Local Levi-Flat hypersurfaces invariants by a codimension one holomorphic foliation

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Abstract. *In this paper we study codimension one holomorphic foliations leaving invariant real analytic hypersurfaces. In particular, we prove that a germ of real analytic Levi-flat hypersurface with sufficiently "small" singular set is given by the zeroes of the imaginary part of a holomorphic function.*

Contents

1. Introduction and statement of the results

Let M be a germ at $(\mathbb{C}^n,0)$ of a real codimension one irreducible analytic set. For the sake of simplicity, we will denote germs and representative of germs by the same letters. Since M is real analytic of codimension one and irreducible, it can be defined by $(F=0)$, where $F\colon (\mathbb{C}^n,0)\to(\mathbb{R},0)$ is an irreducible germ of real analytic function. The singular set of M is defined by $sing(M)=(F=0)\cap(dF=0)$ and its smooth part $(F=0)\setminus(dF=0)$ will be denoted by M^*. Note that $sing(M)$ contains all points $m\in M$ such that M is smooth at m, but the codimension of M at m is at least two. The Levi distribution L on M^* is defined by $L_p := ker(\partial F(p))\subset T_pM^* = ker(dF(p))$, for any $p\in M^*$.

Definition 1. We say that M is *Levi-flat* if the Levi distribution on M^* is integrable.

The integrability condition of L implies that it is tangent to a real codimension one foliation $\mathcal{L}$ of M^*. Since the hyperplanes L_p, $p\in M^*$, are complex, the leaves of $\mathcal{L}$ are complex codimension one holomorphic submanifolds immersed on M^*.

Remark 1. If the hypersurface M is defined by $F=0$ then the Levi distribution L on M can be defined by the real analytic 1-form $i(\partial F-\overline{\partial}F)$, which will be called the

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Levi 1-form of F. The integrability condition is equivalent to $(\partial F - \overline{\partial} F) \wedge \partial\overline{\partial} F|_{M^*} = 0$. Since $dF = \partial F + \overline{\partial} F$, this is also equivalent to

$$\partial F(p) \wedge \overline{\partial} F(p) \wedge \partial\overline{\partial} F(p) = 0 \ , \ \forall p \in M \ .$$

Example 1. If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is holomorphic and non constant then the analytic set defined by $M = (\mathcal{I}m(f) = 0)$ is Levi-flat. The leaves of the Levi foliation on M are the real levels of f.

For instance, if M is smooth a classical result of E. Cartan says that there exists a coordinate system $(z_1, ..., z_n)$ such that $M = (\mathcal{R}e(z_1) = 0)$. More recently Burns and Guong (cf. [B-G]) have proved an analogous result in the case where $F(z_1, ..., z_n) = \mathcal{R}e(z_1^2 + ... + z_n^2) + h.o.t.$ (see also corolary 1).

Definition 2. Let $\mathcal{F}$ and $M = F^{-1}(0)$ be germs at $(\mathbb{C}^n, 0)$ of a codimension one singular holomorphic foliation and of a real Levi-flat hypersurface, respectively. We say that $\mathcal{F}$ and M are tangent, if the leaves of the Levi foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

In this paper we will prove two results concerning the situation of definition 2. Our first result is the following :

Theorem 1. *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of holomorphic codimension one foliation tangent to a germ at $0 \in \mathbb{C}^n$ of real codimension one and irreducible analytic variety M. Then $\mathcal{F}$ has a non-constant meromorphic first integral.*

In the case of dimension two we can precise more :

(a). *If $\mathcal{F}$ is dicritical then it has a non-constant meromorphic first integral f/g, where $f, g \in \mathcal{O}_2$ and $f(0) = g(0) = 0$.*

(b). *If $\mathcal{F}$ is non-dicritical then it has a non-constant holomorphic first integral.*

Recall that a germ of foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is *dicritical* if it has infinitely many analytic separatrices through the origin. Otherwise it is called *non-dicritical*.

Remark 2. The definition of dicritical and non-dicritical codimension one foliations in dimension $n \geq 3$ is more involved than in dimension two. On the other hand, the proof of theorem 1 in dimension $n \geq 3$ will be done by reduction to the case of dimension two, taking 2-plane sections transverse to the foliation. We will see that if the foliation restricted to the 2-plane is non-dicritical then the first integral is in fact holomorphic.

Remark 3. Consider the foliation $\mathcal{F}$ defined by the differential equation $\omega := dP + Q.(x\,dy - y\,dx) = 0$, where $P, Q \in \mathbb{C}[x, y]$ are homogeneous polynomials of degree four. In section 5 we will see that it is possible to choose P and Q in such a way that $\mathcal{F}$ has no non-constant meromorphic first integral, but after one blowing-up at $0 \in \mathbb{C}^2$, say $\pi\colon \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$, the strict transform $\tilde{\mathcal{F}} := \pi^*(\mathcal{F})$ is tangent to a real analytic hypersurface $\tilde{M} \subset \tilde{\mathbb{C}}^2$. In this example the hypersurface $M := \pi(\tilde{M})$ is real analytic outside the origin, but has no analytic equation in a neighborhood of $0 \in \mathbb{C}^2$. The example was motivated by the way we have tried to prove theorem 1 in the non-dicritical case at the begining.

However, the example satisfies the following property : the foliation $\mathcal{F}$ has an affine transversal structure outside the set of separatrices. This can be expressed as follows : there exists a closed meromorphic 1-form $\eta = \frac{3}{2}\frac{dp}{p}$ such that $d\omega = \eta \wedge \omega$ (see [Sc]). In a future paper we hope to study and describe more precisely this type of situation.

The second result concerns the existence of a foliation tangent to the singular Levi-flat. We need a definition.

Definition 3. Let $M = F^{-1}(0)$ be a germ at $0 \in \mathbb{C}^n$ of real analytic Levi-flat hypersurface. We define the complexification $M_{\mathbb{C}}$ of M as $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$, where $F_{\mathbb{C}}$ is the complexification of F in $(\mathbb{C}^{2n}, 0)$, when we consider F as a real analytic germ of analytic function at $(\mathbb{R}^{2n}, 0) \simeq (\mathbb{C}^n, 0)$. We define the *algebraic dimension* of $sing(M)$ as the complex dimension of the singular set of $M_{\mathbb{C}}$. Let $\eta = i(\partial F - \overline{\partial} F)$ be the Levi 1-form of F. We will denote by $\eta_{\mathbb{C}}$ the complexification of η on $(\mathbb{C}^{2n}, 0)$.

In section 2.1 we will precise how we do these complexifications (see also example 2).

Remark 4. Let $M_{\mathbb{C}}^* = (M_{\mathbb{C}} \setminus (dF_{\mathbb{C}} = 0)$. The integrability condition of $\eta = i(\partial F - \overline{\partial} F)|_{M^*}$ implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable. Therefore, the differential equation $\eta_{\mathbb{C}} = 0$ defines a foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called the *complexification* of $\mathcal{L}$.

In a certain sense, the next result asserts that if the singularities of M are sufficiently small (in the algebraic sense) then M is given by the zeroes of the imaginary part of a holomorphic function.

Theorem 2. *Let $M = F^{-1}(0)$ be a germ of an irreducible analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi 1-form $\eta = i(\partial F - \overline{\partial} F)$. Assume that the algebraic dimension of $sing(M)$ is $\leq 2n - 4$. Then there exists a unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, if one of the following conditions is fulfilled :*

(a). *$n \geq 3$ and $cod_{M_{\mathbb{C}}^*}(sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 3$.*
(b). *$n \geq 2$, $cod_{M_{\mathbb{C}}^*}(sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral f such that $M = (\mathcal{I}m(f) = 0)$.

Remark 5. About the uniqueness of $\mathcal{F}_M$, we would like to observe that this is a general fact : if M is a real analytic Levi-flat and there is a holomorphic foliation $\mathcal{F}$ tangent to it, then it is the unique one. This follows from the following facts :

(a). Two holomorphic foliations in a connected open set that coincide in a non-empty open subset are equal. This follows from the definition of holomorphic foliation (cf. [LN-S]).
(b). Given $p \in M^*$ then by E. Cartan's theorem there exists a holomorphic coordinate system $z = (z_1, ..., z_n)$ such that $M = \mathcal{R}e(z_1)$. In this case the unique holomorphic extension of the Levi foliation to a neighborhood V of p is the foliation $\mathcal{G}$ whose leaves are the levels z_1 =constant. In particular, $\mathcal{F}_M|_V = \mathcal{G}$ and so it is the unique one tangent to M.

Remark 6. We would like to observe that the conclusion of theorem 2 is also true for $n = 2$ and $cod_{M_{\mathbb{C}}^*}(sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 3$. However, in this case the conclusion would be that M is smooth. In fact, a non-smooth real analytic Levi-flat hypersurface in $(\mathbb{C}^2, 0)$ defined by $(\mathcal{I}m(f) = 0)$ never satisfies the condition $cod_{M_{\mathbb{C}}^*}(sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 3$. This can be seen clearly in the next example.

Example 2. Let M be the real hypersurface of $\mathbb{C}^2$ given by $F(x,y) = \mathcal{R}e(x^2+y^2) = 0$. Since $F(x,y) = \frac{1}{2}(x^2+y^2+\overline{x}^2+\overline{y}^2)$ its complexification is

$$F_{\mathbb{C}}(x,y,z,w) = \frac{1}{2}(x^2+y^2+z^2+w^2) \implies dF_{\mathbb{C}} = \alpha+\beta \ ,$$

where $\alpha = x\,dx + y\,dy$ and $\beta = z\,dz + w\,dw$. The complexification of the Levi 1-form in this case is $\eta_{\mathbb{C}} = i(\alpha-\beta)$. Since $dF_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 0$, we get

$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 2i\,\alpha|_{M^*_{\mathbb{C}}} = -2i\,\beta|_{M^*_{\mathbb{C}}} \ .$$

From the above relations it can be easily proved that $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = X_1 \cup X_2$, where $X_1 = M_{\mathbb{C}} \cap (x=y=0)$ and $X_2 = M_{\mathbb{C}} \cap (z=w=0)$, so that $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = 2$.

A consequence of theorem 2 is the following nice result due to Burns and Gong (cf. [B-G]) :

Corollary 1. *Let $M = F^{-1}(0)$, where $F\colon (\mathbb{C}^n,0) \to (\mathbb{R},0)$, $n \geq 2$, is a germ of real analytic function such that*

(a). $F(z_1,...,z_n) = \mathcal{R}e(z_1^2+...+z_n^2) + h.o.t.$.
(b). $F^{-1}(0)$ *is Levi-flat.*

Then there exists a germ of biholomorphism $\phi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $\phi(M) = (\mathcal{R}e(x_1^2+...+x_n^2) = 0)$.

In the next example, we will see that there are germs of real analytic Levi-flat hypersurfaces which are not tangent to foliations, even in the case $n=2$. As we will see, these examples are tangent to holomorphic webs.

