

In this paper we improve the algorithms to compute Liouvillian solutions of third order linear differential equations. The paper is organised into four sections. Section 2 gives a survey of the computation of Liouvillian solutions by providing a uniform approach for second and third order equations. In section 3 we show that the imprimitive case can be solved using only linear algebra except if $L(y)$ is the second symmetric power of some second order equation or if the differential Galois group is one of four finite imprimitive groups. The four exceptional imprimitive groups are characterised by the fact that the third symmetric power can be of order 9 or that there are several minimal polynomials of degree 3 of algebraic solutions of the Riccati equation associated to $L(y)$. In the examples of section 4 we show that all exceptional situations can indeed occur for those four groups. In section 4 we consider the classical case $k = \mathbf{C}(x)$ and derive strong necessary conditions for \mathcal{G} to be conjugate to a given finite group. We derive strong necessary conditions for the finite primitive group and for the four finite exceptional imprimitive groups. In particular the conditions will always allow to compute the exceptional imprimitive groups. The approach is based on the ramification data given by the local exponents. Using the character table of the groups we show how to restrict possible ramification data of factors in constructions and in particular of exponential solutions of symmetric powers. The examples in this section show that this approach substantially improves the previous algorithms (Hoeij *et al.*, 1999; Singer and Ulmer, 1993b, 1997). Large parts of the approach generalise to higher order equations.

In the paper k is a differential field whose field of constants \mathcal{C} is algebraically closed of characteristic 0 (e.g. $\mathbf{C}(x)$ with the usual derivation d/dx). For the derivation δ of k and $a \in k$ we write $\delta^j(a) = a^{(j)}$ and also $a^{(1)} = a'$, $a^{(2)} = a''$, \dots . Let

$$L(y) = y^{(n)} + a_{n-2} y^{(n-2)} + \dots + a_1 y' + a_0 y = 0, \quad a_i \in k \quad (1)$$

be a linear differential equation of order n over k . A solution of (1)

1. in k is a *rational solution*,
2. in an algebraic extension of k is an *algebraic solution*,
3. whose logarithmic derivative is in k is an *exponential solution*,
4. belonging to a field obtained from k by successive adjunction of exponentials, integrals and algebraic functions is a *Liouvillian solution*.

Using the variable transformation $\tilde{y} = y \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$, it is always possible to transform a given linear differential equation into the above form without losing the Liouvillian character of the solutions. We refer to (Magid, 1994; van der Put and Singer, 2003) for an introduction to differential Galois theory and for definitions of a *Picard-Vessiot extension* K of k and *differential Galois group* $\mathcal{G}(L)$ for (1). The group $\mathcal{G}(L)$ sends a solution of (1) to another solution of (1). This action gives a faithful representation of $\mathcal{G}(L)$ as a subgroup of $GL(n, \mathcal{C})$,

also denoted $\mathcal{G}(L)$, whose character is denoted χ . The assumption that in (1) the coefficient $a_{n-1} = 0$ implies that $\mathcal{G}(L) \subseteq SL(n, \mathcal{C})$. A linear differential equation $L(y) = 0$ is reducible if there exists a decomposition $L(y) = L_1(L_2(y))$ into two operators of lower order. The equation (1) is reducible if and only if $\mathcal{G} \subseteq SL(n, \mathcal{C})$ is a reducible linear group (cf. (Kolchin, 1948) Section 22 Theorem 1).

DEFINITION 1.1: Let $G \subset GL(V)$ be a linear group acting irreducibly on the vector space V of dimension n over \mathcal{C} . Then G is said to be *imprimitive* if there exist subspaces V_1, \dots, V_k with $k > 1$ such that $V = V_1 \oplus \dots \oplus V_k$ and, for each $\sigma \in G$, the mapping $V_i \rightarrow \sigma(V_i)$ is a permutation of the set $\mathcal{S} = \{V_1, \dots, V_k\}$. The set \mathcal{S} is called a *system of imprimitivity* of G . If all the subspaces V_i are one dimensional, then G is called *monomial*. An irreducible group $G \subseteq GL(V)$ which is not imprimitive is called *primitive*.

Since an imprimitive group G is assumed to act irreducibly on V , G must permute the V_i transitively. In particular, all the V_i have the same dimension and if n is prime the group must be monomial.

Our approach is based on a connection between Liouvillian solutions and polynomial semi-invariants of \mathcal{G} .

DEFINITION 1.2: Let V be a \mathcal{C} -vector space, let Y_1, \dots, Y_n be a basis for V , and let $G \subseteq GL(V)$ be a linear group. Define a group action of $\sigma \in G$ on $I \in \mathcal{C}[Y_1, \dots, Y_n]$ by $\sigma \cdot (I(Y_1, \dots, Y_n)) = I(\sigma(Y_1), \dots, \sigma(Y_n))$. If a homogeneous $I \in \mathcal{C}[Y_1, \dots, Y_n]$ of degree m has the property that

$$\forall \sigma \in G, \quad \sigma \cdot I(Y_1, \dots, Y_n) = \psi_I(\sigma) \cdot I(Y_1, \dots, Y_n), \quad \text{with } \psi_I(\sigma) \in \mathcal{C}$$

then $\psi_I: G \rightarrow \mathcal{C}$ is a one dimensional character of G and I is a semi-invariant of degree m of G with character ψ_I . If ψ_I is the trivial character $\mathbb{1}$ of G , then I is an invariant of degree m of G . If $j \in \mathbf{N}$ is minimal such as $(\psi_I)^j = \mathbb{1}$, then j is the order of the character ψ_I and of the semi-invariant I .

From (Singer and Ulmer, 1997) Theorem 3 we get that the existence of Liouvillian solutions is equivalent to the existence of semi-invariants that factor into linear forms. An algorithm to compute semi-invariants of \mathcal{G} is given in (van Hoeij and Weil, 1997). We will however, in general, use the concept of the *value* of a \mathcal{G} semi-invariant.

DEFINITION 1.3: Let $L(y)$ be a n -th order linear differential equation with Galois group $\mathcal{G} \in GL(n, \mathcal{C})$. To a basis y_1, \dots, y_n of the solution space of $L(y) = 0$ we associate the evaluation morphism

$$\begin{aligned} \Phi: \mathcal{C}[Y_1, \dots, Y_n] &\rightarrow K \\ Y_i &\mapsto y_i \end{aligned}$$

The value of a semi-invariant of $\mathcal{G} \subset GL(n, \mathcal{C})$ is its image in K under the above evaluation morphism.

The morphism Φ has the following properties:

1. Φ restricts to a bijection between linear forms and solutions of $L(y) = 0$.
2. Semi-invariants of finite order j are sent to j -th roots of elements of k . In particular invariants are fixed under the action of \mathcal{G} and are sent to elements of k .

DEFINITION 1.4: *Let $L(y) = 0$ be an n -th order homogeneous linear differential and let y_1, \dots, y_n be a fundamental system of solutions. The differential equation $L^{\otimes m}(y)$ whose solution space, denoted V_m , is spanned by monomial of degree m in y_1, \dots, y_n is called the m -th symmetric power of $L(y) = 0$.*

An algorithm to construct $L^{\otimes m}(y)$ is given in (Singer, 1980). The value of a semi-invariant of degree m and order j is a j -th root of a solution of $L^{\otimes m}(y) = 0$, cf. (Singer and Ulmer, 1993b) Lemma 1.6.

2. Algebraic solution of the Riccati equation

The first three subsection introduce the basic idea and the notation and are well known (cf. (Bronstein, 2001; Hoeij *et al.*, 1999)). If $L(y) = 0$ has a solution whose logarithmic derivative is algebraic with minimal polynomial $Q \in k[U]$ of degree m , then all zeros u_i of Q give a Liouvillian solution $z_i = e^{\int u_i}$. Thus

$$Q = \left(U - \frac{z'_1}{z_1} \right) \left(U - \frac{z'_2}{z_2} \right) \dots \left(U - \frac{z'_m}{z_m} \right).$$

Multiplication of Q by $\prod_{i=1}^m z_i$ gives $P \in K[U]$ of the form

$$\left(\prod_{i=1}^m z_i \right) U^m - (z'_1 z_2 \dots z_m + \dots + z_1 \dots z_{m-1} z'_m) U^{m-1} + \dots + (-1)^m (z'_1 z'_2 \dots z'_m).$$

Note that the coefficients of P will be in k only after division by the leading coefficient $\prod_{i=1}^m z_i$. Our goal in this section is to compute Q or equivalently to compute P .

2.1. Computation of symmetric powers

Let $R_m = k[Z_{1,0}, \dots, Z_{m,n-1}]$ be a polynomial ring in nm variables. Using the differential equation (1) we define a derivation Δ on R by

1. $\Delta(f) = \delta(f)$ for $f \in k$,
2. $\Delta(Z_{i,s}) = Z_{i,s+1}$ if $0 \leq s < n-1$,
3. $\Delta(Z_{i,n-1}) = -(a_{n-2} Z_{i,n-2} + \dots + a_1 Z_{i,1} + a_0 Z_{i,0})$

For convenience we will use the notation Z_i for $Z_{i,0}$ and $Z_i^{(j)}$ for $Z_{i,j}$ ($0 < j < n$). Note that the variables Z_i are by construction solutions of (1). For integers $n > d_1 \geq d_2 \geq \dots \geq d_m \geq 0$ we introduce the notation W_{d_1, \dots, d_m} for the monic polynomial corresponding to

$$\sum_{\sigma \in S_m} Z_{\sigma(1)}^{(d_1)} Z_{\sigma(2)}^{(d_2)} \dots Z_{\sigma(m)}^{(d_m)} \in R_m.$$

EXAMPLE 2.1: For $n = 3$ the sum over the orbit under S_3 of $Z_1' Z_2 Z_3$ is $2Z_1' Z_2 Z_3 + 2Z_1 Z_2' Z_3 + 2Z_1 Z_2 Z_3'$. Thus $W_{1,0,0}$ is $Z_1' Z_2 Z_3 + Z_1 Z_2' Z_3 + Z_1 Z_2 Z_3'$.

If we start with the monomial $\mathcal{S} = Z_1 Z_2 \dots Z_m \in R_m$ and take derivatives, then the resulting expressions will all be symmetric in the m indices and thus can be expressed as sums over k of the $N = \binom{n+m-1}{n-1}$ polynomials $W_{d_1, \dots, d_m} \in R_m$. For $m \geq 3$ we get

$$\begin{aligned} \mathcal{S} &= Z_1 Z_2 \dots Z_m \\ &= W_{0,0,\dots,0} \\ \mathcal{S}' &= Z_1^{(1)} Z_2 \dots Z_m + Z_1 Z_2^{(1)} \dots Z_m + \dots + Z_1 \dots Z_{m-1} Z_m^{(1)} \\ &= W_{1,0,\dots,0} \\ \mathcal{S}'' &= \sum_{1 \leq j \leq m} Z_j^{(2)} \frac{\prod_i Z_i}{Z_j} + 2 \sum_{1 \leq j_1 < j_2 \leq m} Z_{j_1}^{(1)} Z_{j_2}^{(1)} \frac{\prod_i Z_i}{Z_{j_1} Z_{j_2}} \\ &= W_{2,0,\dots,0} + 2W_{1,1,0,\dots,0} \\ \dots &= \dots \end{aligned}$$

Considering j derivatives of \mathcal{S} we get a linear system of the form

$$\begin{pmatrix} \mathcal{S} \\ \mathcal{S}' \\ \dots \\ \mathcal{S}^{(j)} \end{pmatrix} = A_j \begin{pmatrix} W_{0,\dots,0} \\ W_{1,0,\dots,0} \\ \dots \\ W_{n-1,\dots,n-1} \end{pmatrix}$$

If we take the first $N = \binom{n+m-1}{n-1}$ derivatives of \mathcal{S} , then we will have more equations than unknowns on the right side, so the equations will be linearly dependent over k , giving a linear differential equation of degree N for \mathcal{S} . We now consider the linear combination of the $\mathcal{S}^{(i)}$ involving the lowest possible order of derivation, in other words where for the first time A_j has a row rank less than $j + 1$. Then a unique, up to multiple, non zero element $(b_j, b_{j-1}, \dots, b_0) \in k^{j+1}$ exists such as

$$(b_0, b_1, \dots, b_j) \begin{pmatrix} \mathcal{S} \\ \mathcal{S}' \\ \dots \\ \mathcal{S}^{(j)} \end{pmatrix} = (b_0, b_1, \dots, b_j) A_j = 0,$$

i.e. for which $\sum_{i=0}^j b_i \mathcal{S}^{(i)} = 0$. For any m non zero solutions z_1, \dots, z_m of $L(y) = 0$ we consider the differential evaluation morphism

$$\begin{aligned} \Psi_{z_1, \dots, z_m}: R_m &\rightarrow K \\ Z_i^{(j)} &\mapsto z_i^{(j)} \end{aligned}$$

We have $\Psi_{z_1, \dots, z_m} \left(\sum_{i=0}^j b_i \mathcal{S}^{(i)} \right) = \sum_{i=0}^j b_i \left(\prod_{i=1}^m z_i \right)^{(i)} = 0$, showing that $L^{\otimes m}(y) = \sum_{i=0}^j b_i y^{(i)}$. Note that the above procedure is just the algorithm proposed in (Singer, 1980, 1981) to compute $L^{\otimes m}(y)$ and that one could even start with $\mathcal{S} = Y_1^m$ instead of $Y_1 Y_2 \cdots Y_m$ (cf. (Singer and Ulmer, 1993a), Section 3.2.2).

2.2. Computation of $P(U)$

From (Singer and Ulmer, 1997), Theorem 3 and its proof we get that if $g \in K$ is an exponential solution* of $L^{\otimes m}(y)$ and $g = \Psi_{z_1, \dots, z_m}(\mathcal{S})$ for $m \in \mathbf{N}$ and non zero solutions z_i of $L(y) = 0$, then for $P \in K[U]$ given by

$$\begin{aligned} &\left(\prod_{i=1}^m z_i \right) U^m - (z'_1 z_2 \cdots z_m + \cdots + z_1 \cdots z_{m-1} z'_m) U^{m-1} + \cdots + (-1)^m \prod_{i=1}^m z'_i \\ &= \Psi_{z_1, \dots, z_m}(W_{0, \dots, 0}) U^m - \Psi_{z_1, \dots, z_m}(W_{1, 0, \dots, 0}) U^{m-1} + \cdots + (-1)^m \Psi_{z_1, \dots, z_m}(W_{1, 1, \dots, 1}) \end{aligned}$$

we have that

$$Q = \frac{P}{\prod_{i=1}^m z_i} \in k[U].$$

In this case all roots u_i of Q give a Liouvillian solution $e^{\int u_i}$ of $L(y) = 0$.

If $L^{\otimes m}(y)$ is of maximal order $N = \binom{n+m-1}{n-1}$, then the matrix A_{N-1} corresponding to

$$\begin{pmatrix} \Psi_{z_1, \dots, z_m}(\mathcal{S}) \\ (\Psi_{z_1, \dots, z_m}(\mathcal{S}))' \\ \cdots \\ (\Psi_{z_1, \dots, z_m}(\mathcal{S}))^{(N-1)} \end{pmatrix} = \begin{pmatrix} g \\ g' \\ \cdots \\ g^{(N-1)} \end{pmatrix} = A_{N-1} \begin{pmatrix} \Psi_{z_1, \dots, z_m}(W_{0, \dots, 0}) \\ \Psi_{z_1, \dots, z_m}(W_{1, 0, \dots, 0}) \\ \cdots \\ \Psi_{z_1, \dots, z_m}(W_{n-1, \dots, n-1}) \end{pmatrix}$$

will be invertible, allowing us to obtain the remaining coefficients of P or Q from g and its derivatives using linear algebra. We give an explicit example in the next section.

Therefore the computation of $Q \in k[U]$ is reduced to the computation of g and linear algebra if

1. $g \in K$ is an exponential solution of $L^{\otimes m}(y)$ such that $g = \Psi_{z_1, \dots, z_m}(\mathcal{S})$ for m non zero solutions z_i of $L(y) = 0$.

*From (Bronstein, 2001) Theorem 3.8 we get that for reductive groups it is enough to consider rational solutions, but to do this one must work with larger values of m . This has for example be proposed in (Ulmer and Weil, 1996).

2. $L^{\otimes m}(y)$ is of maximal order $N = \binom{n+m-1}{n-1}$.

Since a solution z_i is a linear combination over \mathcal{C} of y_1, \dots, y_n the first condition is equivalent to the fact that $g \in K$ is the value of a semi-invariant of \mathcal{G} that factors into linear forms ((Singer and Ulmer, 1997), Theorem 3 and its proof). Our goal is to look for conditions in order to be in the above situation.

2.3. Second order Example

We consider a second order example $L(y) = y'' + a_0 y = 0$ since for third order the results of the various steps are too long to be included. For $\mathcal{S} = Z_1 Z_2 \in R_2$ we have

$$\begin{aligned} \mathcal{S} &= Z_1 Z_2 \\ \mathcal{S}' &= Z_1' Z_2 + Z_1 Z_2' \\ \mathcal{S}'' &= -2a_0 Z_1 Z_2 + 2 Z_1' Z_2' \\ \mathcal{S}''' &= -2a_0' Z_1 Z_2 - 4a_0 (Z_1' Z_2 + Z_1 Z_2') \end{aligned}$$

This gives the system:

$$\begin{pmatrix} \mathcal{S} \\ \mathcal{S}' \\ \mathcal{S}'' \\ \mathcal{S}''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \\ -2a_0' & -4a_0 & 0 \end{pmatrix} \begin{pmatrix} W_{0,0} \\ W_{1,0} \\ W_{1,1} \end{pmatrix} = A_3 \begin{pmatrix} W_{0,0} \\ W_{1,0} \\ W_{1,1} \end{pmatrix}$$

From the matrix form we see that in this case the symmetric power $L^{\otimes 2}(y)$ will always be of maximal order 3. To compute $L^{\otimes 2}(y)$ we look for an element (b_0, b_1, b_2, b_3) in the kernel of A_3^t ,

$$(2a_0', 4a_0, 0, 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \\ -2a_0' & -4a_0 & 0 \end{pmatrix} = (0, 0, 0, 0),$$

which gives us the following formula:

$$L^{\otimes 2}(y) = 2a_0' y + 4a_0 y' + y'''$$

Since homogeneous forms in two variables, over an algebraic closed field, always factor into linear forms, we get that for second order equations $L(y)$, any non zero exponential solution g of $L^{\otimes 2}(y) = 0$ is a product of solutions, i.e. $g = z_1 z_2$, and will lead to a polynomial P (cf. (Ulmer and Weil, 1996) Theorem 2.1).

Thus for any exponential solution g of $L^{\otimes 2}(y)$ by solving the linear system

$$\begin{pmatrix} g \\ g' \\ g'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a_0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \Psi_{z_1, z_2}(W_{0,0}) \\ \Psi_{z_1, z_2}(W_{1,0}) \\ \Psi_{z_1, z_2}(W_{1,1}) \end{pmatrix}$$

we get

$$Q = U^2 - \frac{g'}{g} U + \frac{a_0 g + \frac{g''}{2}}{g} \in k[U]$$

Note that the “formula” is in fact (up to the computation of g and linear algebra) given by the above two matrices.

Remark: The second order case is well known (cf. (Kovacic, 1986; Ulmer and Weil, 1996)). This case is characterised by the existence of various recursions (cf. (Bronstein *et al.*, 1997)) and simplified by the fact that forms in two variables over an algebraic closed field factor into linear forms. In particular $L^{\otimes m}(y)$ is always of maximal order $m + 1$ and a formula like the above can be given for the cases $m \in \{1, 2, 4, 6, 12\}$ of (Kovacic, 1986). From (Kovacic, 1986) we get that if no solution is found for $m \in \{1, 2, 4, 6, 12\}$, then there are no Liouvillian solutions. For third order equations symmetric powers are not always of maximal order and simple recursions are no longer available but we will show that, up to a few exceptions, a formula similar to the above can be given.

3. Third order equations with an imprimitive group

We now consider a third order equation $L(y) = y''' + a_1 y' + a_0 y$ whose differential Galois group \mathcal{G} is imprimitive. From (Singer and Ulmer, 1993a) Theorem 4.6 we get that this case is characterised by the fact that $L^{\otimes 3}(y) = 0$ has an exponential solution whose square is rational, or equivalently that a (semi-)invariant of degree 3 and order ≤ 2 exists. The new difficulties, compared to second order equations, in the computation of the minimal polynomial Q of degree 3 of an algebraic solution of the Riccati equation are:

1. $L^{\otimes 3}(y)$ is not always of maximal order, in which case the matrix A_9 is not invertible.
2. Not all exponential solutions of $L^{\otimes 3}(y) = 0$ are values of forms that factor into linear forms.

Our goal is to characterise the above exceptional cases and to produce a formula in the general case where the order of $L^{\otimes 3}(y)$ is 10 and where there is, up to multiples, a unique (semi-) invariant of degree 3 and order ≤ 2 (whose value, up to multiple, will be the unique solution of $L^{\otimes 3}(y)$ whose square is in k).

3.1. The order of symmetric powers

LEMMA 3.1: *For a third order linear differential equation $L(y) = y''' + a_1 y' + a_0 y$ over k the order of $L^{\otimes 3}(y)$ is 7, 9 or 10. The order of $L^{\otimes 3}(y)$ is 7 if and only if $2a_0 = a_1'$. In this case $L(y) = \tilde{L}^{\otimes 2}(y)$ where $\tilde{L}(y) = y'' + \frac{a_1}{4}y$.*

Proof: Let Φ_m be the \mathcal{G} -morphism obtained by restriction of the evaluation morphism Φ to the vector space of homogeneous forms of degree m of $\mathcal{C}[Y_1, Y_2, Y_3]$ (cf. (Singer and Ulmer, 1993a) Lemma 3.5). Because Φ_1 is a bijection, (Singer

and Ulmer, 1993a) Lemma 3.5 (4) implies that the kernel of Φ_2 is at most of dimension 1, i.e. $L^{\otimes 2}(y)$ is of order at least 5. If Φ_2 is a bijection, then (Singer and Ulmer, 1993a) Lemma 3.5 (4) implies that $L^{\otimes 3}(y)$ is of order 9 or 10. Suppose now that the kernel of Φ_2 is of dimension 1 and generated by $F_2 \in \mathcal{C}[Y_1, Y_2, Y_3]$. Note that F_2 must be irreducible since otherwise Φ_1 is not a bijection. The homogeneous polynomials of degree 3 divisible by F_2 correspond to the \mathcal{C} -Span of Y_1F_2, Y_2F_2, Y_3F_2 and are in the kernel of Φ_3 , showing that $L^{\otimes 3}(y)$ is of order ≤ 7 . If $L^{\otimes 3}(y)$ is of order ≤ 6 , there must exist an $F_3 \in \mathcal{C}[Y_1, Y_2, Y_3]$ in the kernel of Φ_3 which is not a multiple of F_2 . If F_2 or F_3 is of degree zero in Y_3 , then we get a non constant homogeneous polynomial $F \in \mathcal{C}[Y_1, Y_2]$ whose evaluation via Φ is zero. Since F is homogeneous and \mathcal{C} algebraically closed, F factors as a product of linear forms $\prod_i (\alpha_i Y_1 - \beta_i Y_2)$. The evaluation via Φ gives a linear relation between the basis elements y_1, y_2 and thus a contradiction. Denote $F_{2,3} \in \mathcal{C}[Y_1, Y_2]$ the resultant of F_2 and F_3 with respect to Y_3 . Since F_2, F_3 are relatively prime and are both of degree ≥ 1 in Y_3 , we get that $F_{2,3} \in \mathcal{C}[Y_1, Y_2]$ is homogeneous of degree ≥ 1 . Arguing as above with F , we get a contradiction. Thus $L^{\otimes 3}(y)$ is of order 7 in this case. This proves the first assertion.

We want to characterise the fact that $L^{\otimes 3}(y)$ is of order 7 or equivalently that $L^{\otimes 2}(y)$ is of order 5. This has been done (Singer, 1985) Lemma 3.4, but we will give a proof using the introduced formalism. According to section 2.1 we have that $L^{\otimes 2}(y)$ is of order ≤ 5 if and only if the matrix A_5 is not invertible (i.e. if and only if $(Z_1Z_2Z_3), \dots, (Z_1Z_2Z_3)^{(5)}$ are linearly dependent). Performing a fraction free Gaussian elimination on the matrix $(A_5)^t$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & -2a_0 & -2a'_0 & 2a_1a_0 - 2a''_0 \\ 0 & 1 & 0 & -a_1 & -5a_0 - a'_1 & -7a'_0 + a_1^2 - a''_1 \\ 0 & 0 & 2 & 0 & -8a_1 & -10a'_1 - 10a_0 \\ 0 & 0 & 0 & 6 & 0 & -30a_1 \\ 0 & 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 108(a'_1 - 2a_0) \end{bmatrix}.$$

Therefore $L^{\otimes 3}(y)$ is of order 7 if and only if $a'_1 = 2a_0$. A direct computation shows that in this case $L(y) = \tilde{L}^{\otimes 2}(y)$. \square

3.2. The four exceptional finite imprimitive groups

We now want to characterise the cases where the order of $L^{\otimes 3}(y)$ is 9 or where there exists more than one, up to multiples, (semi-) invariants of degree 3 and order ≤ 2 . In order to avoid repetitions we shall consider the matrices:

$$\sigma_c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_t = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \omega^3 & 0 & 0 \\ 0 & -\omega^3 - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } \sigma_9 = \begin{pmatrix} -\omega^5 - \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & -\omega^5 - \omega^2 \end{pmatrix}$$

where $\omega^6 + \omega^3 + 1 = 0$, and define the groups $G_{27} = \langle \sigma_c, \sigma_3 \rangle$, $G_{54} = \langle \sigma_c, \sigma_3, \sigma_t \rangle$, $G_{81} = \langle \sigma_c, \sigma_9 \rangle$ and $G_{162} = \langle \sigma_c, \sigma_9, \sigma_t \rangle$ whose respective orders correspond to the indices. We note that for each of the four groups all faithful irreducible unimodular characters of degree 3 are conjugate.

We consider an imprimitive group $G \subset SL(3, C)$. This group will, in the basis Y_1, Y_2, Y_3 corresponding to a system of imprimitivity, contain a diagonal element

$$\sigma_d = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \frac{1}{\alpha\beta} \end{pmatrix}$$

and, since the group acts transitively on the system of imprimitivity, an element

$$\begin{pmatrix} 0 & 0 & \mu \\ \nu & 0 & 0 \\ 0 & \frac{1}{\mu\nu} & 0 \end{pmatrix}$$

of order 3 which we can assume to be of the conjugated form

$$\sigma_c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have a morphism $\varphi: G \rightarrow S_3$ whose image $\text{Im}(\varphi)$ is A_3 or S_3 (the transitive subgroups of S_3). Since the Schur representation groups of A_3 and S_3 have no irreducible representation of degree 3, the irreducible group G must contain a non scalar diagonal element. Since conjugation by σ_c permutes the entries of σ_d cyclically, we may assume that in the above σ_d we have $\alpha \neq \beta$. If $\text{Im}(\varphi) = A_3$ then $Y_1 Y_2 Y_3$ is an invariant and if $\text{Im}(\varphi) = S_3$ then $Y_1 Y_2 Y_3$ is a semi-invariant of order 2. The following shows that, with two exceptions, $Y_1 Y_2 Y_3$ is, up to constant multiple, the only (semi-)invariant of degree 3 and order ≤ 2 .