Example 3. Let $f_0, f_1, ..., f_k \in \mathcal{O}_n$, $n \geq 2$, be irreducible germs of holomorphic functions, where $k \geq 2$. Consider the family of hypersurfaces

$$\mathcal{S} := \{G_s := f_0 + s.f_1 + ... + s^k.f_k \mid s \in \mathbb{R}\} \ .$$

By eliminating the real variable s in the system $G_s = \overline{G}_s = 0$, we obtain a real analytic germ $F\colon (\mathbb{C}^n,0) \to (\mathbb{R},0)$ such that any complex hypersurface $(G_s = 0)$ is contained in the real hypersurface $(F=0)$. For instance, in the case $k=2$, we obtain

$$F = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ \overline{f}_0 & \overline{f}_1 & \overline{f}_2 & 0 \\ 0 & \overline{f}_0 & \overline{f}_1 & \overline{f}_2 \end{pmatrix} = \tag{1}$$

$$= f_0^2.\overline{f}_2^2 + \overline{f}_0^2.f_2^2 + f_0.f_2.\overline{f}_1^2 + \overline{f}_0.\overline{f}_2.f_1^2 - |f_1|^2.(f_0.\overline{f}_2 + \overline{f}_0.f_2) - |f_0|^2.|f_2|^2 \ .$$

which comes from the elimination of s in the system

$$f_0 + s.f_1 + s^2.f_2 = \overline{f}_0 + s.\overline{f}_1 + s^2.\overline{f}_2 = 0 \ .$$

We would like to observe that the examples of this type are tangent to singular webs. The web is obtained by the elimination of s in the system given by

$$\begin{cases} f_0 + s.f_1 + s^2.f_2 + ... + s^k.f_k = 0 \\ df_0 + s.df_1 + s^2.df_2 + ... + s^k.df_k = 0 \end{cases}$$

In the case of (1) we get a 2-web given by the implicit differential equation $\Omega = 0$, where

$$\Omega = det \begin{pmatrix} df_0 & df_1 & df_2 & 0 \\ 0 & df_0 & df_1 & df_2 \\ f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \end{pmatrix}$$

This type of example shows that, although $\mathcal{L}$ is a foliation on $M^* \subset M = (F = 0)$, in general it is not tangent to a germ of holomorphic foliation at $(\mathbb{C}^n, 0)$.

In fact, M. Brunella in [B] has proved that in the general situation a germ of real analytic hypersurface is "almost" like that. He proves that there exist a complex manifold Y together with a codimension one divisor D, a real levi-flat singular analytic hypersusface $N \subset Y$, an open subset $N_0 \subset N$, a codimension one singular foliation $\mathcal{F}$ on Y tangent to N and a holomorphic map $\pi \colon (Y, D) \to (\mathbb{C}^n, 0)$ such that

(a). $\pi|_{N_0} \colon N_0 \to M^*$ is an isomorphism.
(b). $\pi|_{\overline{N_0}} \colon \overline{N_0} \to \overline{M^*}$ is a proper map.

In particular, the Levi foliation $\mathcal{L}$ on M^* satisfies $\pi^*(\mathcal{L}) = \mathcal{F}|_{N_0}$, but in general there is no germ of foliation $\mathcal{G}$ at $0 \in \mathbb{C}^n$ such that $\pi^*(\mathcal{G}) = \mathcal{F}$, whereas sometimes there are webs as in example 3.

Let us state a problem motivated by our results.

Problem. Let M be a real analytic germ of a Levi-flat hypersurface at $0 \in \mathbb{C}^n$. Assume that there exists a singular codimension one k-web, $k \geq 2$, such that any leaf of the Levi foliation $\mathcal{L}$ on M^* is also a leaf of the web. Does the web has a non-constant meromorphic first integral as in example 3 ?

By a meromorphic first integral we mean something like $f_0(x) + z.f_1(x) + ... + z^k.f_k(x) = 0$, where $f_0, ..., f_k \in \mathcal{O}_n$.

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2. Preliminaries.

In the proof of theorem 1 we will use Seidenberg's reduction theorem (cf. [Se]). According to this theorem, after a finite number of blowing-ups, we obtain a foliation such that all its singularities are reduced in the sense of [S] and [M-M]. Section 2.2 will be devoted to prove that a reduced foliation in dimension two, tangent to a real analytic hypersurface, has a holomorphic first integral. In section 2.3 we will see some consequences of this fact for the Seidenberg's reduction of a foliation tangent to a real analytic hypersurface. On the other hand, the proof of theorem 2 will be based in the complexification of a Levi-flat real hypersurface and of the Levi foliation. This topic and some basic results, will be discussed in section 2.1.

Let us fix some notations that will be used from now on.

(a). $\mathcal{O}_n$: the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. $\mathcal{O}(U)$ = set of holomorphic functions in the open set $U \subset \mathbb{C}^n$.
(b). $\mathcal{O}_n^* = \{f \in \mathcal{O}_n \,|\, f(0) \neq 0\}$. $\mathcal{O}^*(U) = \{f \in \mathcal{O}(u) \,|\, f(z) \neq 0 \ , \ \forall\, z \in U\}$.

(c). $\mathcal{A}_n$: the ring of germs at $0 \in \mathbb{C}^n$ of complex valued real analytic functions.
(d). $\mathcal{A}_{n\,\mathbb{R}}$: the ring of germs at $0 \in \mathbb{C}^n$ of real valued real analytic functions. Note that $\mathcal{A}_{n\,\mathbb{R}} \subset \mathcal{A}_n$ and $F \in \mathcal{A}_n$ is in $\mathcal{A}_{n\,\mathbb{R}}$ if, and only if, $\overline{F} = F$.
(e). $Diff(\mathbb{C}^n, 0)$: the group of germs at $0 \in \mathbb{C}^n$ of holomorphic diffeomorphisms $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with the operation of composition.
(f). $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

2.1. **The complexification.** This section will be devoted to state some properties of the complexification of a Levi-flat germ.

Given $G \in \mathcal{A}_n$ we can write its Taylor series at 0 as

$$G(z) = \sum_{\mu\,,\,\nu} G_{\mu\,\nu}\, z^{\mu}\, \overline{z}^{\nu} \ , \tag{2}$$

where $G_{\mu\,\nu} \in \mathbb{C}$, $\mu = (\mu_1, ..., \mu_n)$, $\nu = (\nu_1, ..., \nu_n)$, $z^{\mu} = z_1^{\mu_1}...z_n^{\mu_n}$ and $\overline{z}^{\nu} = \overline{z}_1^{\nu_1}...\overline{z}_n^{\nu_n}$. When $G \in \mathcal{A}_{n\,\mathbb{R}}$ then the coefficients $G_{\mu\,\nu}$ satisfy

$$\overline{G}_{\mu\,\nu} = G_{\nu\,\mu} \ .$$

The complexification $G_{\mathbb{C}} \in \mathcal{O}_{2n}$ of G is defined by the series

$$G_{\mathbb{C}}(z, w) = \sum_{\mu\,,\,\nu} G_{\mu\,\nu}\, z^{\mu}\, w^{\nu} \ . \tag{3}$$

If the series in (2) converges in polydisk $D_r^n = \{z \in \mathbb{C}^n \mid |z_j| < r\}$ then the series in (3) converges in the polydisk D_r^{2n}. Moreover, $G(z) = G_{\mathbb{C}}(z, \overline{z})$ for all $z \in D_r^n$.

Remark 7. The complexification does not depends on the coordinate system, in the sense that if $\varphi \in Diff(\mathbb{C}^n, 0)$ then there exists an unique $\varphi_{\mathbb{C}} \in Diff(\mathbb{C}^{2n}, 0)$ such that

$$(G \circ \varphi)_{\mathbb{C}} = G_{\mathbb{C}} \circ \varphi_{\mathbb{C}} \tag{4}$$

In fact, if $\varphi(x) = \sum_{\sigma} \varphi_{\sigma} x^{\sigma}$ is the Taylor series of φ and $\varphi_{\mathbb{C}}(x, y) = (\varphi(x), \overline{\varphi}(y))$, where $\overline{\varphi}(y) = \sum_{\sigma} \overline{\varphi_{\sigma}} y^{\sigma}$ then relation (4) is satisfied for all $G \in \mathcal{A}_n$.

Let $F \in \mathcal{A}_{n\,\mathbb{R}}$, $F(0) = 0$, be irreducible and such that $M = F^{-1}(0)$ is a Levi-flat. If the Taylor series of F is

$$F(z) = \sum_{\mu\,,\,\nu} F_{\mu\,\nu}\, z^{\mu}\, \overline{z}^{\nu}$$

then the complexification $\eta_{\mathbb{C}}$ of its Levi 1-form $\eta = i(\partial F - \overline{\partial} F)$ can be written as

$$\eta_{\mathbb{C}} = i \sum_{j=1}^{n} \left(\frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j - \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j \right) = i \sum_{\mu\,,\,\nu} \left(F_{\mu\,\nu}\, w^{\nu}\, d(z^{\mu}) - F_{\mu\,\nu}\, z^{\mu}\, d(w^{\nu}) \right) \ . \tag{5}$$

The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$. By remark 7, $M_{\mathbb{C}}$ does not depends on the coordinate system. Its smooth part will be denoted by $M_{\mathbb{C}}^* = M_{\mathbb{C}} \setminus (dF_{\mathbb{C}} = 0)$. As we have already remarked, $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable and defines a codimension one foliation $\mathcal{L}_{\mathbb{C}}$ in $M_{\mathbb{C}}^*$.

We will assume that the Taylor series of F converges in the polydisk D_r^n. The first result of this section is the following :

Lemma 2.1. *Let F, M, M^* and $F_{\mathbb{C}}$ be as above. Then for any $z_o \in M^*$ the leaf L_{z_o} of $\mathcal{L}$ through z_o is contained in the hypersurface $\{z \in D_r^n \mid F_{\mathbb{C}}(z, \overline{z}_o) = 0\}$. In particular, L_{z_o} is closed in $(\mathbb{C}^n, 0)$.*

Proof. Fix $z_o \in M^*$ and define $f(z) := F_{\mathbb{C}}(z, \overline{z}_o)$. Note that $f(z_o) = 0$ and $f^{-1}(0)$ is a complex sub-variety of D^n_r. Let S_{z_o} be the irreducible component of $f^{-1}(0) \subset D^n_r$ which contains z_o. We assert that $L_{z_o} \subset S_{z_o} \subset M$ and $dim_{\mathbb{C}}(L_{z_o}) = dim_{\mathbb{C}}(S_{z_o}) = n-1$.

In fact, since z_o is a smooth point of M, by E. Cartan's theorem there exists a local coordinate system $(W, \psi = (x_1, ..., x_n) \in \mathbb{C}^n)$ such that $\psi(z_o) = 0$, $\psi(W) = \mathbb{D}^n$ and $M^* \cap W = (\mathcal{R}e(x_n) = 0)$ (cf. [Ce-S]). In this coordinate system the foliation $\mathcal{L}$ is defined by $dx_n|_{M^* \cap W} = 0$. In particular, $(x_n = 0) \subset L_{z_o}$.