THEOREM 3.2: *Let $L(y) = y''' + a_1 y' + a_0 y$ be a third order linear differential equation with an unimodular imprimitive Galois group \mathcal{G} . Then one of the following must be true*

1. \mathcal{G} has a two dimensional space of invariants of degree 3. In this case $\mathcal{G} \cong G_{27}$.
2. \mathcal{G} has no invariant of degree 3 but a two dimensional space of semi-invariants of degree 3 (for the same character) and order 2. In this case $\mathcal{G} \cong G_{54}$.
3. \mathcal{G} has up to multiples a unique (semi-)invariant of degree 3 and order ≤ 2 . This semi-invariant must factor into linear forms. If \mathcal{G} has another semi-invariant of degree 3 then one of the following holds
 - (a) \mathcal{G} also has (up to multiples) 3 semi-invariants of degree 3 and order 3 corresponding to 3 different characters. In this case $\mathcal{G} \cong G_{81}$

- (b) \mathcal{G} has also (up to multiples) 1 semi-invariant of degree 3 and order 6.
In this case $\mathcal{G} \cong G_{162}$

Proof: We suppose $\mathcal{G} \subset SL(3, \mathcal{C})$, given in a basis Y_1, Y_2, Y_3 corresponding to a system of imprimitivity, contains elements of the form σ_c and σ_d with $\alpha \neq \beta$. We want to find all possible semi-invariants of degree 3 and order ≤ 2 of \mathcal{G} of the form:

$$I = a_1 Y_1^3 + a_2 Y_2^3 + a_3 Y_3^3 + b_1 Y_1^2 Y_2 + b_2 Y_1 Y_3^2 + b_3 Y_2^2 Y_3 + c_1 Y_2^2 Y_1 + c_2 Y_1^2 Y_3 + c_3 Y_2 Y_3^2 + d Y_1 Y_2 Y_3$$

which are not multiples of $Y_1 Y_2 Y_3$. If I is a semi-invariant of \mathcal{G} of order ≤ 2 , $\sigma_c(I) = \pm I$. By equating coefficients we find that $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$ and $c_1 = c_2 = c_3$. We can therefore assume I to be of the form:

$$\tilde{I} = a Y_1^3 + a Y_2^3 + a Y_3^3 + b Y_1^2 Y_2 + b Y_1 Y_3^2 + b Y_2^2 Y_3 + c Y_2^2 Y_1 + c Y_1^2 Y_3 + c Y_2 Y_3^2 + d Y_1 Y_2 Y_3$$

If \tilde{I} is a semi-invariant of \mathcal{G} of order 2 then $\sigma_d(\tilde{I}) = \lambda \tilde{I}$ where $\lambda^2 = 1$. Equating coefficients and multiplying by powers of α and β which are not zero, we obtain a polynomial system in the variables $a, b, c, d, \alpha, \beta, \lambda$. To this we add $\lambda^2 - 1$ and the extra condition $1 - w\alpha\beta(\alpha - \beta)$ with a new variable w in order to guarantee that α, β are non zero and distinct. Computing a Gröbner basis for the corresponding system for a lex degree order in the variables $w, a, b, c, d, \alpha, \beta, \lambda$ we get the following equations not involving w :

$$a(\alpha^2 + \alpha\beta + \beta^2), a(\beta^3 - 1), a(\lambda - 1), b, c, d(\lambda - 1), \lambda^2 - 1$$

If \tilde{I} is not a multiple of $Y_1 Y_2 Y_3$ then we must have $a \neq 0$, from which we get $\beta^3 = 1$. From $\alpha^2 + \alpha\beta + \beta^2$ we get that $\alpha = J\beta$ or $\alpha = J^2\beta$ with $J^2 + J + 1 = 0$. Thus \mathcal{G} contains at most six non scalar diagonal matrices. A group containing such a matrix, say σ_3 and σ_c will contain all other 6 such matrices and must be conjugated to G_{27} . This is because all six matrices can be obtained from σ_3, σ_3^2 and cyclic permutation of the entries via conjugation by σ_c . Therefore if $\text{Im}(\varphi) = A_3$ then $\mathcal{G} \cong G_{27}$. Otherwise G_{27} is a subgroup of index 2 and thus a normal subgroup of \mathcal{G} . Among the 15 non isomorphic groups of order 54 the group G_{54} is the only group having those properties and an irreducible unimodular representation of degree 3. By computing, one finds that G_{27} has a two dimensional space of invariants of degree 3 and that G_{54} has no invariants but a two dimensional space of semi-invariants of degree 3 and order 2. This shows that only the three main cases stated can occur.

Now we suppose that \mathcal{G} is not isomorphic to G_{27} or G_{54} . From the above we see that this implies that \mathcal{G} contains a matrix σ_d with $\beta^3 \neq 1$. If, besides multiples of $Y_1 Y_2 Y_3$, \mathcal{G} has a semi-invariant I of degree 3, then $\sigma_c(I) - \lambda_1 I = 0$ and $\sigma_d(I) - \lambda_2 I = 0$ with $\lambda_1, \lambda_2 \in \mathcal{C}^*$. By equating coefficients and multiplying by powers

of α and β which are not zero, we obtain a polynomial system in the variables $a_i, b_i, c_i, d, \alpha, \beta, \lambda_i$. To this we add the extra condition $1 - w\alpha\beta(\alpha - \beta)\lambda_1\lambda_2(\beta^3 - 1)$ with a new variable w in order to guarantee that $\alpha, \beta, \lambda_1, \lambda_2$ are non zero, $\alpha \neq \beta$ and $\beta^3 \neq 1$. Computing a Gröbner basis for the corresponding system for a lex degree order in the variables $w, \lambda_1, \lambda_2, d, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, \alpha, \beta$ we get (among others) the following equations:

$$a_1^2 - a_2a_3, a_1a_2 - a_3^2, a_1a_3 - a_2^2, a_1(\alpha^2 + \alpha\beta + \beta^2), a_1(\beta^6 + \beta^3 + 1), a_2(\alpha^2 + \alpha\beta + \beta^2), \\ a_2^3 - a_3^3, a_2(\beta^6 + \beta^3 + 1), a_3(\alpha^2 + \alpha\beta + \beta^2), a_3(\beta^6 + \beta^3 + 1), b_1, b_2, b_3, c_1, c_2, c_3$$

Therefore, if I is not a multiple of $Y_1Y_2Y_3$ (i.e. some a_i is non zero), then β is a primitive 9-th root of unity and $\alpha = J\beta$ or $\alpha = J^2\beta$. There are at most twelve matrices of this form. A group containing a matrix of this form, say σ_9 , and σ_c will also contain $\{\sigma_c(\sigma_9)^s\sigma_c, \sigma_c^{-2}(\sigma_9)^s\sigma_c^2 | 1 \leq s \leq 8\}$ and thus all twelve matrices and, by multiplication of those, all elements of G_{27} (which we excluded via $\beta^3 \neq 1$). Thus there is a unique such group which is conjugated to G_{81} . Therefore, if $\text{Im}(\varphi) = A_3$ then $\mathcal{G} \cong G_{81}$ otherwise G_{81} is a subgroup of index 2 and thus a normal subgroup of \mathcal{G} which is of order 162. In this case the group of order 162 is non abelian of exponent 18 and has a center of order 3. From the fifty-five non isomorphic groups of order 162 only G_{162} satisfies those requirements and has a unimodular representation of degree 3. By computing all semi-invariants of degree 3 of G_{81} and G_{162} we get the result. \square

COROLLARY 3.3: *Let $L(y) = y''' + a_1y' + a_0y$ be a third order linear differential equation with unimodular imprimitive Galois group \mathcal{G} . If $L^{\otimes 3}(y)$ is not of order 7, then one of the following holds:*

1. $L^{\otimes 3}(y)$ is of order 9 and \mathcal{G} is isomorphic to G_{27}, G_{54}, G_{81} or G_{162} .
2. $L^{\otimes 3}(y)$ is of order 10 and one of the following holds:
 - (a) $L^{\otimes 3}(y) = 0$ has up to multiple a unique solution whose square is rational. This must be the value of a semi-invariant that factor into linear forms.
 - (b) $L^{\otimes 3}(y) = 0$ has a two dimensional space of rational solutions. In this case $\mathcal{G} \cong G_{27}$. Within the two dimensional space of invariants of degree 3 there are, up to multiples, four invariants that factor into linear forms.
 - (c) $L^{\otimes 3}(y) = 0$ has a two dimensional space of non rational solutions whose square is rational. In this case $\mathcal{G} \cong G_{54}$. Within the two dimensional space of semi-invariants of degree 3 and order 2 there are, up to multiples, four semi-invariants that factor into linear forms.

Proof: The possible orders 7, 9, 10 for $L^{\otimes 3}(y)$ follow from Lemma 3.1. Let Φ_3 be the \mathcal{G} -morphism obtained by restriction of the evaluation morphism Φ to the

vector space W of homogeneous forms of degree 3 of $\mathcal{C}[Y_1, Y_2, Y_3]$ ((Singer and Ulmer, 1993a) Lemma 3.5).

Suppose that $L^{\otimes 3}(y)$ is of order 9. The kernel of Φ_3 is of dimension one generated by a homogeneous $F \in \mathcal{C}[Y_1, Y_2, Y_3]$ of degree 3. Since, with the \mathcal{G} -action defined in Definition 1.2, Φ_3 is a \mathcal{G} -morphism, we have $\forall \sigma \in \mathcal{G}$ that $\Phi_3(\sigma \cdot F) = \sigma(\Phi_3(F)) = 0$ and thus that $\sigma(F) \in \text{Ker}(\Phi_3)$ is a multiple of F . Therefore F is a semi-invariant of \mathcal{G} . Since the value of a semi-invariant which factors into linear forms (i.e. whose value is a product of non zero solutions) cannot be zero, we get that there must be at least 2 semi-invariants of degree 3 of \mathcal{G} . By the previous result the only possibilities are the four given groups.

If $L^{\otimes 3}(y)$ is of order 10, then Φ_3 is a bijection of W on the the solution space of $L^{\otimes 3}(y) = 0$. In this case there is a bijection between the semi-invariants of degree 3 and order ≤ 2 and the solutions of $L^{\otimes 3}(y) = 0$ whose square is rational. If Y_1, Y_2, Y_3 is a basis corresponding to a system of imprimitivity, then $Y_1 Y_2 Y_3$ is a semi-invariant that factors into linear forms and its value is a solution of $L^{\otimes 3}(y) = 0$ whose square is rational. From the previous result we get that only G_{27} and G_{54} can, up to multiples, have more than one semi-invariant of degree 3 and order ≤ 2 and therefore more than a one dimensional space of solutions of $L^{\otimes 3}(y) = 0$ whose square is rational. The two exceptions now follow from the previous theorem. From (Hendriks and van der Put, 1995) or (Ulmer, 1994), Theorem 2.1 we get that G_{27} and G_{54} both have, up to multiples, four semi-invariants of degree 3 and order ≤ 2 that factor into linear forms. \square

In the next section we construct examples of equations whose differential Galois group corresponds to G_{27} , G_{54} , G_{81} and G_{162} such that $L^{\otimes 3}(y)$ is of order 9. For G_{27} and G_{54} this shows that the 4 semi-invariants that factor into linear forms may all have the same value. Therefore, in general, it is impossible to recover the minimal polynomial of an algebraic solution just from the value of a semi-invariant that factors into linear forms (cf. Example 4.11).

COROLLARY 3.4: *Let $L(y) = y''' + a_1 y' + a_0 y$ be an irreducible third order linear with irreducible Galois group $\mathcal{G} \in SL(3, \mathcal{C})$. If $L^{\otimes 3}(y)$ is of order 9, then \mathcal{G} is a finite group isomorphic to G_{27} , G_{54} , G_{81} , G_{162} or $F_{36}^{SL_3}$, the primitive group of order 108.*

Proof: An irreducible group is either imprimitive or primitive. If \mathcal{G} is imprimitive we get the result from the previous Corollary. If \mathcal{G} is primitive then from (Singer and Ulmer, 1993a) Table 2 we get that only for $F_{36}^{SL_3}$ the order of $L^{\otimes 3}(y)$ can be 9. \square

3.3. The generic imprimitive case

Consider $L(y) = y''' + a_1 y' + a_0 y$ and suppose that \mathcal{G} is an imprimitive group not conjugate to G_{27} , G_{54} and given in a monomial basis Y_1, Y_2, Y_3 . If $L^{\otimes 3}(y)$ is

of maximal order 10, then a non zero element in the kernel of the transposed of

$$\begin{pmatrix} \mathcal{S} \\ \mathcal{S}' \\ \dots \\ \mathcal{S}^{(10)} \end{pmatrix} = A_{10} \begin{pmatrix} W_{0,0,0} \\ W_{1,0,0} \\ \dots \\ W_{2,2,2} \end{pmatrix}$$

gives the coefficients of $L^{\otimes 3}(y) = 0$ (cf. Section 2.1). Since \mathcal{G} is not conjugated to G_{27} , G_{54} , there exists up to multiples a unique non zero exponential solution $g \in K$ of $L^{\otimes 3}(y) = 0$ with $g^2 \in k$ which must be the value of $Y_1 Y_2 Y_3$ and therefore $g = z_1 z_2 z_3$ for some solutions z_i of $L(y) = 0$ (cf. previous section and (Singer and Ulmer, 1993a), Theorem 4.6). The square matrix A_9 in

$$\begin{pmatrix} g \\ g' \\ \dots \\ g^{(9)} \end{pmatrix} = A_9 \begin{pmatrix} \Psi_{z_1, z_2, z_3}(W_{0,0,0}) \\ \Psi_{z_1, z_2, z_3}(W_{1,0,0}) \\ \dots \\ \Psi_{z_1, z_2, z_m}(W_{2,2,2}) \end{pmatrix}$$

is invertible, allowing us to express the coefficients of $Q \in k[U]$ using g and its derivatives (cf. Section 2.2).

The matrices A_9 and A_{10} have to be computed only once. The point in the above is that it is enough to know that $g \neq 0$ is a product of solutions but that the actual decomposition is not needed. This follows also from ((Hoeij *et al.*, 1999), Theorem 2.1) and the method described there would also produce the above formula in this case.

EXAMPLE 3.5: Consider the equation $L(y) = 0$ over $\mathbf{C}(x)$ given by

$$\frac{d^3 y}{dx^3} + \frac{123x^2 - 103x + 108}{144x^2(x^2 - 2x + 1)} \frac{dy}{dx} - \frac{748x^3 - 937x^2 - 648 + 1605x}{864x^3(x^3 - 3x^2 + 3x - 1)} y = 0$$

Factorisation shows that L is an irreducible operator. The equation $L^{\otimes 3}(y) = 0$ is of order 10 and has (up to multiples) a unique exponential solution $g = x^{7/2} - 2x^{5/2} + x^{3/2}$ with $g^2 \in \mathbf{C}(x)$. Since the semi-invariant I whose value is g must factor into linear forms, our generic formula with $m = 3$ can be applied. Solving the 10×10 system for $g = x^{7/2} - 2x^{5/2} + x^{3/2}$ we get

$$\begin{aligned} \Psi_{z_1, z_2, z_3}(W_{0,0,0}) &= x^{7/2} - 2x^{5/2} + x^{3/2} \\ \Psi_{z_1, z_2, z_3}(W_{1,0,0}) &= \frac{7}{6}x^{5/2} - \frac{5}{3}x^{3/2} + \frac{1}{2}x^{1/2} \\ \Psi_{z_1, z_2, z_3}(W_{1,1,0}) &= \frac{187x^2 - 159x + 36}{144x^{1/2}} \\ \Psi_{z_1, z_2, z_3}(W_{1,1,1}) &= \frac{1156x^3 - 1467x^2 + 675x - 108}{864x^{3/2}(x-1)} \end{aligned}$$

This gives us the following minimal polynomial $Q \in k[U]$:

$$U^3 - \frac{7x-3}{2x(x-1)}U^2 + \frac{187x^2-159x+36}{48x^2(x-1)^2}U - \frac{1156x^3-1467x^2+675x-108}{864x^3(x-1)^3}$$

Note that the above solutions z_i are in fact $e^{\int u_i}$ where the u_i are the roots of Q , but that the computation of the z_i is not needed in order to compute Q .

3.4. Algorithm for third order equations: reducible and imprimitive case

The following algorithm decides for an imprimitive group \mathcal{G} if the formula can be used or if the group is among the four finite imprimitive groups. The steps must be performed in the given order:

1. The reducible case

If $L(y) = 0$ is reducible, then a basis of Liouvillian solutions can be computed using a factorisation of $L(y)$ and the reduction method of d'Alembert (cf. (Singer and Ulmer, 1993b) Section 2).

2. If $2a_0 = a_1'$

We then set $\tilde{L}(y) = y'' + \frac{a_1}{4}y$ and look for Liouvillian solutions of $\tilde{L}(y) = 0$. If $\tilde{L}(y) = 0$ has two Liouvillian solutions z_1, z_2 , then z_1^2, z_1z_2, z_2^2 will be a basis of Liouvillian solutions of $L(y) = 0$. Otherwise $L(y) = 0$ does not have Liouvillian solutions but can be solved in terms of solutions of $\tilde{L}(y) = 0$ since y_1^2, y_1y_2, y_2^2 will be a basis of solutions of $L(y) = 0$.

3. The imprimitive case

Compute the exponential solutions of $L^{\otimes 3}(y) = 0$.

- (a) If $L^{\otimes 3}(y) = 0$ has a two dimensional space of rational solutions, then $\mathcal{G} \cong G_{27}$.
- (b) If $L^{\otimes 3}(y) = 0$ has a two dimensional space of solutions which are square roots of rational functions, then $\mathcal{G} \cong G_{54}$.
- (c) If $L^{\otimes 3}(y)$ is of order 10 and has an (up to multiple) unique non zero solution g whose square is rational, then \mathcal{G} is an imprimitive group. There exists a Liouvillian solution and we are in the **Generic case where the formula holds**. Find the remaining coefficients of $P(U)$ of degree 3 by solving the linear system where a_0, a_1 have been replaced by their values.
- (d) If $L^{\otimes 3}(y)$ is of order 9 and has an (up to multiple) unique non zero solution g whose square is rational, then \mathcal{G} is isomorphic to G_{27}, G_{54}, G_{81} or G_{162} .

4. Otherwise $\mathcal{G} \subseteq SL(3, \mathbb{C})$ is a primitive linear group

4. The finite groups

Throughout this section we will assume that $k = \mathbf{C}(x)$, $\delta = \frac{d}{dx}$ and that the algorithm of the previous section showed that \mathcal{G} is one of the four exceptional finite imprimitive groups or belongs to the finite list of primitive unimodular group ((Singer and Ulmer, 1993a) Section 2.2). In this case all Liouvillian solutions of $L(y) = 0$ are algebraic. Therefore $L(y)$ has a Liouvillian solution if and only if \mathcal{G} is one of the four exceptional imprimitive groups or one of the eight finite primitive subgroups of $SL(3, \mathbf{C})$ denoted A_5 , $A_5 \times C_3$, $G_{168}^{SL_3}$, $G_{168}^{SL_3} \times C_3$, $A_6^{SL_3}$, $H_{216}^{SL_3}$, $H_{72}^{SL_3}$ and $F_{36}^{SL_3}$ whose matrix generators can be found in (Singer and Ulmer, 1993b) Section 4.1.2. Section 2.3 of (Singer and Ulmer, 1993a) shows that it is always possible to compute \mathcal{G} , if \mathcal{G} belongs to a finite list of groups using the decompositions of symmetric powers. Therefore, after running the algorithm of the previous section, we will always be able to compute \mathcal{G} . The determination of \mathcal{G} is often simplified by the use of necessary conditions on the exponents of $L(y)$ (cf. (Kovacic, 1986; Singer and Ulmer, 1995)) and is in general much easier than the computation of the minimal polynomial $F \in \mathbf{C}(x)[Y]$ of a solution of $L(y) = 0$. The procedure to compute $F \in \mathbf{C}(x)[Y]$ given in (Singer and Ulmer, 1993b) is based on the computation of values of semi-invariants of \mathcal{G} and involves solving high symmetric powers[†]. In (Fakler, 1999; Hoeij *et al.*, 1999; van Hoeij and Weil, 1997; Weil, 1995) the approach was simplified using syzygies, systems and standard constructors (Hessian, Jacobian, bordered Hessian) for the invariants. The computation of $F \in \mathbf{C}(x)[Y]$ is still difficult and the method using the above constructors only works up to third order equations.

In the following we show how strong necessary conditions can be derived for each of the twelve finite groups and how the computation of $F \in \mathbf{C}(x)[Y]$ can be simplified. We present the results for third order equations, but the generalisation to higher order is straightforward.

4.1. Ramification data

Let $L(y)$ be an n -th order linear differential equation over $\mathbf{C}(x)$ with finite differential Galois group $\mathcal{G} \subseteq GL(n, \mathbf{C})$. According to (Singer and Ulmer, 1995) Lemma 2.2 (3) all singular points c of $L(y)$ must be regular singular points. From (Singer and Ulmer, 1995) Lemma 2.4 we get that the monodromy matrix $M_{c,L}$ corresponding to a small loop γ around c containing no other singularities

[†]Note that in (van Hoeij, 2002) it is shown how to construct the minimal polynomial of a solution from the minimal polynomial of the Riccati equation.

of $L(y)$ is conjugate to

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}\alpha_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & e^{2\pi\sqrt{-1}\alpha_n} \end{pmatrix} \quad (2)$$

where $\alpha_1, \dots, \alpha_n \in \mathbf{Q}$ are the exponents of $L(y)$ at c . The eigenvalues of $M_{c,L}$ determine the exponents at c up to integers. In other words, writing $\alpha_i = \frac{a_i}{b_i} + n_i$ where $n_i \in \mathbf{Z}$ and $\frac{a_i}{b_i} \in]0, 1]$, then $\frac{a_i}{b_i}$ is the part of the exponent that is uniquely determined by $M_{c,L}$.

DEFINITION 4.1: *With the above notation we define $\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\}$ as the ramification data of $L(y)$ at the singularity c or, equivalently, of $M_{c,L} \in \mathcal{G}$.*

Note that ramification data of conjugate elements of \mathcal{G} coincide.

EXAMPLE 4.2: *To illustrate the computation of the ramification data we have reproduced a small part of the character table of G_{162} in form of an output of the computer algebra system MAGMA:*

<i>Class</i>	1	2	3	4	5	6	7	8	9	10	...
<i>Size</i>	1	9	1	1	6	18	18	18	9	9	...
<i>Order</i>	1	2	3	3	3	3	3	3	6	6	...
$p = 2$	1	1	4	3	5	6	8	7	3	4	...
$p = 3$	1	2	1	1	1	1	1	1	2	2	...
$\chi_{1,1}$	1	-1	1	1	1	1	1	1	-1	-1	...
$\chi_{1,2}$	1	-1	1	1	1	1	$-1 - J$	J	-1	-1	...
χ_2	2	0	2	2	2	-1	$1 + J$	$-J$	0	0	...
χ	3	-1	$-3 - 3J$	$3J$	0	0	0	0	$-J$	$1 + J$...
χ_6	6	0	6	6	-3	0	0	0	0	0	...
...

where $J^2 + J + 1 = 0$ and “ $p = j$ ” is the p -power map which indicates for $\sigma \in \mathcal{G}$ in the i -th conjugacy class to which class σ^j belongs. Consider a third order differential equation $L(y)$ with $\mathcal{G} \cong G_{162}$ whose character is χ . If the monodromy matrix $M_{c,L}$ at a given singularity c belongs to the class number 10, then, according to the character table we get $\text{trace}(M_{c,L}) = 1 + J$. Using

the p -power maps we also get $\text{trace}((M_{c,L})^2) = 3J$ and $\text{trace}((M_{c,L})^3) = -1$. Therefore if $M_{c,L}$ is conjugate to

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

then $\sum_{i=1}^3 \alpha_i = 1 + J$, $\sum_{i=1}^3 \alpha_i^2 = 3J$ and $\sum_{i=1}^3 \alpha_i^3 = -1$. Using the newton formulae we get α_i are roots of

$$X^3 - (J+1)X^2 - JX - 1 = \left(X - e^{2\pi\sqrt{-1}\frac{2}{3}}\right) \left(X - e^{2\pi\sqrt{-1}\frac{1}{6}}\right) \left(X - e^{2\pi\sqrt{-1}\frac{1}{6}}\right).$$

The ramification data of the character χ for a monodromy matrix in the conjugacy class number 10 is $\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\}$. Clearly such a computation is possible for any conjugation class and any character.

In the application we don't know the class of $M_{c,L}$, but only few conjugacy classes correspond to a given ramification data. For example the conjugacy class 10 is the only possible for the ramification data $\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\}$, while for $\{\frac{1}{3}, \frac{2}{3}, 1\}$ there are 4 possible conjugacy classes in G_{162} .

Suppose $L(y)$ is of order n and denote $\text{Sol}(L)$ and $\text{Sol}(L^{\otimes m})$ the solution space of $L(y) = 0$ and $L^{\otimes m} = 0$. From a basis Y_1, \dots, Y_n of $\text{Sol}(L)$ we get a basis $Y_1^m, Y_1^m Y_2, \dots, Y_n^m$ of $\text{Sym}^m(\text{Sol}(L))$ and the symmetrization of the representation ρ gives a morphism

$$\begin{aligned} \rho_m: \text{GL}(\text{Sol}(L)) &\rightarrow \text{GL}(\text{Sym}^m(\text{Sol}(L))) \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} &\mapsto \begin{pmatrix} a_{1,1}^m & a_{1,1}^{m-1} a_{1,2} & \cdots & a_{1,n}^m \\ m a_{1,1}^{m-1} a_{2,1} & \cdots & \cdots & m a_{n-1,n} a_{1,n}^{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1}^m & a_{n,1}^{m-1} a_{n,2} & \cdots & a_{n,n}^m \end{pmatrix}. \end{aligned}$$

Let Φ_m be the \mathcal{G} -morphism obtained by restriction of the evaluation morphism Φ to the vector space of homogeneous forms of degree m of $\mathcal{C}[Y_1, \dots, Y_n]$ ((Singer and Ulmer, 1993a) Lemma 3.5). By composition we get a representation ψ of \mathcal{G} on $\text{Sol}(L^{\otimes m})$. If L_i is a right factor of $L^{\otimes m}$, then $\text{Sol}(L_i)$ is a \mathcal{G} -submodule of $\text{Sol}(L^{\otimes m})$.

$$\begin{array}{ccccc}
\mathcal{G} & \xrightarrow{\rho} & \mathrm{GL}(\mathrm{Sol}(L)) & \xrightarrow{\rho_m} & \mathrm{GL}(\mathrm{Sym}^m(\mathrm{Sol}(L))) \\
& & & & \downarrow \Phi_m \\
& & & & \mathrm{GL}(\mathrm{Sol}(L^{\otimes m})) \\
& \searrow \psi & & & \downarrow \psi_i \\
& & & & \mathrm{GL}(\mathrm{Sol}(L_i))
\end{array}$$

We denote resp. χ , $\chi^{\otimes m}$, $\chi_{L^{\otimes m}}$ and χ_{L_i} the character of resp. ρ , $(\rho_m \circ \rho)$, ψ and $(\psi_i \circ \psi)$.

LEMMA 4.3: *Let $L(y)$ be a irreducible linear differential equation whose differential Galois group \mathcal{G} is finite. Then, using the above notation, we have*

1. $\chi_{L^{\otimes m}}$ is a summand of $\chi^{\otimes m}$.
2. If c is a non apparent singularity of $L^{\otimes m}(y)$, then c is a singularity of $L(y)$. There exists a $\sigma \in \mathcal{G}$ such that the monodromy matrices of L and of $L^{\otimes m}$ at c are conjugated respectively to $\rho(\sigma)$ and $\psi(\sigma)$.
3. Suppose that $L^{\otimes m}(y)$ is the least common left multiple of irreducible right factors $L_1(y), \dots, L_t(y)$. If $L_i(y)$ is a right factor of $L^{\otimes m}(y)$ then χ_{L_i} is a summand of $\chi^{\otimes m}$. There exists a $\sigma \in \mathcal{G}$ such that the monodromy matrices of L and of L_i at c are conjugated respectively to $\rho(\sigma)$ and $(\psi_i \circ \psi)(\sigma)$.

Proof: 1. The first assertion follows from the complete reducibility of \mathcal{G} .