On the other hand, since $dF(z_o) \neq 0$ and $d\,\mathcal{R}e(x_n) \neq 0$, there exists a real function G on W such that $G(p) \neq 0$ for all $p \in M^* \cap W$ and $F \circ \psi^{-1}(x) = G(x).\mathcal{R}e(x_n)$. From this relation we get

$$2\,G(x)\,(x_n + \overline{x}_n) = F_{\mathbb{C}}(\psi^{-1}(x), \overline{\psi^{-1}(x)}) \ .$$

If we set $H_{\mathbb{C}}(x, y) = F_{\mathbb{C}}(\psi^{-1}(x), \overline{\psi^{-1}(\overline{y})})$ then, by complexification we get $2\,G_{\mathbb{C}}(x, y)\,(x_n + y_n) = H_{\mathbb{C}}(x, y)$, where $G_{\mathbb{C}}$ is the complexification of G. Since $\psi^{-1}(0) = z_o$ we get

$$f \circ \psi^{-1}(x) = H_{\mathbb{C}}(x, 0) = 2\,G_{\mathbb{C}}(x, 0).x_n \implies$$

$$f^{-1}(0) \cap W = S_{z_o} \cap W = (x_n = 0) \subset L_{z_o} \subset M \implies S_{z_o} \subset M \ ,$$

because S_{z_o} and M are analytic. This proves also that S_{z_o} is a complex hypersurface of D^n_r. In particular, $dim_{\mathbb{C}}(S_{z_o}) = dim_{\mathbb{C}}(L_{z_o}) = n-1$ and this implies that $L_{z_o} \subset S_{z_o}$, because L_{z_o} is connected. Hence, $L_{z_o} = S_{z_o} \cap M^*$ and L_{z_o} is closed in M^*. □

Remark 8. In lemma 4.3 of section 4.2 we will prove that, for $n = 2$, all leaves of the complexification $\mathcal{L}_{\mathbb{C}}$ are closed in $M^*_{\mathbb{C}} \setminus sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$. This fact will be used in the proof of corollary 1. We would like to remark that the proof of lemma 4.3 can be adapted to the general case $n \geq 2$.

Another fact that will be used is the following :

Lemma 2.2. *Let $h \in \mathcal{O}_n$, $h \neq 0$, $h(0) = 0$. Suppose that h is not a power in $\mathcal{O}_n$. Then $\mathcal{I}m(h)$ and $\mathcal{R}e(f)$ are irreducible in $\mathcal{A}_{n\,\mathbb{R}}$.*

Proof. We will prove that $\mathcal{I}m(f)$ is irreducible. Since $\mathcal{R}e(f) = \mathcal{I}m(i.f)$ we will get also that $\mathcal{R}e(f)$ is irreducible. Let $h(z) = \sum_\mu h_\mu\, z^\mu$ be the Taylor series of h and $\overline{h}(w) := \sum_\mu \overline{h_\mu}\, w^\mu$, $w \in (\mathbb{C}^n, 0)$. Note that $\mathcal{I}m(h) = \frac{i}{2}(h(z) - \overline{h}(\overline{z}))$. Therefore, the complexification $H_{\mathbb{C}}$ of $H := \frac{2}{i}\mathcal{I}m(h)$ can be written as

$$H_{\mathbb{C}}(z, w) = h(z) - \overline{h}(w) \ .$$

Suppose by contradiction that $\mathcal{I}m(h)$ is reducible in $\mathcal{A}_{n\,\mathbb{R}}$. In this case, we can write $H(z) = \phi(z).\psi(z)$, where $\phi, \psi \in \mathcal{A}_{n\,\mathbb{R}}$ and $\phi(0) = \psi(0) = 0$. Let $\phi_{\mathbb{C}}$ and $\psi_{\mathbb{C}}$ be the complexifications of ϕ and ψ, respectively. Since $H = \phi.\psi$, by complexification of the Taylor series in both members, we get $H_{\mathbb{C}}(z, w) = \phi_{\mathbb{C}}(z, w).\psi_{\mathbb{C}}(z, w)$, so that $H_{\mathbb{C}}$ is reducible.

Now, since the germ h is not a power, there exist $\epsilon > 0$ and a representative of h, denoted by the same letter, holomorphic in a polydisk D^n_r around 0, such that the fiber $h^{-1}(c)$ is connected for all c with $0 < |c| < \epsilon$ (cf. [M-M]). This implies that, if $w_o \in D^n_r$ is such that $0 < |\overline{h}(w_o)| < \epsilon$ then

$$H_{w_o}(z) := H_{\mathbb{C}}(z, w_o) = h(z) - \overline{h}(w_o)$$

is irreducible in D_r^n. Let $0<\delta\le r$ be such that $|\overline{h}(w)|<\epsilon$ if $|w|<\delta$ and set $V:=D_\delta^n\setminus \overline{h}^{-1}(0)$. We have concluded that if $w\in V$ then $H_{w_o}(z)$ is irreducible in D_δ^n.

For a fixed $w_o\in D_\delta^n$ set $\phi_{w_o}(z)=\phi_{\mathbb{C}}(z,w_o)$ and $\psi_{w_o}(z)=\psi_{\mathbb{C}}(z,w_o)$. From $H_{\mathbb{C}}=\phi_{\mathbb{C}}.\psi_{\mathbb{C}}$ we get

$$H_w=\phi_w.\psi_w$$

and this implies that for all $w\in V$, either $\phi_w\in\mathcal{O}^*(D_\delta^n)$, or $\psi_w\in\mathcal{O}^*(D_\delta^n)$. Since 0 is in the closure of the open set V, we can assume that there exists a sequence $(w_n)_{n\ge1}$ in V such that $\lim\limits_{n\to\infty} w_n=0$ and $\psi_{w_n}\in\mathcal{O}^*(D_\delta^n)$, for instance. Since $\psi_{w_n}\underset{n}{\to}\psi_0$ in compact sets, this is possible only if $\psi_0\equiv0$, because $\psi_0(0)=\psi_{\mathbb{C}}(0,0)=0$.

On the other hand, the complexification $\psi_{\mathbb{C}}$ of $\psi\in\mathcal{A}_{n\,\mathbb{R}}$ satisfies $\psi_{\mathbb{C}}(z,w)=\overline{\psi_{\mathbb{C}}(\overline{w},\overline{z})}$, and so

$$\psi_0\equiv0\implies\overline{\psi_{\mathbb{C}}(0,\overline{z})}=\psi_{\mathbb{C}}(z,0)=0\ ,\ \forall z\implies\psi_z(0)=0\ ,\ \forall z\ ,$$

a contradiction with $\psi_{w_n}(0)\neq0$. This contradiction proves that $H_{\mathbb{C}}$ is irreducible and the lemma. □

Another result that will be used is the following :

Lemma 2.3. *If F is irreducible in $\mathcal{A}_{n\mathbb{R}}$ and $M=F^{-1}(0)$ has real codimension one then $F_{\mathbb{C}}$ is irreducible in $\mathcal{O}_{2n}$.*

Proof. Write the decomposition into irreducible factors of $F_{\mathbb{C}}=g_1^{k_1}...g_r^{k_r}$, $g_j\in\mathcal{O}_{2n}$, $k_j\ge1$, $g_j(0)=0$, $1\le j\le r$. This implies $F=G_1^{k_1}...G_r^{k_r}$, where $G_j(z)=g_j(z,\overline{z})$ and $G_j\in\mathcal{A}_n$, $1\le j\le r$. Note that $g_j=G_{j\mathbb{C}}$, $1\le j\le r$. Since $F=\overline{F}$ we get

$$F=G_1^{k_1}...G_r^{k_r}=\overline{G_1}^{k_1}...\overline{G_r}^{k_r}\implies G_{1\mathbb{C}}^{k_1}...G_{r\mathbb{C}}^{k_r}=(\overline{G_1})_{\mathbb{C}}^{k_1}...(\overline{G_r})_{\mathbb{C}}^{k_r}\ .$$

This implies that $r=2a+b$ and, after reordering the indexes, we can assume that there exist units $U_1,...,U_a,V_1,...,V_b\in\mathcal{O}_{2n}$ such that $G_{2j-1\mathbb{C}}=U_j.(\overline{G_{2j}})_{\mathbb{C}}$, $k_{2j-1}=k_{2j}$, if $1\le j\le a$ and $G_{2a+j\mathbb{C}}=V_j.(\overline{G_{2a+j}})_{\mathbb{C}}$, if $1\le j\le b$. In particular, we can write $F=G.H$, where

$$G=(G_1.\overline{G_1})^{k_1}...(G_a.\overline{G_a})^{k_a}\ ,\ H=U.G_{2a+1}^{k_{2a+1}}...G_r^{k_r}\ ,$$

and U is an unit in $\mathcal{A}_n$. Note that $H\in\mathcal{A}_{n\mathbb{R}}$ because $F,G\in\mathcal{A}_{n\mathbb{R}}$. Since F is irreducible in $\mathcal{A}_{n\mathbb{R}}$ we have two possibilities :

1^{st}. $a=1$, $k_1=1$, $b=0$ and $H(0)\neq0$. This implies $F=H.|G_1|^2=H.(\mathcal{R}e(G_1)^2+\mathcal{I}m(G_1)^2)$, and so $M=(\mathcal{R}e(G_1)=\mathcal{I}m(G_1)=0)$. But, this would imply that M has real codimension ≥2, a contradiction.

2^{nd}. $a=0$, $b=r$ and $(\overline{G_j})_{\mathbb{C}}=V_j.G_{j\mathbb{C}}$ for all $j=1,...,r$. In this case, we get $\overline{G_j}=U_j.G_j$, where $U_j(z)=V_j(z,\overline{z})$ is an unit. This implies that $|U_j|=1$, and so $U_j=e^{i\alpha_j}$, $\alpha_j\in\mathcal{A}_{n\mathbb{R}}$. If we set $h_j:=e^{i\alpha_j/2}.G_j$, then we get $\overline{h}_j=h_j$ and $h_j\in\mathcal{A}_{n\mathbb{R}}$. Therefore, we can write $F=U.h_1^{k_1}...h_b^{k_b}$, where $U\in\mathcal{A}_{n\mathbb{R}}$, $U(0)\neq0$. Since F is irreducible in $\mathcal{A}_{n\mathbb{R}}$ we get $b=1$ and $k_1=1$. Hence, $F_{\mathbb{C}}$ is irreducible in $\mathcal{O}_{2n}$. □

Now, assume that $M=F^{-1}(0)\subset(\mathbb{C}^2,0)$ is a Levi-flat hypersurface and there exists a germ of holomorphic foliation $\mathcal{F}$ tangent to M and defined by a holomorphic vector field $X=P\,\partial_x+Q\,\partial_y$, with an isolated singularity at $0\in\mathbb{C}^2$, where $P,Q\in\mathcal{O}_2$. Define

$$X(F):=P\,\frac{\partial F}{\partial x}+Q\,\frac{\partial F}{\partial y}\ .$$

Remark 9. $X(F)|_{M^*} = 0$. In fact, since $\mathcal{F}$ is tangent to M, for all $p \in M^*$ we have $X(p) \in ker(\partial F(p))$. This is equivalent to $X(F)(p) = 0$. As a consequence of the next lemma we will get $F \,|\, X(F)$.

Lemma 2.4. *Let $G \in \mathcal{A}_2$ be such that $G|_{M^*} \equiv 0$. Then $F \mid G$, that is there exists $H \in \mathcal{A}_2$ such that $G = H.F$. In particular, $F \mid X(F)$.*

Proof. Let $F_{\mathbb{C}}$ and $G_{\mathbb{C}}$ be the complexifications of F and G, respectively, and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^4, 0)$. It follows from lemma 2.3 that $F_{\mathbb{C}}$ is irreducible in $\mathcal{O}_4$. We assert that $G_{\mathbb{C}}|_{M_{\mathbb{C}}} \equiv 0$. In fact, fix $q_o = (x_o, y_o) \in M^*$. Since $dF(q_o) \neq 0$ and $G|_{M^*} \equiv 0$ there exists a neighborhood V of q_o in $\mathbb{C}^2$ such that $\tilde{H} := (G/F)|_V$ is real analytic in V. Let $\tilde{H}_{\mathbb{C}}$ be the complexification of $\tilde{H}$, which is holomorphic in some neighborhood W of $(x_o, y_o, \overline{x}_o, \overline{y}_o) \in \mathbb{C}^4$.

It follows from $G = \tilde{H}.F$ that $G_{\mathbb{C}} = \tilde{H}_{\mathbb{C}}.F_{\mathbb{C}}$. Therefore, $G_{\mathbb{C}}|_{W \cap M_{\mathbb{C}}} \equiv 0$. Since $M_{\mathbb{C}}$ is irreducible, this implies that $G_{\mathbb{C}}|_{M_{\mathbb{C}}} \equiv 0$.

Finally, $G_{\mathbb{C}}|_{M_{\mathbb{C}}} \equiv 0$ implies that there exists $H_{\mathbb{C}} \in \mathcal{O}_4$ such that $G_{\mathbb{C}} = H_{\mathbb{C}}.F_{\mathbb{C}}$. Hence, $G = H.F$, where $H(x,y) = H_{\mathbb{C}}(x, y, \overline{x}, \overline{y})$. ☐

2.2. **The case of reduced singularities in dimension two.** Let M and $\mathcal{F}$ be germs at $(\mathbb{C}^2, 0)$ of a real analytic Levi-flat hypersurface and of a holomorphic foliation, respectively, where $\mathcal{F}$ is tangent to M. We will assume that :

(I). $\mathcal{F}$ is defined by a germ at $0 \in \mathbb{C}^2$ of holomorphic vector field X with an isolated singularity at 0.

(II). M is irreducible and defined by $(F = 0)$, where $F \in \mathcal{A}_{2\mathbb{R}}$ is irreducible.

Let us assume that 0 is a reduced singularity of X, in the sense of Seidenberg. Denote the eigenvalues of $DX(0)$ by λ_1, λ_2. We have two possibilities :

(a). $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$. In this case, X has exactly two analytic separatrices through p, both smooth. It can be written in a suitable coordinate system (u,v), as

$$X = \lambda_1.u(1 + R_1(u,v))\partial_u + \lambda_2.v(1 + R_2(u,v))\partial_v \ , \tag{6}$$

where $R_1(0,0) = R_2(0,0) = 0$. The separatrices are $S_1 := (v = 0)$ and $S_2 := (u = 0)$.

(b). $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case, X has a saddle-node at p. We will suppose, without loss of generality, that $\lambda_1 = 1$. It can be written in a suitable coordinate system (u,v), as

$$X = u^{m+1}\,\partial_u + [v(1 + \lambda.u^m) + h.o.t.]\partial_v \ , \tag{7}$$

where $\lambda \in \mathbb{C}$, $m \geq 1$ (cf. [M-R-1]). In this case, X has one or two analytic separatrices through the origin.

Lemma 2.5. *Suppose that X has a reduced singularity at $0 \in \mathbb{C}^2$ and is tangent to $M = F^{-1}(0)$. Then $\lambda_1, \lambda_2 \neq 0$, $\lambda_2/\lambda_1 \in \mathbb{Q}_-$ and X has a holomorphic first integral. In particular, in a suitable coordinate system (x,y) around $0 \in \mathbb{C}^2$, $X = \phi.Y$, where $\phi(0) \neq 0$ and*

$$Y = q.x\,\partial_x - p.y\,\partial_y \ , \ gcd(p,q) = 1 \ .$$

In this coordinate system, $f(x,y) := x^p.y^q$ is a first integral of X.

Proof. In a certain sense, for the dynamical systems experts, the lemma is almost immediate : lemma 2.1 implies that $\mathcal{F}$ has a non-enumerable number of closed leaves (those contained in M^*). However, there are some difficulties when λ_1, λ_2 are resonant or in the presence of small divisors. So, we will give a detailed proof.

Assume first that X is like in (6). Let X and M have representatives in a polydisc $Q := (|u|, |v| < \epsilon)$ and denote by $\mathcal{F}$ the foliation defined by X. Suppose further that all leaves of X in M^* are closed (lemma 2.1). We denote the leaf of $\mathcal{F}$ through $q \in Q \setminus \{0\}$ by L_q. The foliation $\mathcal{F}$ has two analytic separatrices through 0, $(u = 0)$ and $(v = 0)$. The union of the separatrices of $\mathcal{F}$ will be denoted by $Sep(\mathcal{F})$. The first remark is that $M \supset Sep(\mathcal{F})$.

In fact, let $(q_n)_{n\geq 1}$ be a sequence in $Q \setminus Sep(\mathcal{F})$ such that $\lim\limits_{n\to\infty} q_n = 0$. It is well known that $\overline{\cup_n L_{q_n}} \supset Sep(\mathcal{F})$. Since M is closed, $0 \in M$ and $dim_{\mathbb{R}}(M) = 3 > dim_{\mathbb{R}}(Sep(\mathcal{F})) = 2$, there exists a sequence $(q_n)_{n\geq 1}$ in $M \setminus Sep(\mathcal{F})$ such that $\lim\limits_{n\to\infty} q_n = 0$. The fact that X is tangent to M implies that $L_{q_n} \subset M$ for all $n \geq 1$. Hence, $M \supset Sep(\mathcal{F})$.

Let $\Sigma := \{(u, c)\,|\,|u| < \epsilon\}$, $0 < |c| < \epsilon$. Note that Σ is transverse to X in a neighborhood of $(0, c)$. We will consider Σ parametrized by $u \mapsto (u, c)$. Let $h \in Diff(\Sigma, 0)$ be the holonomy map of the separatrix $S_1 := (u = 0)$, relative to a closed path in S_1 going around $0 \in S_1$ once. Denote by $U \subset \Sigma$ the domain of h. Given $q \in M \cap U$ we have $h(q) \in M \cap \Sigma$, because $h(q) \in L_q$ and $L_q \subset M$. In particular, $h(M \cap U) \subset M \cap \Sigma$. Note that $M \cap \Sigma \neq \emptyset$ is a real analytic curve.

Claim 2.1. *If* $\lambda_1, \lambda_2 \neq 0$ *then* $\lambda_2/\lambda_1 \in \mathbb{Q}_-$.

Proof. In fact, in this case, $h'(0) = \mu$, $\mu = e^{2\pi i \lambda_1/\lambda_2}$, and we can write $h(u) = \mu.u + h.o.t.$. Denote by Γ the reduced germ at $0 \in \Sigma$ of the real analytic curve $M \cap \Sigma$. The germ Γ has a finite number of irreducible components, say $\Gamma_1, ..., \Gamma_\ell$. We need a definition.

Definition 4. Let $\overline{\gamma} \subset (\mathbb{C}, 0)$ be a germ of real analytic irreducible curve. We define the *tangent cone*, $C(\overline{\gamma})$, of $\overline{\gamma}$ as follows : since $\overline{\gamma}$ is irreducible it admits a Puiseux's parametrization $\gamma\colon (\mathbb{R}, 0) \to (\mathbb{C}, 0)$, $\gamma(t) = t^m.u(t)$, $m \geq 1$, $u(0) \in \mathbb{C}^*$. We set

$$C(\overline{\gamma}) = \frac{u(0)}{|u(0)|}.\mathbb{R} = \left\{ s.\frac{u(0)}{|u(0)|} \;|\; s \in \mathbb{R} \right\} .$$

As the reader can check, $C(\overline{\gamma})$ does not depends on the Puiseux's parametrization. Moreover, if $f\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ of biholomorphism then

$$C(f(\overline{\gamma})) = f'(0).C(\overline{\gamma})) = \frac{f'(0)}{|f'(0)|}.C(\overline{\gamma}) .$$

Let $\mathcal{D} = \{C(\Gamma_1), ..., C(\Gamma_\ell)\}$. Since $h(M \cap U) \subset M \cap \Sigma$, for all $j \in \{1, ..., \ell\}$ there exists an unique $i(j)$ such that $h(\Gamma_j) = \Gamma_{i(j)}$. In particular, $C(\Gamma_{i(j)}) = C(h(\Gamma_j)) = \frac{\mu}{|\mu|}.C(\Gamma_{i(j)})$, so that

$$\frac{\mu}{|\mu|}\mathcal{D} = \mathcal{D} \implies \mathcal{D} = \bigcup_{m\in\mathbb{Z}} \frac{\mu^m}{|\mu^m|}.\mathcal{D}$$

This implies that $\frac{\mu}{|\mu|}$ is a root of unity, because otherwise $\mathcal{D}$ would be infinite. Hence, $\mathcal{R}e(\lambda_1/\lambda_2) \in \mathbb{Q}$. Let us prove that $\mathcal{I}m(\lambda_1/\lambda_2) = 0$. Set $\lambda_1/\lambda_2 = a + i\,b$, $a \in \mathbb{Q}$.

For each $t \in [0, +\infty)$, set $\Sigma_t = \{(u, e^{-t}.c) \,|\, u \in \mathbb{C}\}$. Then Σ_t is transverse to $\mathcal{F}$ in a neighborhood of $p_t := (0, e^{-t}.c)$ and $\Sigma_0 = \Sigma$. Let $h_t \colon (\Sigma, p_0) \to (\Sigma_t, p_t)$ be the holonomy transformation of $\mathcal{F}$ relative to the open curve $t \mapsto p_t$. It can be shown that

$$h_t'(0) := \mu_t = e^{-(a+ib)t} \ .$$

Since $h_t(\Sigma_1 \cap M) \subset \Sigma_t \cap M$, h_t must send each branch Γ_j of $\Sigma \cap M$ to a branch Γ_{jt} of $\Sigma_t \cap M$. The tangent cone of Γ_{jt} at p_t is given by $C(\Gamma_{jt}) = \frac{\mu_t}{|\mu_t|}.C(\Gamma_j) = e^{-ibt}.C(\Gamma_j)$.

Since M is real analytic, there exists the limit $\lim\limits_{t \to +\infty} C(\Gamma_{jt})$. Hence, $b = 0$ and $\lambda_1/\lambda_2 \in \mathbb{Q}$. This implies that $\lambda_1/\lambda_2 \in \mathbb{Q}_-$, because the singularity is reduced. □

Claim 2.2. *The foliation $\mathcal{F}$ is linearizable.*

Proof. Let us assume first that $\lambda_1, \lambda_2 \neq 0$, so that $\lambda_2/\lambda_1 = -p/q$, $p, q \in \mathbb{N}$, by claim 2.1. After multiplying X by a constant, we can suppose that its linear part at 0 is $S = q\,u\,\partial_u - p\,v\,\partial_v$ and that the separatrices of X through 0 are $(u = 0)$ and $(v = 0)$.