2. Since $y_1^m, y_1^{m-1}y_2, \dots, y_n^m$ span the solution space of $L^{\otimes m}(y)$, a non apparent singularity of $L^{\otimes m}(y)$ must be a singularity of $L(y)$. We now suppose that the basis y_1, \dots, y_n chosen so that the monodromy matrix $M_{c,L}$ of L at c is diagonal and given by (2). Note that there exists $\sigma \in \mathcal{G}$ such that $\rho(\sigma) = M_{c,L}$. For this basis a continuation along a small loop around c transforms y_i into $e^{2\pi\sqrt{-1}\alpha_i}y_i$ and its affect on the generating set $y_1^m, y_1^{m-1}y_2, \dots, y_n^m$ corresponds to the action of $\rho_m(M_{c,L}) = \rho_m(\rho(\sigma))$ on the basis $Y_1^m, Y_1^{m-1}Y_2, \dots, Y_n^m$ of $\mathrm{Sym}^m(V)$. Since the monodromy matrix $M_{c,L^{\otimes m}}$ is conjugated to the image of $\rho_m(\rho(\sigma)) = \psi(\sigma)$ in $\mathrm{GL}(\mathrm{Sol}(L^{\otimes m}))$ we get the result.
3. Since \mathcal{G} is completely reducible we get from (Singer, 1996) that $L^{\otimes m}(y)$ is the least common left multiple of irreducible right factors $L_1(y), \dots, L_t(y)$ whose solution spaces by the above are \mathcal{G} -submodules of $\mathrm{Sym}^m(V)$. The result now follows like in the previous cases.

□

The fact that the monodromy matrices are images of elements of the same conjugation class of \mathcal{G} allows to link the exponents of $L(y)$ and those of a right factor L_i of $L^{\otimes m}$ as shown in the following example:

EXAMPLE 4.4: *We continue with the previous example. Suppose that $L(y)$ is a third order differential equation with $\mathcal{G} \cong G_{162}$ and that χ is the character of the representation of \mathcal{G} on the solution space $V = \mathbf{C}^3$ of $L(y) = 0$. The character $\chi_{1,2}$ is a summand of $\chi^{\otimes 3}$ and thus possibly (the order of $L^{\otimes 3}(y)$ may be less than 10) the character of a right factor $L_1(y)$ of order 1 of $L^{\otimes 3}(y)$. Suppose that at a singularity c of $L(y)$ the ramification data (given by the exponents) is $\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\}$, then the ramification data of $L_1(y)$ at c must be $\{\frac{1}{2}\}$. If at a singularity c of $L(y)$ the ramification data (given by the exponents) is $\{\frac{1}{3}, \frac{2}{3}, 1\}$, then the ramification data of $L_1(y)$ at c must be $\{\frac{1}{3}\}$, $\{\frac{2}{3}\}$ or 1.*

We will illustrate the use of the ramification data in the case of the imprimitive group G_{162} in the following subsection and keep the same notation for the other groups.

4.2. The group G_{162}

4.2.1. Ramification table

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional imprimitive representation of G_{162} with character χ . Decomposing the character $\chi^{\otimes 3}$ (cf. (Singer and Ulmer, 1993a), Section 2.3) we get $\chi^{\otimes 3} = \chi_{1,1} + \chi_{1,2} + \chi_2 + \chi_6$, where $\chi_{1,1}$ is a linear character of order 2, $\chi_{1,2}$ is a linear character of order 6, χ_2 is an irreducible character of degree 2 and χ_6 is an irreducible character of degree 6.

In the table below the top part contains all possible ramification data of G_{162} for χ . For each ramification data of χ we included all possible ramification data for some characters in the decomposition of $\chi^{\otimes 3}$ (which are possible characters of factors of $L^{\otimes 3}(y)$). Note that a ramification data like $\{\frac{1}{3}, \frac{2}{3}, 1\}$ of χ may behave in different ways for possible characters of factors of $L^{\otimes 3}(y)$, in which case all possible cases have been included. The identity, corresponding to an apparent singularity, has not been included since it gives no information. Ramification data of χ , which are identical for several conjugation classes of \mathcal{G} and which give the same ramification data for all characters in the table will be merged into one single column.

χ	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{5}{9}$	$\frac{7}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{1}{9}$
	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{5}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{7}{9}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{13}{18}$	$\frac{1}{18}$	$\frac{17}{18}$	$\frac{17}{18}$	$\frac{13}{18}$	$\frac{11}{18}$
	1	$\frac{2}{3}$	$\frac{1}{3}$	1	1	1	1	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{5}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{7}{9}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{1}{18}$	$\frac{7}{18}$	$\frac{11}{18}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{5}{18}$
$\chi_{1,1}$	$\frac{1}{2}$	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\chi_{1,2}$	$\frac{1}{2}$	1	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{5}{6}$
χ_2	$\frac{1}{2}$	1	1	1	$\frac{2}{3}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{5}{6}$
	1	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$
χ_6	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1	1	1	1	1
	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
	1	1	1	$\frac{1}{3}$	1	1	1	1	1	1	1	1	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	1	1	1	$\frac{1}{3}$	1	1	1	1	1	1	1	1	1	1	1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The information given in a ramification table of the above type gives stronger necessary conditions on the exponents of $L(y)$ for \mathcal{G} to be the group under consideration, than those of the type given in ((Singer and Ulmer, 1995), Necessary conditions for case 3).

EXAMPLE 4.5: *The following differential operator was constructed using the method given in (van der Put and Ulmer, 2000):*

$$L(y) = \frac{d^3y}{dx^3} - \frac{8x-3}{x(x-1)} \frac{d^2y}{dx^2} + \frac{3264x^2 - 2452x + 729}{108x^2(x-1)^2} \frac{dy}{dx} - \frac{71280x^3 - 80410x^2 + 47466x - 10935}{1458x^3(x-1)^3} y$$

The operator is irreducible and $L^{\otimes 3}(y)$ is of order 9. From Corollary 3.4 we get that \mathcal{G} is either G_{27} , G_{54} , G_{81} , G_{162} or F_{36} . Since $L^{\otimes 3}(y) = 0$ has no rational solution the groups G_{27} and G_{81} are not possible. At 0, 1 and ∞ the exponents are $\{2, \frac{3}{2}, \frac{5}{2}\}$, $\{\frac{22}{9}, \frac{47}{18}, \frac{53}{18}\}$ and $\{-4, -\frac{10}{3}, -\frac{11}{3}\}$, corresponding to the ramification indices $\{1, \frac{1}{2}, \frac{1}{2}\}$, $\{\frac{4}{9}, \frac{11}{18}, \frac{17}{18}\}$ and $\{1, \frac{2}{3}, \frac{1}{3}\}$. Since neither G_{54} , nor F_{36} contain elements of order 18 (see ramification table of those groups), the group must be G_{162} .

4.2.2. Ramification data and exponential solutions

For G_{162} we want to find the value (up to multiples) of the two semi-invariants of degree 3. The fact that one value will be a square root with character $\chi_{1,1}$

(the value of a semi-invariant that factors into linear forms) and the other value will be a cube root will allow us to identify them. If $L^{\otimes 3}(y)$ is of order 9, then the cube root must vanish, since a product of non zero solutions cannot be zero.

To compute an exponential solution g of $L^{\otimes 3}(y) = 0$ which is a j -th root of a rational function (i.e. which is the value of a semi-invariant of degree 3 and order j) we write g in the form

$$g = \frac{f^{\frac{1}{j}}}{\prod_{c_i} (x - c_i)^{-e_i}}$$

where the product is taken over all finite singular points c_i of $L^{\otimes 3}(y)$ (which are also singular points of $L(y)$), e_i are exponents of $L^{\otimes 3}(y)$ at c_i and $f \in \mathbf{C}[x]$. Since g is a j -th root, the exponents e_i above must be of the form $\frac{m_i}{j}$ with $m_i \in \mathbf{Z}$. The degree m of f must be of the form

$$m = -j \left(\sum_{i=1}^s e_i + e_{\infty} \right) \in \mathbf{N} \quad (3)$$

where e_{∞} is an exponent of $L^{\otimes 3}(y)$ at infinity (cf. (Singer and Ulmer, 1995) Theorem 3.5).

EXAMPLE 4.6: *In example 4.5 the ramification data at 0, 1 and ∞ is $\{1, \frac{1}{2}, \frac{1}{2}\}$, $\{\frac{4}{9}, \frac{11}{18}, \frac{17}{18}\}$ and $\{1, \frac{2}{3}, \frac{1}{3}\}$. From the ramification table of G_{162} we get that the only possible ramification data at 0, 1, ∞ of a factor of order 1 of $L^{\otimes 3}(y) = 0$ with character $\chi_{1,1}$, must be $\{\frac{1}{2}\}, \{\frac{1}{2}\}, \{1\}$. This shows that the exponents at 0, 1, ∞ of an exponential solution g of $L^{\otimes 3}(y) = 0$ whose square is rational, must be of the form $[\frac{1}{2} + n_1, \frac{1}{2} + n_2, n_3]$ with $n_i \in \mathbf{Z}$. Therefore g must be of the form*

$$g = \frac{f^{\frac{1}{2}}}{x^{-\frac{1}{2}-n_1}(x-1)^{-\frac{1}{2}-n_2}}$$

with $f \in \mathbf{C}[x]$ such that the degree m of the polynomial f is a positive integer of the form $m = -2(\frac{1}{2} + n_1 + \frac{1}{2} + n_2 + e_{\infty})$ for some integer exponent e_{∞} at ∞ of $L^{\otimes 3}(y) = 0$. In the above example the actual exponents of $L^{\otimes 3}(y)$ at 0, 1 and ∞ are

$$\begin{aligned} & \left\{ 5, 6, 7, 8, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2} \right\} \\ & \left\{ 8, \frac{15}{2}, \frac{17}{2}, \frac{22}{3}, \frac{23}{3}, \frac{25}{3}, \frac{47}{6}, \frac{49}{6}, \frac{53}{6} \right\} \\ & \left\{ -9, -10, -11, -12, -\frac{29}{3}, -\frac{31}{3}, -\frac{32}{3}, -\frac{34}{3}, -\frac{35}{3} \right\}. \end{aligned}$$

Among them we must find exponents at 0, 1, ∞ of the form $\frac{1}{2} + n_1, \frac{1}{2} + n_2, n_3$ with the property $-2(\frac{1}{2} + n_1 + \frac{1}{2} + n_2 + n_3) = m \in \mathbf{N}$. It turns out that in this example, the only possible exponents for g at 0, 1 and ∞ are $[\frac{9}{2}, \frac{15}{2}, -12]$ giving $m = 0$. Therefore the exponential solution must be $g = x^{9/2}(x-1)^{15/2}$.

In general the exponents will not determine the solution completely, but will narrow down the possible ramifications at the singularities and thus simplify the computation of an exponential solution.

If we don't know priorly that $\mathcal{G} \cong G_{162}$, then we could suppose this to be true and do the above in order to find g . If none of the possible exponents produces a solution g of $L^{\otimes 3}(y) = 0$, then we can conclude that $\mathcal{G} \not\cong G_{162}$. This gives further necessary conditions for G_{162} (cf. Example 4.9).

4.2.3. Ramification data and factors in symmetric powers

We will use the following facts about Fuchsian linear differential equations $L(y)$ of order n .

- The number of apparent singularities can be bounded by

$$\frac{1}{2}(\mu - 2)n(n - 1) - \sum_{j=1}^{\mu} \sum_{i=1}^n e_{j,i} \quad (4)$$

where n is the order of the equation under consideration, μ the number of non apparent singularities (possibly including ∞) and the $e_{i,j}$ are the exponents at the non apparent singularities ((Cormier *et al.*, 2002), relation (5)).

- Fuchs's relation states that the sum of all exponents at all ν singularities of $L(y)$ (including apparent singularities and possibly including ∞) must be

$$\frac{1}{2}(\nu - 2)n(n - 1) \quad (5)$$

The integer exponents at the apparent singularities can be bounded using this relation.

- If $L(y)$ has ν singular points (including apparent singularities and possibly including ∞), then $L(y)$ is determined by its exponents up to

$$\frac{1}{2}(n - 1)(n\nu - 2n - 2) \quad (6)$$

accessory parameters ((Ince, 1956) Section 15.4).

Suppose $L(y)$ is a third order equation with imprimitive differential Galois group $\mathcal{G} \cong G_{162} \in SL(3, \mathbf{C})$. We want to compute a second order right factor $L_2(y)$ of $L^{\otimes 3}(y)$ whose character is χ_2 . Again we use the information of the ramification table to get the ramification type of the solutions of $L_2(y)$, corresponding to the character χ_2 , at those singularities which are also singularities of $L(y)$. New apparent singularities (whose exponents are all positive integers) may also appear in $L_2(y)$. An example where apparent singularities appear is given in Example 4.13.

EXAMPLE 4.7: In example 4.5 the ramification data at $0, 1$ and ∞ is $\{1, \frac{1}{2}, \frac{1}{2}\}$, $\{\frac{4}{9}, \frac{11}{18}, \frac{17}{18}\}$ and $\{1, \frac{2}{3}, \frac{1}{3}\}$. From the ramification table of G_{162} we get that the only possible ramification data at $0, 1$ of a right factor $L_2(y)$ is $\{\frac{1}{2}, 1\}$, $\{\frac{5}{6}, \frac{1}{3}\}$ and at ∞ it could be $\{1, 1\}$, $\{\frac{2}{3}, \frac{1}{3}\}$, $\{1, \frac{1}{3}\}$ or $\{1, \frac{2}{3}\}$. Thus the form of the exponents of $L_2(y)$ is $\{n_{0,1}, 1/2 + n_{0,2}\}$ at 0 , $\{\frac{1}{3} + n_{1,1}, \frac{5}{6} + n_{1,2}\}$ at 1 and $\{n_{2,1}, n_{2,2}\}$, $\{\frac{2}{3} + n_{2,1}, \frac{1}{3} + n_{2,2}\}$, $\{1 + n_{2,1}, \frac{1}{3} + n_{2,2}\}$ or $\{1 + n_{2,1}, \frac{2}{3} + n_{2,2}\}$ at ∞ where $n_{i,j} \in \mathbf{Z}$. According to the list of exponents of $L^{\otimes 3}(y)$ given in the previous section, the smallest possible exponents at $0, 1, \infty$ which satisfy these conditions are $\{5, 9/2\}$, $\{22/3, 47/6\}$, $\{-12, -35/3\}$. Relation (4) implies that for these exponents (and thus also for any larger exponents) there are no apparent singularities in $L_2(y)$. Since the smallest possible exponents satisfy Fuchs's relation (5), the exponents cannot be larger. A second order equation with 3 singular points (here $0, 1, \infty$) is uniquely determined by its exponents and can be obtained using the formula from (Ince, 1956) Section 15.4:

$$L_2(y) = \frac{d^2 y}{dx^2} - \frac{17(8x-3)}{6x(x-1)} \frac{dy}{dx} + \frac{2520x^2 - 1891x + 405}{18(x-1)^2 x^2} y$$

The character table will restrict the possible ramification data at the singularities of L_2 and therefore simplifies the factorisation of $L^{\otimes 3}(y)$. But in general, several possible sets of exponents might satisfy Fuchs's relation and apparent singularities might appear. The accessory parameters and the apparent singularities must satisfy polynomial relations obtained by setting the remainder of a right division of $L^{\otimes 3}(y)$ by $L_2(y)$ to zero. The gain of the approach depends on the order of the factor we are looking for, the number of singularities and on arithmetic conditions of the exponents. But in any case the information should be used in the algorithms.