In this case, the holonomy transformation $h \colon (\Sigma, 0) \to (\Sigma, 0)$, $\Sigma = (v = c)$, $c \neq 0$, is of the form

$$h(u) = \zeta.u + h.o.t. \ , \ \zeta = e^{2i\pi q/p} \quad \Longrightarrow \quad h^p(u) = u + h.o.t. \ .$$

It is well known that the germ of foliation $\mathcal{F}$ is linearizable if, and only if, h is linearizable (cf. [M-M] or [L]). Suppose by contradiction that h is non-linearizable. In this case, $h^p \neq id$, so that $h^p(u) = u(1 + a.u^k) + o(u^{k+2})$, where $a \neq 0$. For a germ in $Diff(\mathbb{C}, 0)$ tangent, but not equal, to the identity, it is known that all the pseudo-orbits near the origin accumulate in the origin. This would imply that all leaves near the separatrix $(u = 0)$ accumulate at $(u = 0)$, and so they cannot be closed.

Since the eigenvalues of X are not in the Poincaré domain ($\lambda_2/\lambda_1 < 0$), the saturation by $\mathcal{F}$ of a small neighborhood V of $(0, c)$ in Σ, $sat_{\mathcal{F}}(V)$, is such that $sat_{\mathcal{F}}(V) \cup (v = 0)$ contains a neigborhood W of the origin. This implies that $\mathcal{F}$ has only two closed leaves near the origin : $(u = 0)$ and $(v = 0)$.

On the other hand, lemma 2.1 implies that all leaves of $\mathcal{F}$ contained in M^* are closed, a contradiction, because the origin is in the closure of M^*. Therefore, $\mathcal{F}$ is linearizable.

It remains to consider the case of a saddle-node. We will prove that in this case there is no germ of real analytic variety M, with $dim_{\mathbb{R}}(M) = 3$, such that X is tangent to M. We can be reduce this proof to the previous case by doing a blowing-up at $0 \in \mathbb{C}^2$. In fact, if we write X as in (7) and consider the blowing-up $\pi(t, v) = (t.v, v)$, then

$$Z := \pi^*(X) = t\,(1 + (\lambda - 1)(tu)^m + h.o.t)\,\partial_t - v\,(1 + \lambda(tu)^m + h.o.t)\,\partial_v \ .$$

It follows that $DZ(0)$ has eigenvalues 1 an -1. On the other hand, it is known that the holonomy of the separatrix $(t = 0)$ is tangent but not equal to the identity. Therefore, it is not linearizable. Hence, the germ of Z at $(t = v = 0)$ is not linearizable and Z cannot have a real analytic germ of variety $\tilde{M}$, with $dim_{\mathbb{R}}(\tilde{M}) = 3$, tangent to it. Finally, if X had a germ $M = (F = 0)$, tangent to it, then Z would have $(F \circ \pi = 0)$ tangent to it. This finishes the proof of lemma 2.5. □

2.3. **The hypersurface after the resolution.** As we said at the begining, we will use Seidenberg's resolution theorem [Se]. Denote by $\pi\colon (Y,D) \to (\mathbb{C}^2,0)$ a sequence of blowing-ups, begining by one at $0 \in \mathbb{C}^2$, such that all singularities of the foliation $\tilde{\mathcal{F}} := \pi^*(\mathcal{F})$ are reduced. Let $D = \cup_{j=1}^r D_j$ be the decomposition of the divisor $D = \pi^{-1}(0)$ into irreducible components. Each D_j will be called a divisor of π. We will say that D_j is dicritical, if it is not invariant by $\tilde{\mathcal{F}}$. Otherwise, it will be called non-dicritical. We will say that $\mathcal{F}$ is dicritical, if D contains at least one dicritical divisor. Otherwise, we will say that it is non-dicritical.

By doing additional blowing-ups if necessary, we can suppose that $\tilde{\mathcal{F}}$ has no singularities in the dicritical divisors and that the dicritical divisors are isolated : if D_i and D_j are dicritical divisors with $i \neq j$ then $D_i \cap D_j = \emptyset$ (cf. [C-LN-S]). Note that, if D_j is a dicritical divisor then $\tilde{\mathcal{F}}$ is transverse to D_j.

We will denote by $\tilde{M}$ the strict transform of $M = F^{-1}(0)$ by π :

$$\tilde{M} = \overline{\pi^{-1}(M \setminus \{0\})} \ .$$

Note that $\tilde{M}$ is an irreducible real analytic hypersurface $\tilde{\mathcal{F}}$-invariant.

Lemma 2.6. *In the above situation we have two possibilities :*

(a). *$\mathcal{F}$ is non-dicritical. In this case, $\tilde{M} \supset D$.*

(b). *$\mathcal{F}$ is dicritical. In this case, $\tilde{M}$ cuts at least some dicritical divisor D_i. Moreover, $\tilde{M} \cap D_i$ is a real analytic curve.*

Proof. Let $I = \{j \,|\, D_j \text{ is dicritical}\}$ and $J = \{1,...,r\} \setminus I$. Recall that if $\#I \geq 2$ then the $D_i \cap D_j = \emptyset$ if $i,j \in I$ and $i \neq j$. Since D is connected, we can decompose it as

$$D = (\cup_{i \in I} D_i) \cup (\cup_{j=1}^k E_j) \ , \tag{8}$$

where each E_j is a maximal connected union of non-dicritical divisors. If $J = \emptyset$ then $\#I = 1$ and up to a biholomorphism $\mathcal{F}$ is the radial foliation, given by $x\,dy - y\,dx = 0$. If $I, J \neq \emptyset$ then each E_j cuts at least one D_i, $i \in I$.

Since $0 \in M$ there exists a divisor D_α such that $\tilde{M} \cap D_\alpha \neq \emptyset$. Suppose first that D_α is non-dicritical. We assert that $D_\alpha \subset \tilde{M}$.

In fact, let $q \in D_\alpha \cap \tilde{M}$. Since $dim_{\mathbb{R}}(\tilde{M}) = 3 > dim_{\mathbb{R}}(D_\alpha) = 2$, there exists a sequence $(q_n)_{n\geq 1}$ in $\tilde{M} \setminus D_\alpha$ such that $\lim\limits_{n\to\infty} q_n = q$. If we denote by L_n the leaf of $\tilde{\mathcal{F}}$ through q_n, then $L_n \subset \tilde{M}$. Suppose first that $q \notin sing(\tilde{\mathcal{F}})$. In this case, $\tilde{M} \supset \overline{\cup_n L_n} \supset D_\alpha$, because $D_\alpha \setminus sing(\mathcal{F})$ is a leaf of $\mathcal{F}$. On the other hand, if $q \in sing(\mathcal{F})$ then $\tilde{\mathcal{F}}$ has a holomorphic first integral in a neighborhood of q, by lemma 2.5. This implies that $\overline{\cup_n L_n}$ contains both separatrices of $\tilde{\mathcal{F}}$ through q. One of these separatrices is contained in D_α. This implies that $\tilde{M} \supset \overline{\cup_n L_n} \supset D_\alpha$.

By the same reason, $\tilde{M}$ contains all separatrices of all singularities of $\tilde{\mathcal{F}}$ in D_α. In particular, $\tilde{M}$ contains all non-dicritical divisors D_β such that $D_\beta \cap D_\alpha \neq \emptyset$. By connexity, $\tilde{M} \supset E_j$, where E_j is the unique component in (8) which contains D_α. This proves assertion (a) : if $\mathcal{F}$ is non-dicritical then $\tilde{M} \supset D$. On the other hand, if $\mathcal{F}$ is dicritical then $\tilde{M}$ cuts at least one dicritical divisor. In fact, let $q \in \tilde{M} \cap D$. If $q \in E_j$ for some j then the previous argument shows that $\tilde{M} \supset E_j$. But E_j cuts at least one dicritical divisor, say D_i, and so $\tilde{M} \cap D_i \neq \emptyset$ for some $i \in I$. This implies that $\tilde{M}$ is transverse to D_i because $\tilde{M}$ and $\tilde{\mathcal{F}}$ are tangent and $\tilde{\mathcal{F}}$ is transverse to

D_i. Since $dim_{\mathbb{R}}(\tilde{M}) = 3$ and $dim_{\mathbb{R}}(D_i) = 2$, it follows that $\tilde{M} \cap D_i$ is a real analytic curve. □

3. Theorem 1.

3.1. **Proof in dimension two : the dicritical case.** First of all, let us fix a notation. Given $G \in \mathcal{A}_2$ we can write its Taylor series as

$$G(x,y) = \sum_{j,k,\ell,m} a_{j,k,\ell,m}\, x^j.y^k.\overline{x}^\ell.\overline{y}^m \ ,\ a_{j,k,\ell,m} \in \mathbb{C} \ .$$

If we set $g_{\ell,m}(x,y) = \sum_{j,k} a_{j,k,\ell,m}\, x^j.y^k$ then

$$G(x,y) = \sum_{\ell,m} g_{\ell,m}(x,y)\, \overline{x}^\ell.\overline{y}^m \ . \tag{9}$$

Note that $g_{\ell,m}$ is holomorphic for all $\ell, m \geq 0$.

By lemma 2.4 we have $X(F) = H.F$, where $H \in \mathcal{A}_2$. If we write as in (9), $F(x,y) = \sum_{\ell,m} f_{\ell,m}(x,y)\, \overline{x}^\ell.\overline{y}^m$ and $H(x,y) = \sum_{\ell,m} h_{\ell,m}(x,y)\, \overline{x}^\ell.\overline{y}^m$, then we get

$$X(F) = \sum_{\ell,m} X(f_{\ell,m})\, \overline{x}^\ell.\overline{y}^m \ ,$$

because $X(\overline{x}) = X(\overline{y}) = 0$. From $X(F) = H.F$ we get

$$X(F) = \sum_{\ell,m} X(f_{\ell,m}) \overline{x}^\ell.\overline{y}^m = \sum_{\ell,m} \left(\sum_{\substack{\alpha+\gamma=\ell \\ \beta+\delta=m}} h_{\alpha,\beta}.f_{\gamma,\delta} \right) \overline{x}^\ell.\overline{y}^m \tag{10}$$

Let D_i be a dicritical divisor such that $\tilde{M} \cap D_i$ is a real analytic curve, say Γ. Fix a smooth point $q \in \Gamma$ and a parametrization $\gamma\colon I \to D_i$, $I = (-\epsilon, \epsilon)$, of a neighborhood of q in Γ such that $\gamma(0) = q$ and $\gamma'(t) \neq 0$, for all t.

Without loss of generality we can assume that q is a smooth point of D. Choose a holomorphic chart $(U, \psi = (u,v))$ around q such that :

(i). $q = (u = v = 0)$, $U \cap D = U \cap D_i = (v = 0)$ and $\psi(U) = \{(u,v) \in \mathbb{C}^2 \,|\, |u|\,,\, |v| \, < \delta\}$.

(ii). $\tilde{\mathcal{F}}|_U$ is defined by $du = 0$, that is, their leaves in U are the curves $(u = c, |v| < \delta)$, $|c| < \delta$. This is possible because $\tilde{\mathcal{F}}$ is transverse to D_i.

(iii). The germ of γ at $0 \in I$ can be written as $\gamma(t) = t$, with $t \in (\mathbb{R}, 0)$. This is possible because γ is real analytic and $\gamma'(0) \neq 0$.

It follows from $\pi(D_i) = \{0\}$ and $D_i \cap U = (v = 0)$ that the map π can be written in the chart (u,v) as $\pi(u,v) = (x(u,v), y(u,v))$, where

$$x(u,v) = v^m.g(u,v) \text{ and } y(u,v) = v^n.h(u,v) \ ,$$

$m, n \in \mathbb{N}$ and $g(u,0), h(u,0) \not\equiv 0$. After taking a smaller U we can suppose that $g \equiv 1$, or $h \equiv 1$.

In fact, since $U \cap D = (v = 0)$, either $g(0,0) \neq 0$, or $h(0,0) \neq 0$. Suppose for instance that $g(0,0) \neq 0$. Let ϕ be a branch of the m^{th} root of g, defined in some neighborhood of q. Then the map

$$\Phi(u,v) := (u, \tilde{v}) = (u, v.\phi(u,v))$$

is a biholomorphism in a neighborhood of 0 and satisfies

$$\pi \circ \Phi^{-1}(u,\tilde{v}) = (\tilde{v}^m, \tilde{v}^n.\tilde{h}(u,\tilde{v})) \ ,$$

as the reader can check. In the new chart $(u,\tilde{v})$ properties (i), (ii) and (iii) are still true and π is as asserted. Therefore, from now on, we will suppose $\pi(u,v) = (v^m, v^n.h(u,v))$. We would like to observe that $h_u(u,v) \neq 0$ for $v \neq 0$, because π is a biholomorphism outside $D_i \cap U = (v=0)$.

Remark 10. Let us write $G \in \mathcal{A}_2$ as in (9), $G(x,y) = \sum_{k,\ell} g_{k,\ell}(x,y).\overline{x}^k.\overline{y}^\ell$. Let $r(G) = min\{m.k + n.\ell \,|\, g_{k,\ell} \not\equiv 0\}$ and $J(G) = \{(k,\ell) \,|\, m.k + n.\ell = r(G)\}$. A straightforward computation, shows that

$$G \circ \pi(u,v) = \sum_{k,\ell} g_{k,\ell}(v^m, v^n.h(u,v)).\overline{v}^{m.k+n.\ell}.\overline{h(u,v)}^{\ell} =$$

$$(11) \qquad = \sum_{s \geq r} \overline{v}^s.\left(\sum_{m.k+n.\ell=s} g_{k,\ell}(v^m, v^n.h(u,v)).\overline{h(u,v)}^{\ell} \right) \ .$$

On the other hand, we can write $\overline{h}$ as $\overline{h(u,v)} = \sum_j h_j(\overline{u}).\overline{v}^j$. If we substitute this expression in the series (11), we see that $G \circ \pi$ can be written as

$$G \circ \pi(u,v) = \sum_{s \geq r(G)} G_s(u,v,\overline{u}).\overline{v}^s \ .$$

The fact that h is holomorphic and the definitions of r and J imply that

$$(12) \qquad G_r(u,v,\overline{u}) = \sum_{(k,\ell)\in J(G)} g_{k,\ell}(v^m, v^n.h(u,v)).\overline{h(u,0)}^{\ell} \ ,$$

as the reader can check.

Let $r(F)$ and $J(F)$ be as in remark 10. It follows from (10) and (12) that the term in $\overline{v}^r$ in the Taylor series of $(X(F) - H.F) \circ \pi$ is

$$(13) \qquad \sum_{m.k+n.\ell=r(F)} \overline{h(u,0)}^{\ell} \, [X(f_{k,\ell}) - h_{0,0}.f_{k,\ell}]_{(v^m, v^n.h(u,v))} \equiv 0.$$

From (13) and $h(u,0) \not\equiv 0$ we get

$$(14) \qquad X(f_{k,\ell}) - h_{0,0}.f_{k,\ell} \equiv 0 \ , \ \forall\, (k,\ell) \in J \ .$$

There are two possibilities :

(I). There are two different pairs $(\alpha,\beta), (\gamma,\delta) \in J$ such that $f_{\gamma,\delta}/f_{\alpha,\beta}$ is non-constant.

(II). There exists a pair $(\alpha,\beta) \in J$ such that $f_{\alpha,\beta} \not\equiv 0$ and for any other pair $(\gamma,\delta) \in J$ then $f_{\gamma,\delta} = a_{\gamma,\delta}.f_{\alpha,\beta}$, where $a_{\gamma,\delta} \in \mathbb{C}$.

In case (I), $f_{\gamma,\delta}/f_{\alpha,\beta}$ is a meromorphic first integral of X, because

$$\frac{X(f_{\gamma,\delta}/f_{\alpha,\beta})}{f_{\gamma,\delta}/f_{\alpha,\beta}} = \frac{X(f_{\gamma,\delta})}{f_{\gamma,\delta}} - \frac{X(f_{\alpha,\beta})}{f_{\alpha,\beta}} = 0 \ .$$

We assert that case (II) is not possible.

In fact, since $\tilde{M}$ is $\tilde{\mathcal{F}}$-invariant, it follows from (ii) and (iii) that $\tilde{M} \cap U$ is parametrized by $\Lambda\colon (-\delta,\delta)\times D_\delta \to U$, where $D_\delta = \{v \in \mathbb{C} \,|\, |v| < \delta\}$ and $\Lambda(t,v) = (t,v)$. In particular, we have

$$F \circ \pi \circ \Lambda \equiv 0$$

From (9) we get

$$F \circ \pi \circ \, \Lambda(t,v) = \sum_{k,\ell} f_{k,\ell}(v^m, v^n.h(t,v))\, \overline{v}^{m.k+n.\ell}.\overline{h(t,v)}^{\ell} \equiv 0 \tag{15}$$

From (12) the term in $\overline{v}^r$ in the series (15) is

$$G_r(t,v) = \sum_{m.k+n.\ell=r} f_{k,\ell}(v^m, v^n.h(t,v))\, .\, \overline{h(t,0)}^{\ell}\ . \tag{16}$$

Relation (15) implies that $G_r(t,v) \equiv 0$, for all $(t,v) \in (-\delta,\delta)\times D_\delta$.

On the other hand, in case (II), we can write

$$\Phi(x,y,\overline{x},\overline{y}) := \sum_{(k,\ell)\in J} f_{k,\ell}(x,y).\overline{x}^k.\overline{y}^\ell = f_{\alpha,\beta}(x,y).p(\overline{x},\overline{y})\ ,$$

where $p(s,t) = \sum_{(\gamma,\delta)\in J} a_{\gamma,\delta}\, s^\gamma.t^\delta$ is a non-zero polynomial. For $t \in (-\delta,\delta)$ we have

$$\Phi(v^m, v^n.h(t,v), \overline{v}^m, \overline{v}^n.\overline{h(t,0)}) = \overline{v}^r.G_r(t,v) \equiv 0\,.$$

Therefore the identity

$$f_{\alpha,\beta}(v^m, v^n.h(t,v)).p(\overline{v}^m, \overline{v}^n.\overline{h(t,0)}) \equiv 0$$

implies that, either $f_{\alpha,\beta}(v^m, v^n.h(t,v)) \equiv 0$, or $p(\overline{v}^m, \overline{v}^n.\overline{h(t,0)}) \equiv 0$. Since $t \mapsto h(t,v)$ is non-constant and $f_{\alpha,\beta}$ is holomorphic, we get $f_{\alpha,\beta}(v^m, v^n.h(t,v)) \not\equiv 0$, because $(f_{\alpha,\beta} = 0)$ has a finite number of irreducible components. Hence, $p(\overline{v}^m, \overline{v}^n.\overline{h(t,0)}) \equiv 0$. However, this implies that for all $t \in (-\delta,\delta)$ the polynomial $q_t(z) := p(z^m, z^n.\overline{h(t,0)}) \equiv 0$, so that $\overline{h(t,0)} \equiv 0$, which contradicts $h(u,0) \not\equiv 0$. □

3.2. **Proof in dimension two : the non-dicritical case.** Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^2$ of a non-dicritical foliation tangent to a real analytic subset M of codimension one. In this case, $\mathcal{F}$ has a finite number of analytic closed leaves which accumulate at $0 \in \mathbb{C}^2$: its separatrices.

Consider a resolution $\pi\colon (\tilde{\mathbb{C}}^2, D) \to (\mathbb{C}^2, 0)$ of the foliation $\mathcal{F}$. Set $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ and let $\tilde{M}$ be the strict transform of M by π :

$$\tilde{M} = \overline{\pi^{-1}(M)\setminus\{0\}}\ .$$

Since $\mathcal{F}$ is non-dicritical, all irreducible components of D are $\mathcal{F}$-invariant. Moreover, $\tilde{M} \supset D$ by lemma 2.6. In particular, $\tilde{M}$ contains all singularities of $\tilde{\mathcal{F}}$ in D.

It follows from lemma 2.5 that all singularities of $\tilde{\mathcal{F}}$ have a local first integral : if $q \in sing(\tilde{\mathcal{F}}) \subset D$ then there exists a local coordinate system $(W,(u,v))$ such that $\tilde{\mathcal{F}}|_W$ has a first integral of the form $u^m.v^n$, where $m,n \in \mathbb{N}$ and $gcd(m,n) = 1$. We will call this type of singularity a *saddle with a first integral.*

We will use the following result (cf. [L] page 162) :

Theorem 3.1. *Let $\mathcal{F}$ be a non-dicritical foliation and $\pi\colon (\tilde{\mathbb{C}}^2, D) \to (\mathbb{C}^2, 0)$ be a minimal resolution and $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$. Assume that all singularities of $\tilde{\mathcal{F}}$ in D are saddles with a first integral. Fix a transversal Σ through a point $p \in D \cap \Sigma$, which is not a singularity of $\tilde{\mathcal{F}}$. Then :*

(a). *The transversal is complete, in the sense that there is a neighborhood U_o of p in Σ such that for any smaller neighborhood $p \in U \subset U_o$ then $V_U := int\left(\overline{sat_{\tilde{\mathcal{F}}}(U)}\right)$ is a neighborhood of D, where int denotes the interior and*

$$sat_{\tilde{\mathcal{F}}}(U) := \cup_{q \in U} L_q \ , \ L_q = \text{leaf of } \tilde{\mathcal{F}} \text{ through } q \ .$$

(b). *There exists a finite ramified covering $\Pi\colon (\mathbb{D},0) \to (\Sigma,0)$ and a subgroup $G \subset Diff(\mathbb{C},0)$ which covers the pseudo-group of holonomy of the germ $\tilde{\mathcal{F}}_D$ of $\tilde{\mathcal{F}}$ at D.*

For a precise definition of the pseudo-group of holonomy of the germ $\tilde{\mathcal{F}}_D$ in assertion (b) of theorem 3.1, we send the reader to the reference [L]. The group G is usually called the *global holonomy group* of $\tilde{\mathcal{F}}$. In the reference [L] the author proves that it can be defined in a more general situation, namely when all the singularities in the corners of D are saddles with a first integral and the others are either hyperbolic or saddles. As an application, he gives a nice proof of a theorem due to Mattei and Moussu about the existence of a non-constant holomorphic first integral for a non-dicritical germ of foliation with closed leaves (cf. [M-M]). In particular, he proves the following :

Corollary 3.1. *In the situation of theorem 3.1 the foliation $\mathcal{F}$ has a first integral if, and only if, the group G is finite.*

When the foliation has not necessarily a non-constant first integral, in the situation of theorem 3.1 the group G is finitely generated, $G =< f_1, ..., f_r >$, where each generator has finite order : $f_j^{n_j} = id$. In this case, it is known that G is finite if, and only if, G is abelian. Therefore, if $\mathcal{F}$ has no non-constant first integral we have $G^1 = [G,G] \neq \{id\}$. If $\tilde{f} \in G^1 \setminus \{id\}$ then $\tilde{f}(z) = z(1 + a.z^k) + o(z^{k+2})$, $a \neq 0$, for some $k \geq 1$. Let $f\colon W \to \mathbb{C}$ be a representative of $\tilde{f}$, where $0 \in W$ and $f(0) = 0$.