4.2.4. Precomputation of the invariants

We now investigate the computation of the minimal polynomial $F \in \mathbf{C}(x)[Y]$ of a solution of $L(y)$ for $\mathcal{G} \cong G_{162}$ along the lines of (Singer and Ulmer, 1993b) Section 4.3. The new contribution presented here is the use of invariants of factors of some constructions $L^{\otimes m}(y)$ to improve the computation of values of invariants which are needed to express the minimal polynomial of a solution of $L(y)$.

The group G_{162} has a unique normal subgroup N of order 27 which is the kernel of χ_2 . Thus the group G_{162}/N of order 18 is the differential Galois group of the, up to multiples, unique second order right factor $L_2(y)$ of $L^{\otimes 3}(y)$. Our strategy will be to compute $L_2(y)$ and its invariants and to relate them to invariants of \mathcal{G} . We denote Y_1, Y_2, Y_3 a basis of $V = \mathbf{C}^3$ corresponding to the representation of G_{162} given in section 3.2. The subspace of $Sym^3(V)$ corresponding to the representation $\rho_2(G_{162})$ of degree 2 of G_{162} with character χ_2 is spanned by $X_1 = Y_1^3 - Y_3^3$ and $X_2 = Y_2^3 - Y_3^3$. In the basis X_1, X_2 the group $\rho_2(G_{162})$ is

generated by

$$\rho_c = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \rho_t = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \rho_9 = \begin{pmatrix} -\omega^3 - 1 & 0 \\ 0 & -\omega^3 - 1 \end{pmatrix}.$$

There is, up to multiples, a unique semi-invariant $X_1X_2 - X_1^2 - X_2^2$ of degree 2 and a unique invariant $X_1^2X_2 - X_1X_2^2$ of degree 4 of $\rho_2(G_{162})$. They correspond to a semi-invariant and an invariant

$$\begin{aligned} S_3 &= Y_1^3Y_2^3 + Y_1^3Y_3^3 + Y_2^3Y_3^3 - Y_2^6 - Y_3^6 - Y_1^6 \\ I &= Y_1^6Y_2^3 - Y_1^6Y_3^3 - Y_1^3Y_2^6 + Y_1^3Y_3^6 + Y_2^6Y_3^3 - Y_2^3Y_3^6 \end{aligned}$$

of G_{162} . Using the invariants I , $I_1 = (Y_1Y_2Y_3)^2$, $I_2 = (Y_1^3 + Y_2^3 + Y_3^3)^6$ and $I_3 = S_3^3$ of \mathcal{G} it is possible to decompose the minimal polynomial of Y_1 which is of degree 54 (cf. (Singer and Ulmer, 1993b) Section 4.3). This Gröbner basis was computed by Jean-Charles Faugère. For simplicity (to keep the result small) we give the minimal polynomial $F \in \mathbf{C}(x)[Y]$ of $y_1 = \Phi(Y_1)$ under the assumption that $\Phi(I_2) = 0$ (i.e. that $L^{\otimes 3}(y)$ is of order 9):

$$\begin{aligned} &(-43046721\Phi(I_3)^2\Phi(I_1)^9(4\Phi(I_3) + 27\Phi(I)^2 - 243\Phi(I_1)^3)(4\Phi(I_3) + 27\Phi(I)^2 - 2187\Phi(I_1)^3)) \\ &+ (-154222052436\Phi(I_3)^2\Phi(I)^6\Phi(I_1)^3 - 8368\Phi(I_3)^6 - 3891342396588661\Phi(I)^2\Phi(I_1)^{15}) \\ &- 502200\Phi(I_3)^5\Phi(I)^2 + 266330673\Phi(I_3)^3\Phi(I)^6 - 237240810644040\Phi(I_3)\Phi(I)^4\Phi(I_1)^9 + 22427685\Phi(I_3)^4\Phi(I)^4 \\ &- 466454268756\Phi(I_3)\Phi(I)^8\Phi(I_1)^3 - 2711943423\Phi(I)^{12} - 7206189623312715\Phi(I)^4\Phi(I_1)^{12} \\ &- 5122559799\Phi(I_3)\Phi(I)^{10} - 533791823949090\Phi(I)^6\Phi(I_1)^9 - 1955311207983\Phi(I_3)^3\Phi(I_1)^9 \\ &+ 18815668605\Phi(I_3)^4\Phi(I_1)^6 - 366112362105\Phi(I)^{10}\Phi(I_1)^3 - 22400712\Phi(I_3)^5\Phi(I_1)^3 \\ &- 19770067553670\Phi(I)^8\Phi(I_1)^6 - 5764951698650172\Phi(I_3)\Phi(I_1)^{15} - 155618674601031\Phi(I_3)^2\Phi(I_1)^{12} \\ &+ 2691256590\Phi(I_3)^4\Phi(I)^2\Phi(I_1)^3 + 5386154535\Phi(I_3)^3\Phi(I)^4\Phi(I_1)^3 - 5396379991281\Phi(I_3)^2\Phi(I)^4\Phi(I_1)^6 \\ &- 15669609098094\Phi(I_3)\Phi(I)^6\Phi(I_1)^6 - 1654754654242179\Phi(I_3)\Phi(I)^2\Phi(I_1)^{12} \\ &- 53787136169826\Phi(I_3)^2\Phi(I)^2\Phi(I_1)^9 - 604648592073\Phi(I_3)^3\Phi(I)^2\Phi(I_1)^6 - 60584274\Phi(I_3)^2\Phi(I)^8 - \mathbf{Y}^{18} \\ &+ (5621779728\Phi(I_3)^4\Phi(I)^2 + 7060738412025\Phi(I)^8\Phi(I_1)^3 + 381279874249350\Phi(I)^6\Phi(I_1)^6 \\ &+ 1171595587360\Phi(I_3)\Phi(I)^6\Phi(I_1)^3 + 138976514163888075\Phi(I)^2\Phi(I_1)^{12} + 5795454088590120\Phi(I_3)\Phi(I)^2\Phi(I_1)^9 \\ &+ 239052790620\Phi(I_3)^3\Phi(I)^2\Phi(I_1)^3 + 40356042657174\Phi(I_3)^2\Phi(I)^2\Phi(I_1)^6 + 3806406304425\Phi(I_3)^2\Phi(I)^4\Phi(I_1)^3 \\ &+ 172789538094\Phi(I_3)^3\Phi(I_1)^6 + 398225646438210\Phi(I_3)\Phi(I)^4\Phi(I_1)^6 + 29854214153724105\Phi(I_3)\Phi(I_1)^{12} \\ &+ 118923741216\Phi(I_3)^2\Phi(I)^6 + 125911658925\Phi(I_3)\Phi(I)^8 + 10294556604732450\Phi(I)^4\Phi(I_1)^9 \\ &+ 289490274893025\Phi(I_3)^2\Phi(I_1)^9 - 13434828480\Phi(I_3)^4\Phi(I_1)^3 + 40299171030\Phi(I_3)^3\Phi(I)^4 \\ &+ 750473176484995605\Phi(I_1)^{15} + 278139744\Phi(I_3)^5 + 52301766015\Phi(I)^{10} - \mathbf{Y}^{36} \\ &+ (43046721\Phi(I_3)^2(4\Phi(I_3) + 27\Phi(I)^2 - 2187\Phi(I_1)^3)(4\Phi(I_3) + 27\Phi(I)^2 - 243\Phi(I_1)^3)) \mathbf{Y}^{54} \end{aligned}$$

There is a unique relation between those four invariants which, under the assumption that $\Phi(I_2) = 0$, is: $4\Phi(I_3) + 27\Phi(I)^2 + 729\Phi(I_1)^3 = 0$. In this case the invariants of $\rho_2(G_{162})$ will give us enough invariants (of high degree) of G_{162} in order to find $F \in \mathbf{C}(x)[Y]$.

EXAMPLE 4.8: *In Example 4.5, computing the values of the corresponding (semi-)invariants of $L_2(y) = 0$ (where the ramification data could again be used) we get the values $\Phi(I) = \alpha_1x^{14}(x-1)^{22}$, $\Phi(I_1) = \alpha_2x^9(x-1)^{15}$, $\Phi(I_2) = 0$ and $\Phi(I_3) = \alpha_3(x-1)^{44}x^{18}$. Inserting those values into the above relation (this is the method proposed in (Fakler, 1999)) we get among the solutions $\alpha_1 = 1, \alpha_2 =$*

$-1/3, \alpha_3 = -27/4$ which gives, in this example, a solution whose minimal polynomial

$$Y^{54} + \frac{(2x-11)(x-1)^{44}x^{27}}{18}Y^{36} + \frac{(16x^2+184x+43)(x-1)^{88}x^{54}}{3888}Y^{18} + \frac{(x-1)^{135}x^{81}}{19683}$$

is defined over $\mathbf{Q}(x)$ and has geometric Galois group G_{162} .

The precomputation using the invariants has to be done only once. The use of the operator $L_2(y)$ allows to filter specific invariants and therefore to avoid finding specific values of invariants within a large vector space of rational functions. For G_{162} the next available semi-invariants, besides the two semi-invariants of degree 3 leading to the invariants I_1 and I_2 , are contained in the sixth symmetric power, where besides $(Y_1^3+Y_2^3+Y_3^3)^2$, there is a two dimensional space of semi-invariants of order 3. Using those semi-invariants would have introduced more parameters than using L_2 .

4.3. The group G_{81}

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional imprimitive representation of G_{81} with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $\mathbf{1} + \sum_{i=1}^3 \chi_{1,i} + \sum_{i=1}^2 \chi_{3,i}$, where $\mathbf{1}$ is the trivial character and the $\chi_{1,i}$ are distinct one dimensional characters of order 3 and $\chi_{3,i}$ are distinct characters of degree 3. The ramification table is

	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{8}{9}$
χ	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{8}{9}$
	$\frac{2}{3}$	$\frac{1}{3}$	1	1	1	1	1	1	1	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{2}{9}$
$\chi_{1,1}$	1	1	1	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\chi_{1,2}$	1	1	1	$\frac{2}{3}$	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\chi_{1,3}$	1	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

The value of the (up to constant multiples) unique invariant of degree 3 that factors into linear forms corresponds with the (up to constant multiples) unique rational solution of $L^{\otimes 3}(y)$. If we denote Y_1, Y_2, Y_3 a basis of $V = \mathcal{C}^3$ corresponding to the given matrices of G_{81} , then the subspace of invariants of degree 3 of G_{81} is generated by $I_1 = Y_1Y_2Y_3$ and the 3 spaces of semi-invariants of degree 3 are generated by $S_1 = Y_1^3 + Y_2^3 + Y_3^3$, $S_2 = Y_3^3 + Y_1^3\omega^3 + (-\omega^3 - 1)Y_2^3$ and $S_3 = Y_3^3 + (-\omega^3 - 1)Y_1^3 + \omega^3Y_2^3$. There is one relation

$$S_3^3 - 3S_3S_2S_1 + S_2^3 + S_1^3 - 27I_1^3 = 0$$

and the minimal polynomial $F \in \mathbf{C}(x)[Y]$ of Y_1 is

$$Y^{27} + (-S_3 S_2 S_1 - 3I_1^3) Y^{18} + \left(\frac{S_3^2 S_2^2 S_1^2}{9} - \frac{S_3 S_2^4 S_1}{9} - \frac{S_3 S_2 S_1^4}{9} \right. \\ \left. + S_3 S_2 S_1 I_1^3 + \frac{S_2^6}{27} + \frac{S_2^3 S_1^3}{27} - S_2^3 I_1^3 + \frac{S_1^6}{27} - S_1^3 I_1^3 + 3I_1^6 \right) Y^9 - I_1^9$$

EXAMPLE 4.9: We consider the following differential equation $L(y) = 0$:

$$\frac{d^3 y}{dx^3} + \frac{7x-3}{x(x-1)} \frac{d^2 y}{dx^2} + \frac{2(138x^2 - 115x + 7)}{27x^2(x-1)^2} \frac{dy}{dx} + \frac{10(162x^2 - 28x - 7)}{729x^3(x-1)^2} y = 0.$$

Since $L(y)$ is irreducible, $L^{\otimes 3}(y)$ is of order 10 and $L^{\otimes 3}(y) = 0$ has, up to multiples, a unique rational solution $\frac{1}{x(x-1)^2}$, the group \mathcal{G} is imprimitive. If \mathcal{G} is one of our exceptional groups it must be G_{81} . We then look for 3 exponential solutions of $L^{\otimes 3}(y)$ under the assumption that $\mathcal{G} = G_{81}$ using the above table. The ramification data of $L(y)$ at $0, 1, \infty$ is $\{\frac{4}{9}, \frac{7}{9}, \frac{7}{9}\}, \{\frac{2}{3}, \frac{1}{3}, 1\}, \{\frac{2}{3}, \frac{1}{3}, 1\}$. From the table we get the ramification $\frac{1}{3}$ at 0 . The exponents of $L^{\otimes 3}(y)$ at $0, 1$ and ∞ are

$$\{0, -1, 1, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, \frac{4}{3}, -\frac{5}{3}, \frac{7}{3}\} \\ \{0, -1, 1, -2, \frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{5}{3}\} \\ \{3, 4, 5, 6, \frac{10}{3}, \frac{11}{3}, \frac{13}{3}, \frac{14}{3}, \frac{16}{3}, \frac{17}{3}\}$$

Relation (3) implies that the only possible exponents at $[0, 1, \infty]$ of an exponential solution of $L^{\otimes 3}(y) = 0$ are $[-\frac{5}{3}, -2, \frac{11}{3}]$, $[-\frac{5}{3}, -\frac{5}{3}, \frac{10}{3}]$, $[-\frac{5}{3}, -\frac{4}{3}, 3]$, $[-\frac{5}{3}, -\frac{5}{3}, 3]$, $[-\frac{5}{3}, -2, 3]$, $[-\frac{5}{3}, -2, \frac{10}{3}]$. It is easy to verify the first three, corresponding to

$$\frac{1}{x^{5/3}(x-1)^{5/3}}, \quad \frac{1}{x^{5/3}(x-1)^2}, \quad \frac{1}{x^{5/3}(x-1)^{4/3}},$$

being solutions of $L^{\otimes 3}(y) = 0$. From Theorem 3.2 we get that $\mathcal{G} = G_{81}$.