It is known that there exists a neighborhood U of 0, $U \subset W$, such that for any $z \in U$ then, either its positive orbit is well defined and $\lim\limits_{n\to+\infty} f^n(z) = 0$, or its negative orbit is well defined and $\lim\limits_{n\to-\infty} f^n(z) = 0$ (cf. [L]). This implies that for any $q \in \Pi(U) \subset \Sigma$ then the $\tilde{\mathcal{F}}$-leaf of q accumulates in the divisor D, and so it cannot be analytic. By using this and (a) of theorem 3.1 we get the following :

Corollary 3.2. *In the situation of theorem 3.1, if $\mathcal{F}$ has no non-constant holomorphic first integral then there exists a neighborhood $V_o \subset B$ of $0 \in B$ such that for any smaller neighborhood $0 \in V \subset V_o$ and any $q \in V$ then the leaf of $\mathcal{F}|_V$ through q accumulates in 0. In particular, $\mathcal{F}|_V$ has a finite number of analytic leaves : its separatrices.*

Now, if V is small, it follows from lemma 2.1 that all leaves of $\mathcal{F}|_V$ through points of $M^* \cap V$ are closed in V. This implies that they cannot accumulate at the origin. Since $M^* \cap V$ contains infinitely many leaves of $\mathcal{F}_V$, it follows from corollary 3.2 that $\mathcal{F}$ has a non-constant holomorphic first integral. This proves theorem 1 in dimension two in the non-dicritical case. □

3.3. **Proof in dimension** $n \geq 3$**.** Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a holomorphic codimension one foliation, tangent to a germ at $0 \in \mathbb{C}^n$ of real analytic hypersurface M. We are going to prove that $\mathcal{F}$ has a non-constant meromorphic first integral.

Let us give an idea of the proof. First of all, we will prove that there is a holomorphic embedding $i\colon (\mathbb{C}^2,0)\to(\mathbb{C}^n,0)$ with the following properties :

(i). $i^{-1}(M)$ has real codimension one in $(\mathbb{C}^2,0)$.
(ii). $sing(i^*(\mathcal{F}))=\{0\}$.
(iii). $i^*(\mathcal{F})$ is tangent to $i^{-1}(M)$.

Set $E:=i(\mathbb{C}^2,0)$. The above conditions and theorem 1 in dimension two imply that $\mathcal{F}|_E$ has a non-constant meromorphic first integral (or holomorphic, if $i^*(\mathcal{F})$ is non-dicritical), say f. After that we will use a result of [C-LN-S-1] to prove that f can be extended to a meromorphic germ $f_1\in\mathcal{O}_n$, which is a first integral of $\mathcal{F}$.

Let us suppose that $\mathcal{F}$ is defined by $\omega=0$, where ω is a germ at $0\in\mathbb{C}^n$ of an integrable holomorphic 1-form with $cod_{\mathbb{C}^n}(sing(\omega))\geq 2$. We say that a holomorphic embedding $i\colon(\mathbb{C}^2,0)\to(\mathbb{C}^n,0)$ is *transverse* to ω if $cod_{\mathbb{C}^2}(sing(i^*(\omega)))=2$, which means in fact that, as a germ of set, we have $sing(i^*(\omega))=\{0\}$. Note that the concept is independent of the particular germ of holomorphic 1-form which represents $\mathcal{F}$. Therefore, we will say that the embedding i is transverse to $\mathcal{F}$ if it is transverse to some holomorphic 1-form ω representing $\mathcal{F}$.

According to [M-M], the set of holomorphic embeddings $i\colon(\mathbb{C}^2,0)\to(\mathbb{C}^n,0)$ transverse to $\mathcal{F}$ is non-empty. Let us fix one embedding $i\colon(\mathbb{C}^2,0)\to(\mathbb{C}^n,0)$, transverse to $\mathcal{F}$. By the inverse mapping theorem, we can assume that i is linear.

Let $\mathcal{L}(2,n)$ be the set of linear mappings from $\mathbb{C}^2$ to $\mathbb{C}^n$ and set $\mathcal{L}_{\mathcal{F}}=\{i\in\mathcal{L}(2,n)\mid i \text{ is an embedding transverse to } \mathcal{F}\}$. In [M-M] it is proved that $\mathcal{L}_{\mathcal{F}}$ is open and dense in $\mathcal{L}(2,n)$, if it is not empty, which is our case.

Now, $i_o\in\mathcal{L}(2,n)$ be a linear embedding transverse to $\mathcal{F}$ and $E_o=i_o(\mathbb{C}^2)$. Consider coordinates $(x,y)\in\mathbb{C}^2\times\mathbb{C}^{n-2}$ such that $E_o=(y=0)$. Given $r,s>0$ set $B(r,s)=\{(x,y)\in\mathbb{C}^2\times\mathbb{C}^{n-2}\mid |x|<r \text{ and } |y|<s\}$ and $B_r=\{(x,0)\in E_o\,|\,|x|<r\}$. Fix $r_o>0$ such that $\mathcal{F}$ has a representative in $B(r_o,r_o)$.

Lemma 3.1. *In the above situation, there exists $0<r<r_o$ with the following property : for any $0<\epsilon\leq r$ there exists $0<\delta\leq r_o$ such that for any $p\in B(\epsilon,\delta)$ then the leaf L_p of $\mathcal{F}$ through p, contains a point $q=(x,0)\in E_o$ such that $|q|=|x|=\epsilon$. Moreover, L_p cuts E_o transversely in all points of $L_p\cap B_\epsilon$.*

Proof. Given $y\in\mathbb{C}^{n-2}$ and $r>0$ set $E_y=\{(x,y)\,|\,x\in\mathbb{C}^2\}$ and $S_r(y)=\{(x,y)\in E_y\,|\,|x|=r\}$. Let ω be a holomorphic 1-form representing $\mathcal{F}$ on $B(r_o,r_o)$,

$$\omega=P_1(x,y)\,dx_1+P_2(x,y)\,dx_2+\sum_{j=1}^{n-2}Q_j(x,y)\,dy_j\ .$$

Since i_o is transverse to $\mathcal{F}$ we have $cod_{E_o}(\omega|_{E_o})=2$. In particular, there exists $0<r<r_o$ such that $sing(\omega|_{E_o})\cap\{(x,0)\in E_o\,|\,|x|\leq r\}=\{0\}$, which means that $(P_1(x,0)=P_2(x,0)=0)\cap\overline{B}_r=\{0\}$. In particular, $\mathcal{F}$ is transverse to E_o in all points of $\overline{B}_r\setminus\{0\}$.

Now, fix $0<\epsilon\leq r$. There exists $0<\delta_1(\epsilon)<r_o$ such that

$$|P_1(x,y)|+|P_2(x,y)|>0\ ,\ \forall\,(x,y)\in(|x|=\epsilon)\cap(|y|\leq\delta_1(\epsilon))\ ,$$

which implies that $sing(\omega|_{E_{y_o}})\cap S_\epsilon(y_o)=\emptyset$, for all y_o with $|y_o|\leq\delta_1(\epsilon)$. Since $\mathcal{F}$ is transverse to E_o in the compact set $S_\epsilon(0)=\partial B_\epsilon$, there exists $0<\delta(\epsilon)\leq\delta_1(\epsilon)$ such that for any point $p=(x,y)$ with $|x|=\epsilon$ and $|y|<\delta(\epsilon)$ then L_p cuts $S_\epsilon(0)$ and so cuts E_o.

If we fix $p = (x, y)$, where $|x| \leq \epsilon$ and $|y| \leq \delta(\epsilon)$ then the leaf L_p cuts $S_\epsilon(0) = \partial B_\epsilon$, because :

(i). $L_p \cap E_y$ is a union of leaves of $\mathcal{F}|_{E_y}$. In particular, $L_p \cap E_y$ contains the leaf of $\mathcal{F}|_{E_y}$ through $p = (x, y)$.
(ii). $L_p \cap E_y$ cuts $S_\epsilon(y)$ by the maximum principle.
(iii). If $q \in L_p \cap E_y \cap S_\epsilon(y)$ then $L_q = L_p$ and L_q cuts $S_\epsilon(0) \subset E$.

Since $\mathcal{F}$ is transverse to E_o in the set $B_r \setminus \{0\}$, the intersection of L_p with E_o is transverse on this set. This finishes the proof of lemma 3.1. □

Corollary 3.3. *In the above situation there exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{F}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.*

Proof. Lemma 3.1 has the following consequence : for any 2-plane E_o through $0 \in \mathbb{C}^n$, transverse to $\mathcal{F}$, and any $0 < \epsilon < r_o$ then $M \cap \partial B_\epsilon \neq \emptyset$, that is there exists $p \in E_o \cap M$ with $|p| = \epsilon$. We keep the notations of lemma 3.1.

In fact, fix $0 < \epsilon < r_o$ and let $\delta(\epsilon) > 0$ be as in the assertion of lemma 3.1. Since 0 is in the closure of M^* there exists $q = (x, y) \in B(\epsilon, \delta(\epsilon)) \cap M^*$. By the lemma, L_q contains a point $p = (x, 0)$ with $|x| = \epsilon$. Since M is levi-flat, $L_q \subset M^*$, and so $p \in M^*$ and $|p| = \epsilon$.

The above argument implies that the real dimension of the germ at $0 \in E_o$ of $E_o \cap M$ is always ≥ 1. Since $\mathcal{L}_\mathcal{F}$ is open and dense in $\mathcal{L}(2, n)$, by transversality theory, there exists a linear embedding $i \in \mathcal{L}_\mathcal{F}$ such that $E = i(\mathbb{C}^2)$ is transverse to M^* and to $\mathcal{F}$ simultaneously. This implies corollary 3.3. □

Let E be a 2-plane as in corollary 3.3. By the two dimensional case $\mathcal{F}|_E$ has a non-constant meromorphic first integral (holomorphic in the non-dicritical case), say f. Since the embedding $E \to \mathbb{C}^n$ is transverse to $\mathcal{F}$ and $sing(\mathcal{F}|_E) = \{0\}$, by lemma 3.1 we can take representatives in some $B(r_o, r_o)$ in such a way that there exist $0 < r_1 < r_2 < r_o$ and $0 < \delta < r_o$ with the following properties

(a). If $C := B_{r_2} \setminus \overline{B_{r_1}} \subset E$ then $sing(\mathcal{F}) \cap C = \emptyset$.
(b). If $V := \{(x, y) \,|\, x \in C \text{ and } |y| < \delta\}$ then any leaf of $\mathcal{F}|_V$ cuts C transversely.