Unfortunately the exponents do not tell which value belongs to which semi-invariant. One can consider the six possible combinations and use the relation to obtain a finite set of possible multiples for each case. However the verification modulo a polynomial of degree 27 would be a difficult task.

EXAMPLE 4.10: For the following equation we also have $\mathcal{G} \cong G_{81}$, but this time $L^{\otimes 3}(y)$ is of order 9:

$$\frac{d^3 y}{dx^3} + \frac{2(5x-2)}{x(x-1)} \frac{d^2 y}{dx^2} + \frac{2(327x^2 - 262x + 28)}{27x^2(x-1)^2} \frac{dy}{dx} + \\ \frac{4(2268x^3 - 2729x^2 + 609x + 14)}{729x^3(x-1)^3} y(x)$$

4.4. The group G_{54}

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional imprimitive representation of G_{54} with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $2 \cdot \chi_1 + \sum_{i=1}^4 \chi_{2,i}$, where χ_1 is a one dimensional character and $\chi_{2,i}$ are distinct irreducible characters of degree 2. The ramification table is

χ	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{6}$
	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{6}$
	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	1	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$
χ_1	$\frac{1}{2}$	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
χ_1	$\frac{1}{2}$	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
$\chi_{2,1}$	$\frac{1}{2}$	1	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
	1	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\chi_{2,2}$	$\frac{1}{2}$	1	1	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
	1	1	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\chi_{2,3}$	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$
	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	1
$\chi_{2,4}$	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	1	1

We denote Y_1, Y_2, Y_3 a basis of $V = \mathcal{C}^3$ corresponding to the given representation of G_{54} . Note that there are four possible monomial bases, since there are 4 semi-invariants with character χ_1 that factor into linear forms (Corollary 3.3). The group G_{54} has 4 normal subgroups N_i of order 27 which are each in the kernel of one of the $\chi_{2,i}$. The group G_{54} has a two dimensional representation $\rho_2(G_{54})$ corresponding to $\chi_{2,1}$ on the submodule of $Sym^3(V)$ with basis $X_1 = Y_1^3 - Y_3^3, X_2 = Y_2^3 - Y_3^3$. The group $\rho_2(G_{54}) \cong G_{54}/N_i$ is of order 9. The other 3 two dimensional submodules of $Sym^3(V)$ are defined in a similar way using the other 3 monomial bases. The representation $\rho_2(G_{54})$ has two invariants $X_1^2 - X_1X_2 + X_2^2$ and $X_1^2X_2 - X_1X_2^2$ and a semi-invariant $3X_1^2X_2 + 3X_1X_2^2 - 2X_1^3 - 2X_2^3$. Those give respectively

$$\begin{aligned}
I_1 &= Y_1^6 - Y_1^3Y_2^3 - Y_1^3Y_3^3 + Y_2^6 - Y_2^3Y_3^3 + Y_3^6 \\
I_2 &= Y_1^6Y_2^3 - Y_1^6Y_3^3 - Y_1^3Y_2^6 + Y_1^3Y_3^6 + Y_2^6Y_3^3 - Y_2^3Y_3^6 \\
S_1 &= -2Y_1^9 + 3Y_1^6Y_2^3 + 3Y_1^6Y_3^3 + 3Y_1^3Y_2^6 - 12Y_1^3Y_2^3Y_3^3 + 3Y_1^3Y_3^6 - 2Y_2^9 \\
&\quad + 3Y_2^6Y_3^3 + 3Y_2^3Y_3^6 - 2Y_3^9
\end{aligned}$$

Denoting $F_1 = Y_1Y_2Y_3$ and $F_2 = Y_1^3 + Y_2^3 + Y_3^3$ a basis of the semi-invariants of

order 2 we get the following minimal polynomial $F \in \mathbf{C}(x)[Y]$ for Y_1 :

$$Y^{18} + \left(-\frac{1}{3}F_2^2 - \frac{2}{3}I_1\right)Y^{12} + \left(-2F_1^3F_2 + \frac{1}{9}F_2^4 - \frac{2}{9}F_2^2I_1 + \frac{1}{9}I_1^2\right)Y^6 + F_1^6$$

and the relations $S_1 + 27F_1^3 - F_2^3 + 3F_2I_1 = 0$ and

$$F_1^6 - \frac{2}{27}F_1^3F_2^3 + \frac{2}{9}F_1^3F_2I_1 + \frac{1}{729}F_2^6 - \frac{2}{243}F_2^4I_1 + \frac{1}{81}F_2^2I_1^2 + \frac{1}{27}I_2^2 - \frac{4}{729}I_1^3 = 0$$

EXAMPLE 4.11: *The following equation is an example for G_{54} where all 4 semi-invariants that factor into linear forms evaluate to the same value. Let $L(y)$ be*

$$\begin{aligned} \frac{d^3y}{dx^3} + \frac{3(3x^2 - 1)}{x(x-1)(x+1)} \frac{d^2y}{dx^2} + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2} \frac{dy}{dx} \\ + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3} y(x) \end{aligned}$$

Since the third symmetric power is of order 9 and has no non trivial rational solutions, \mathcal{G} is either G_{54} , G_{162} or F_{36} . The ramification data at $0, 1, -1, \infty$ is

$$\left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}.$$

The exponents of $L^{\otimes 3}(y)$ at $0, 1, -1, \infty$ are

$$\begin{aligned} \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{4, 5, 6, 7, 9/2, 11/2, 13/2, 15/2, 17/2\} \end{aligned}$$

From example 4.17 we get that \mathcal{G} cannot be G_{162} . Assuming that the differential Galois group is G_{54} we get from the above table that the ramification of an exponential solution of $L^{\otimes 3}(y)$ must be $1/2$ at all singular points since the monodromy is of order 6 at those points. From the actual exponents of $L^{\otimes 3}(y)$ and Relation (3) we get that (up to multiple) the only possibility is

$$\frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{3}{2}}(x+1)^{\frac{3}{2}}}$$

which can be verified to be a solution. Thus \mathcal{G} , having a semi-invariant of degree 3 and order 2, must be imprimitive and we get that $\mathcal{G} \cong G_{54}$. Note that in this example all 4 semi-invariants that factor into linear forms have the same value.

The factorisation of $L^{\otimes 3}(y)$ contains 4 factors of order 2 and is difficult to compute using the ramification data. There are hundreds of possibilities for the exponents of a factor and up to 4 apparent singularities. The equation $L^{\otimes 3}(y)$ is an interesting example for factorisation algorithms since 4 apparent singularities will appear. It shows that the ramification can give almost no information in some cases (but it gave useful information for the exponential solution).

4.5. The group G_{27}

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional imprimitive representation of G_{27} with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $2 \cdot \mathbf{1} + \sum_{i=1}^8 \chi_{1,i}$, where $\mathbf{1}$ is the trivial character and the $\chi_{3,i}$ are distinct one dimensional characters of order 3. For this group the ramification table gives no information for the computation of exponential solutions, since all possible ramification type do appear:

χ	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{2}{3}$	$\frac{1}{3}$	1	1	1	1	1	1	1	1
χ_1	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1
χ_2	1	1	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1
χ_3	1	1	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$
χ_4	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$
χ_5	1	1	$\frac{2}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
χ_6	1	1	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$
χ_7	1	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
χ_8	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$

However we can find the value of at least two invariants that factor into linear forms without actually having to factor semi-invariants. The eigenvectors of any non central element of order 3 will be a system of imprimitivity. To see this, note that the exponent of G_{27} is 3, there are 4 normal subgroups of order 9 and the disjoint union of the non central elements of those subgroups give all non central elements of G_{27} . Any system of generators must contain at least 2 non central elements (with 3 distinct eigenvalues) corresponding to 2 different systems of imprimitivity. Therefore, at each singular point where no two exponents differ by integers, the product of the eigenvectors is the value under some Φ of an invariant that factors into linear forms and power series computation allows to find the value in each case. If $L^{\otimes 3}(y)$ is of order 10, this would allow to use the generic formula in this case. In any case it is useful information for the algorithm in (Hoeij *et al.*, 1999).

EXAMPLE 4.12: *The following example has been constructed by Frits Beukers. Consider $P = (Y^3 + x^2)^3 - x^6 - (x + 1)^3 \in \mathbf{Q}(x)[Y]$ and denote $L_P(y)$ the minimal differential operator whose solution space is spanned by the roots of P (cf. (Cormier *et al.*, 2002)). Then $L_P(y)$ is an irreducible third order equation whose*

differential Galois group is irreducible and unimodular (even if the coefficient of $\frac{d^2y}{dx^2}$ is not zero, which can be archived using a transformation). There is a two dimensional space of rational solutions of $L^{\otimes 3}(y) = 0$ spanned by x^2 and $x + 1$ and thus \mathcal{G} is G_{27} . Note that in this case $L^{\otimes 3}(y)$ is of order 9. There will be 4 invariants that factor into linear forms and whose values will be in the space spanned by x^2 and $x + 1$. The singular point $x = J$ (where $J^2 + J + 1 = 0$) has the ramification data $\{1, \frac{1}{3}, \frac{2}{3}\}$ and thus the product of the 3 solutions corresponding to those exponents will be rational and correspond to the value of an invariant, that factors into linear forms. Using power series computation we get that this value is up to multiple $1 + x + x^2$. The same situation occurs at the singular point ρ where $\rho^4 - \rho^3 + 2\rho + 1 = 0$. For the singular point $x = \rho$ we get that two values of an invariant that factors into linear forms are up to multiple $x^2 + J(x + 1)$. This gives 3 values of forms that factor into linear forms.

Another equation whose differential Galois group is G_{27} can be found by computing an irreducible factor (there will be two having both the Galois group G_{27}) of the third symmetric power of an equation whose differential Galois group is G_{81} (cf. examples 4.9 and 4.10). We note that for this factorisation the ramification data can again be used efficiently.

4.6. The group $H_{216}^{SL_3}$

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional representation of $H_{216}^{SL_3}$ with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $\chi_2 + \chi_8$, where the χ_i are irreducible characters of degree i . The ramification table is:

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{3}$
χ	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{5}{6}$
	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1	1	1	$\frac{1}{6}$	$\frac{5}{6}$
χ_2	$\frac{1}{2}$	1	1	1	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{1}{2}$	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{1}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{4}{9}$
χ	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{11}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{7}{18}$	$\frac{5}{18}$	$\frac{11}{18}$
	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{11}{12}$	$\frac{1}{12}$	$\frac{5}{18}$	$\frac{13}{18}$	$\frac{7}{18}$	$\frac{13}{18}$	$\frac{17}{18}$	$\frac{17}{18}$
χ_2	1	1	1	1	1	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$

The equation $L^{\otimes 3}(y)$ must be of order 10. Denote Y_1, Y_2, Y_3 the basis of $V = \mathcal{C}^3$

corresponding to the matrices of $H_{216}^{SL_3}$ given in (Singer and Ulmer, 1993b). The sub-representation $\rho_2(H_{216}^{SL_3})$ of $Sym^3(V)$ corresponding to χ_2 is defined on the span of $Y_1Y_2Y_3$ and $Y_1^3 + Y_2^3 + Y_3^3$ and is generated by

$$\begin{pmatrix} 2/3\omega^3 + 1/3 & -2/9\omega^3 - 1/9 \\ -4\omega^3 - 2 & -2/3\omega^3 - 1/3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega^3 \end{pmatrix}$$

The group $\rho_2(H_{216}^{SL_3})$ has, up to multiples, one invariant $I_1 = 27X_1^4 - X_1X_2^3$ of degree 4, one invariant $I_2 = X_1^6 + \frac{5}{54}X_1^3X_2^3 - \frac{1}{5832}X_2^6$ of degree 6 and one semi-invariant $S_1 = 216X_1^3X_2 + X_2^4$ (corresponding to invariants of degree 12 and 18 and a semi-invariant of degree 12 of $H_{216}^{SL_3}$).

EXAMPLE 4.13: Consider the following equation $L(y) =$

$$\frac{d^3y}{dx^3} + \frac{4(2x-1)}{x(x-1)} \frac{d^2y}{dx^2} + \frac{5(1216x^2 - 1219x + 192)}{432x^2(x-1)^2} \frac{dy}{dx} + \frac{5(-2477 + 4864x)}{5832x^2(x-1)^2} y(x)$$

This equation is irreducible and $L^{\otimes 3}(y)$ is of order 10. The exponents of $L(y)$ at 0, 1 and ∞ are $\{0, -\frac{1}{3}, -\frac{2}{3}\}$, $\{0, -\frac{1}{4}, -\frac{3}{4}\}$ and $\{\frac{10}{9}, -\frac{16}{9}, \frac{19}{9}\}$ and those of $L^{\otimes 3}(y)$ are:

$$\begin{aligned} &\{0, -1, 1, -2, 2/3, -1/3, 1/3, -2/3, -4/3, -5/3\} \\ &\{0, -1, -1/2, -3/2, -1/4, 1/4, -3/4, -5/4, -7/4, -9/4\} \\ &\{4, 5, 6, 10/3, 13/3, 14/3, 16/3, 17/3, 19/3, 22/3\} \end{aligned}$$

Using the relation (3) we get that there is no solution of $L^{\otimes 3}(y) = 0$ which is rational or whose square is a rational function. Thus \mathcal{G} is a primitive group. From the exponents we get that the group cannot be $F_{36}^{SL_3}$. In order to test if \mathcal{G} is conjugated to $H_{216}^{SL_3}$ or $H_{72}^{SL_3}$ we look for a second order factor $L_2(y)$ of $L^{\otimes 3}(y)$ under the assumption that the group is $H_{216}^{SL_3}$. The exponents show that there could be an apparent singularity in $L_2(y)$ and that in this case the exponents must be $\{-2, -\frac{4}{3}\}$, $\{-\frac{9}{4}, -\frac{7}{4}\}$ and $\{4, \frac{10}{3}\}$. In this case $L_2(y)$ must be of the form

$$\begin{aligned} &\frac{d^2y}{dx^2} + \frac{-25x^2 + 28x\alpha + 10x - 13\alpha}{3x(x-1)(\alpha-x)} \frac{dy}{dx} + \\ &\frac{-640x^3 + 317x^2\alpha + 579x^2 - 256x\alpha - 128x + 128\alpha - 48qx^2 + 48qx}{48x^2(x-1)^2(\alpha-x)} y(x) \end{aligned}$$

where α is an apparent singularity and q a parameter. Dividing $L^{\otimes 3}(y)$ from the right by $L_2(y)$ and setting the coefficients of the remainder to zero gives a system of polynomial equations from which we get $\alpha = -7/20$ and $q = 937/192$. From Table 2 in (Singer and Ulmer, 1993a) we get \mathcal{G} being either $H_{216}^{SL_3}$ or $H_{72}^{SL_3}$. The resulting differential equation $L_2(y)$ has a one dimensional space of invariants of degree 4 whose value, up to multiple, is

$$\frac{4 - 121x - 2200x^2 + 22000x^3}{(x-1)^9x^8}.$$

Therefore the differential Galois group is $H_{216}^{SL_3}$ (Lemma 4.15). From $L_2(y)$ we also get the values of I_2 and S_1 , up to multiples, are

$$\frac{-8 + 359x - 24500x^2 + 470000x^3 - 2600000x^4 + 8000000x^5}{x^{12}(x-1)^{13}},$$

$$\frac{-77 + 1760x - 22000x^2 + 40000x^3}{(x-1)^9 x^{22/3}}$$

This illustrates again how to filter invariants using factors and how to avoid the construction of high symmetric powers.

4.7. The group $H_{72}^{SL_3}$

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional representation of $H_{72}^{SL_3}$ with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $\chi_2 + \chi_8$, where the χ_i are irreducible characters of degree i . The ramification table is:

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
χ	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{11}{12}$	$\frac{7}{12}$
	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{12}$	$\frac{1}{12}$
χ_2	$\frac{1}{2}$	1	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
	$\frac{1}{2}$	1	1	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The equation $L^{\otimes 3}(y)$ must be of order 10. Denote Y_1, Y_2, Y_3 the basis of $V = \mathcal{C}^3$ corresponding to the matrices of $H_{72}^{SL_3}$ given in (Singer and Ulmer, 1993b). The sub-representation $\rho_2(H_{72}^{SL_3})$ of $Sym^3(V)$ corresponding to χ_2 is defined on the span of $Y_1 Y_2 Y_3$ and $Y_1^3 + Y_2^3 + Y_3^3$ and is a non abelian group of order 8 generated by

$$\left(\begin{array}{cc} \frac{2\omega^3+1}{3} & \frac{-2\omega^3-1}{9} \\ -4\omega^3-2 & \frac{-2\omega^3-1}{3} \end{array} \right), \quad \left(\begin{array}{cc} \frac{2\omega^3+1}{3} & \frac{\omega^3-1}{9} \\ 2\omega^3+4 & \frac{-2\omega^3-1}{3} \end{array} \right).$$

The group $\rho_2(H_{72}^{SL_3})$ has a two dimensional space of invariants of degree 4 spanned by $I_1 = X_1^4 - \frac{X_1 X_2^3}{27}$, $I_2 = X_1^3 X_2 + \frac{X_2^4}{216}$ and 2 semi-invariants of degree 2.

EXAMPLE 4.14: Consider the following operator constructed in (van Hoeij, 2002):

$$L(y) = \frac{d^3 y}{dx^3} + \frac{21x^2 - 24x - 1}{(3x^2 + 1)(x - 1)} \frac{d^2 y}{dx^2} + \frac{4437x^3 - 5973x^2 + 171x - 683}{48(3x^2 + 1)^2(x - 1)} \frac{dy}{dx}$$

$$+ \frac{13338x^4 - 22647x^3 + 1983x^2 - 7297x - 737}{216(x - 1)(3x^2 + 1)^3} y(x)$$

This operator is irreducible, $L^{\otimes 3}(y)$ is of order 10 and there is no semi-invariant whose square is a rational function. Thus \mathcal{G} is a primitive group. Denote α a root of $3x^2 + 1$. Then $c = 1$ is an apparent singularity and the exponents of $L(y)$ at α , $-\alpha$ and ∞ are $\{-\frac{1}{3}, -\frac{7}{12}, -\frac{1}{12}\}$, $\{-\frac{1}{3}, -\frac{7}{12}, -\frac{1}{12}\}$ and $\{\frac{4}{3}, \frac{13}{12}, \frac{19}{12}\}$ and those of $L^{\otimes 3}(y)$ are:

$$\begin{aligned} &\{0, -1, -1/2, -3/2, 3/4, -1/4, 1/4, -3/4, -5/4, -7/4\} \\ &\{0, -1, 1, -2, 2/3, -1/3, 1/3, -2/3, -4/3, -5/3\} \\ &\{4, 5, 7/2, 9/2, 13/4, 15/4, 17/4, 19/4, 21/4, 23/4\} \end{aligned}$$

Using the relation (3) we get that there is no solution of $L^{\otimes 3}(y) = 0$ which is rational or whose square is a rational function. Thus \mathcal{G} is a primitive group. We want to show that \mathcal{G} is H_{72} . We suppose that this is the case and, according to Table 2 in (Singer and Ulmer, 1993a), we look for a second order factor $L_2(y)$ of $L^{\otimes 3}(y)$ under the assumption the group being $H_{72}^{SL_3}$. The exponents show there are no apparent singularity in $L_2(y)$ and the exponents at α , $-\alpha$ and ∞ must be $\{-5/4, -7/4\}$, $\{-5/4, -7/4\}$ and $\{15/4, 13/4\}$. In this case $L_2(y)$ is uniquely determined by its exponents and must be

$$\frac{d^2y}{dx^2} + \frac{24x}{3x^2 + 1} \frac{dy}{dx} + \frac{15(117x^2 + 11)}{16(3x^2 + 1)^2} y(x)$$

Dividing $L^{\otimes 3}(y)$ from the right by $L_2(y)$ we get that L_2 is a right factor. From Table 2 in (Singer and Ulmer, 1993a), we see that \mathcal{G} must be $F_{36}^{SL_3}$, $H_{216}^{SL_3}$ or $H_{72}^{SL_3}$. By testing that L_2 is irreducible or that L_2 has no solution whose 4-th power is a rational function, we get that \mathcal{G} must be $H_{216}^{SL_3}$ or $H_{72}^{SL_3}$. Since L_2 has two invariants of degree 4 whose values belong to the \mathbf{C} -space spanned by

$$\frac{1}{(3x^2 + 1)^7}, \frac{x}{(3x^2 + 1)^7}$$

Lemma 4.15 shows that \mathcal{G} must be $H_{72}^{SL_3}$.

4.8. The group $F_{36}^{SL_3}$

Suppose the differential Galois group \mathcal{G} of $L(y) = 0$ being a unimodular three dimensional representation of $F_{36}^{SL_3}$ with character χ . Decomposing the character $\chi^{\otimes 3}$ we get $\sum_{i=1}^2 \chi_{1,i} + \sum_{i=1}^2 \chi_{4,i}$, where the $\chi_{1,i}$ are distinct one dimensional characters of order 4, $\chi_{3,i}$ are distinct characters of degree 3 and $\chi_{4,i}$ are distinct characters of degree 4. The ramification table is:

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{5}{12}$
χ	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{11}{12}$	$\frac{1}{12}$	$\frac{11}{12}$
$\chi_{1,1}$	$\frac{1}{2}$	1	1	1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\chi_{1,2}$	$\frac{1}{2}$	1	1	1	1	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

An example of a differential equation $L(y)$ where $\mathcal{G} \cong F_{36}^{SL_3}$ and $L^{\otimes 3}(y)$ is of order 9 can be found in (Geiselman and Ulmer, 1997).

4.9. Computing \mathcal{G} from a finite list

From (Singer and Ulmer, 1993a) Section 2.3 we get that it is possible to compute \mathcal{G} using factorisations of symmetric powers if we know that the group belongs to a finite list of groups. In this section we

1. Simplify the method proposed in (Singer and Ulmer, 1993a) Section 2.3 using the precomputation of the invariants in the previous subsections to distinguish between $H_{216}^{SL_3}$ and $H_{72}^{SL_3}$.
2. In the algorithm from section 3.4 we added 4 finite groups to the list of primitive subgroups of $SL(3, \mathbb{C})$, we will show how to decide if \mathcal{G} is isomorphic to one of those 4 groups using only the ramification data of those groups.

4.9.1. The groups $H_{216}^{SL_3}$ and $H_{72}^{SL_3}$

From the (Singer and Ulmer, 1993a) Table 2 we get that a factorisation of $L^{\otimes 5}(y)$ is needed in order to distinguish $H_{216}^{SL_3}$ and $H_{72}^{SL_3}$. From the result of the previous sections we get:

LEMMA 4.15: *Let $L(y) = y''' + a_1y' + a_0y$ be a third order linear differential equation with a unimodular primitive Galois group \mathcal{G} . If $L^{\otimes 3}(y)$ has an irreducible right factor L_2 of order 2, then*

1. *if $L_2(y)$ has a two dimensional space of invariants of degree 4, then $\mathcal{G} \cong H_{72}^{SL_3}$.*
2. *else $\mathcal{G} \cong H_{216}^{SL_3}$.*

The proof follows from the precomputation of the invariants for those groups. Since the use of the ramification data also simplifies the factorisation of $L^{\otimes 3}(y)$, this is a useful way to distinguish the groups. The criterion has been used in Example 4.14 and 4.13.

4.9.2. The 4 exceptional finite imprimitive groups

In the algorithm from section 3.4 we have to consider the following possibility: $L^{\otimes 3}(y)$ is of order 9 and \mathcal{G} is imprimitive and isomorphic to G_{27} , G_{54} , G_{81} or G_{162} . From the ramification tables of those groups it is possible to decide which group it is:

LEMMA 4.16: *Let $L(y) = y''' + a_1y' + a_0y$ be a third order linear differential equation with unimodular imprimitive Galois group \mathcal{G} . If $L^{\otimes 3}(y)$ is of order 9, then*

1. *If $L^{\otimes 3}(y)$ has, up to multiples, one rational solution then*

- If $L^{\otimes 3}(y)$ has (up to multiples) 8 solutions which are cube roots of rational functions, then $\mathcal{G} \cong G_{27}$.
 - otherwise $\mathcal{G} \cong G_{81}$
2. If $L^{\otimes 3}(y)$ has, up to multiples, one solution which is the square root of a rational function then
- If $L(y)$ has a ramification point of order 9 or a ramification point of order 18 or if $L^{\otimes 3}(y)$ has a ramification point of order 3 at which there are exactly 4 integer exponents, then $\mathcal{G} \cong G_{162}$.
 - otherwise $\mathcal{G} \cong G_{54}$.

Proof: Since \mathcal{G} is imprimitive and $L^{\otimes 3}(y)$ is of order 9 we know that \mathcal{G} is isomorphic to G_{27} , G_{54} , G_{81} or G_{162} . The cases above follow from the character decomposition and the ramification tables given in the corresponding subsections, together with the fact, that for G_{162} , if no monodromy matrix at some singularity is of order 9 or 18, then there must be a monodromy matrix whose conjugacy class corresponds to the bold exponents in the table of G_{162} . This follows from the fact that for G_{162} the remaining conjugacy classes do not generate the group. Since $L^{\otimes 3}(y)$ is of order 9, one semi-invariant has to vanish and a one dimensional character will be missing in the character decomposition of $\mathcal{G}(L^{\otimes 3})$. For G_{162} the vanishing semi-invariant corresponds to the character $\chi_{1,2}$ in the table and the bold conjugacy class are those corresponding to a ramification point of order 3 where 4 integer exponents will remain. This does not happen for G_{54} . \square

EXAMPLE 4.17: Consider the equation $L(y)$ of example 4.11:

$$\begin{aligned} \frac{d^3 y}{dx^3} + \frac{3(3x^2 - 1)}{x(x-1)(x+1)} \frac{d^2 y}{dx^2} + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2} \frac{dy}{dx} \\ + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3} y(x) \end{aligned}$$

Since the third symmetric power is of order 9 and has no non trivial rational solutions, \mathcal{G} is either G_{54} , G_{162} or F_{36} . The ramification data at $0, 1, -1, \infty$ is

$$\left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}, \left\{ \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right\}.$$

The exponents of $L^{\otimes 3}(y)$ at $0, 1, -1, \infty$ are

$$\begin{aligned} \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{0, -1, 1, -2, 1/2, -1/2, 3/2, -3/2, 5/2\} \\ \{4, 5, 6, 7, 9/2, 11/2, 13/2, 15/2, 17/2\} \end{aligned}$$

If the group is imprimitive (i.e. $\mathcal{G} \not\cong F_{36}$), then, since there is no ramification point of order 3 (at which there are exactly 4 integer exponents), the group must be G_{54} .

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