As a consequence, the first integral f can be extended to a first integral $\tilde{f}$ of $\mathcal{F}|_V$. On the other hand, V is a Hartogs domain in $\mathbb{C}^n$ with holomorphic closure $\hat{V} = B(r_2, \delta)$. By Levi's extension theorem $\tilde{f}$ can be extended to a meromorphic function f_1 on $B(r_2, \delta)$ (cf. [Si]). This extension gives a meromorphic first integral of $\mathcal{F}$. This proves theorem 1.

4. Theorem 2 and corollary 1.

4.1. **Proof of theorem 2.** Let us give an outline of the proof. In [Ce-LN] Malgrange's theorem on the existence of a holomorphic first integral for a germ of foliation $\mathcal{G}$ at $0 \in \mathbb{C}^n$ such that $cod_{\mathbb{C}^n}(sing(\mathcal{G})) \geq 3$ is generalized for codimension one holomorphic foliations on an irreducible analytic subset of $\mathbb{C}^N$ of dimension ≥ 3. In this way, hypothesis (a) of theorem 2 was chosen in such a way that we can use the results of [Ce-LN] for the complexifications $\mathcal{L}_\mathbb{C}$ and $M_\mathbb{C}$ of $M = F^{-1}(0)$ and $\mathcal{L}$, respectively. So, both hypothesis, (a) or (b), imply that we have a holomorphic first integral of $\mathcal{L}_\mathbb{C}$ on $M_\mathbb{C}$, say $g_\mathbb{C}$. The hypothesis on the algebraic dimension of $sing(M)$ implies that $M_\mathbb{C}$ is normal. In particular, $g_\mathbb{C}$ can be extended to a holomorphic germ $f_\mathbb{C} \in \mathcal{O}_{2n}$. The idea is to prove that this extension can be done in a special way, namely, with the property that it is the "complexification" of some holomorphic germ $f \in \mathcal{O}_n$ such that $f(M) \subset (\mathbb{R}, 0)$. The function $f|_{M^*}$ will be a holomorphic first integral of $\mathcal{L}$, so that the foliation $\mathcal{F}_M$ defined by $df = 0$ will be tangent to M. Finally, $f(M) \subset (\mathbb{R}, 0)$ will imply that $M = (\mathcal{I}m(f) = 0)$.

Let $\eta = i(\partial F - \overline{\partial} F)$ and $\eta_{\mathbb{C}}$ be its complexification on $(\mathbb{C}^{2n}, 0)$. Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$, respectively.

Lemma 4.1. *Under hypothesis (a) of theorem 2 there exist germs $g, h \in \mathcal{A}_{n\,\mathbb{R}}$ such that $g(0) = 0$, $h(0) = 1$ and $\eta|_{M^*} = h.dg|_{M^*}$. In particular, we get the following :*

(a). *$g|_{M^*} : M^* \to \mathbb{R}$ is a submersion constant along the leaves of $\mathcal{L}$.*
(b). *The leaves of $\mathcal{L}$ on M^* are the hypersurfaces $M^* \cap g^{-1}(c)$, $c \in (\mathbb{R}, 0)$.*
(c). *$\mathcal{L}$ has only a finite number of leaves adherent to the origin : those contained in $g^{-1}(0) \cap M$.*

Proof. By lemma 2.3 the hypersurface $M_{\mathbb{C}}$ is irreducible. So we can apply corollary 1 of the main theorem of [Ce-LN] to the foliation $\mathcal{L}_{\mathbb{C}}$. It follows from this result that there are germs $H_{\mathbb{C}}, G_{\mathbb{C}} \in \mathcal{O}_{2n}$ such that $G_{\mathbb{C}}(0) = 0$, $H_{\mathbb{C}}(0) = 1$ and $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}} = H_{\mathbb{C}}.dG_{\mathbb{C}}|_{M_{\mathbb{C}}}$. Define $H, G \in \mathcal{A}_n$ by $H(z) := H_{\mathbb{C}}(z, \overline{z})$ and $G(z) := G_{\mathbb{C}}(z, \overline{z})$. Since $\eta = \eta_{\mathbb{C}}|_{(w=\overline{z})}$ we get $\eta|_{M^*} = H.dG|_{M^*}$, where $H(0) = 1$.

Now, set $H = h_1 + i\,h_2$ and $G = g_1 + i\,g_2$ where $h_1, h_2, g_1, g_2 \in \mathcal{A}_{n\mathbb{R}}$. Note that $h_1(0) = H(0) = 1$ and $h_2(0) = 0$. Since $H\,dG = (h_1\,dg_1 - h_2\,dg_2) + i(h_1\,dg_2 + h_2\,dg_1)$ and $\overline{\eta} = \eta$ we get by restriction to M^* that

$$(h_1\,dg_2 + h_2\,dg_1)|_{M^*} = 0 \implies \eta|_{M^*} = \left.\frac{h_1^2 + h_2^2}{h_1}\,dg_1\right|_{M^*} .$$

In particular, $\eta|_{M^*} = h\,dg|_{M^*}$, where $g = g_1$, $h = (h_1^2 + h_2^2)/h_1$, $g(0) = 0$ and $h(0) = 1$.

Now, for any $p \in M^*$ we have $h(p)\,dg(p)|_{T_pM^*} = \eta(p)|_{T_pM^*} \neq 0$. Since h is an unit we get that $dg(p)|_{T_pM^*} \neq 0$ if $p \in M^*$ is near the origin. This proves (a). We leave the proofs of (b) and (c) to the reader. □

Consider $g \in \mathcal{A}_{n\,\mathbb{R}}$ given by lemma 4.1 and its complexification $g_{\mathbb{C}}$. Since $g|_{M^*}$ is a first integral of $\mathcal{L}$ we get $dg \wedge \eta|_{M^*} = 0$. The complexification of the Taylor series of dg and η in the relation, implies that $dg_{\mathbb{C}} \wedge \eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 0$. In particular, $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ is a non-constant holomorphic first integral of the complexified Levi foliation $\mathcal{L}_{\mathbb{C}}$ on $M^*_{\mathbb{C}}$.

Now, let us assume hypothesis (b) of theorem 2 : $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral $g_{\mathbb{C}}$ and $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$. Since $dim(sing(M_{\mathbb{C}})) \leq 2n-4$, the hypersurface $M_{\mathbb{C}}$ is normal and so there exists $G_{\mathbb{C}} \in \mathcal{O}_{2n}$ such that $G_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = g_{\mathbb{C}}$. Note that

$$dG_{\mathbb{C}} \wedge \eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 0 \ . \tag{17}$$

Let $G \in \mathcal{A}_n$ be defined by $G(z) := G_{\mathbb{C}}(z, \overline{z})$. It follows from (17) that $dG \wedge \eta|_{M^*} = 0$. If we set $G = G_1 + i\,G_2$, where $G_1, G_2 \in \mathcal{A}_{n\,\mathbb{R}}$ then we get $dG_j \wedge \eta|_{M^*} = 0$, $j = 1, 2$, because $\overline{\eta} = \eta$. Since G is non-constant, one of the germs, G_1 or G_2, is non-constant. Therefore, $\mathcal{L}$ has a real non-constant analytic first integral. To be coherent with the notations of lemma 4.1 we will denote this first integral by g and by $g_{\mathbb{C}}$ its complexification. We can assume that $g_{\mathbb{C}}$ has the generic fiber connected (cf. [M-M]).

Recall that $F_{\mathbb{C}}$ was defined from the Taylor series of F : if $F(z) = \sum_{\mu,\nu} F_{\mu\,\nu}\, z^\mu\, \overline{z}^\nu$ then $F_{\mathbb{C}}(z,w) = \sum_{\mu,\nu} F_{\mu\,\nu}\, z^\mu\, w^\nu$. Since $\eta = i(\partial F - \overline{\partial} F)$, we get by complexification that

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}) \ ,$$

where

$$\partial_z F_{\mathbb{C}} = \sum_{\mu,\nu} F_{\mu\,\nu}\, w^\nu\, d(z^\mu) \text{ and } \partial_w F_{\mathbb{C}} = \sum_{\mu,\nu} F_{\mu\,\nu}\, z^\mu\, d(w^\nu) \ . \tag{18}$$

The next result will provide a "good" extension of $g_{\mathbb{C}}$.

Lemma 4.2. *Let $g \in \mathcal{A}_{n\,\mathbb{R}}$ be a real analytic function such that $g|_{M^*}$ is a non-constant first integral of the Levi foliation $\mathcal{L}$. Let $g_{\mathbb{C}} \in \mathcal{O}_{2n}$ be its complexification, so that $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ is a first integral of $\mathcal{L}_{\mathbb{C}}$. Assume that $n \geq 2$ and $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$. Then there is an unique $f_{\mathbb{C}} \in \mathcal{O}_{2n}$ such that $f_{\mathbb{C}}|_{M_{\mathbb{C}}} = g_{\mathbb{C}}|_{M_{\mathbb{C}}}$ and $\frac{\partial f_{\mathbb{C}}}{\partial w_j} \equiv 0$ for all $j \in \{1,...,n\}$.*

Proof. Let $\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}})$, where $\partial_z F_{\mathbb{C}}$ and $\partial_w F_{\mathbb{C}}$ are as in (18). Set $\omega := i^{-1}\,\eta_{\mathbb{C}}$. Note that $\omega|_{M^*_{\mathbb{C}}}$ defines $\mathcal{L}_{\mathbb{C}}$. With the notations of (18) we have $dF_{\mathbb{C}} = \partial_z F_{\mathbb{C}} + \partial_w F_{\mathbb{C}}$, so that

$$\omega|_{M^*_{\mathbb{C}}} = 2\,\partial_z F_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = -2\,\partial_w F_{\mathbb{C}}|_{M^*_{\mathbb{C}}} \ . \tag{19}$$

Take representatives of $F_{\mathbb{C}}$ and $\eta_{\mathbb{C}}$ in some ball B_ρ around 0 of $\mathbb{C}^{2n}$ such that $dim(sing(dF_{\mathbb{C}})) \leq 2n-4$ and $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$ in B_ρ. Set $X := sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$.

Claim 4.1. *The function $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}\setminus X}$ has an unique holomorphic extension to some neighborhood U of $M^*_{\mathbb{C}} \setminus X$ in $\mathbb{C}^{2n}$, say G, such that*

$$\frac{\partial G}{\partial w_j} \equiv 0 \ , \ \forall\, j \in \{1,...,n\} \ . \tag{20}$$

Proof. Let us write the integrability condition for the 1-form $\theta := \omega|_{M^*_{\mathbb{C}}\setminus X}$. Fix a point $p_o = (z_1^o,...,z_n^o,w_1^o,...,w_n^o) \in M^*_{\mathbb{C}} \setminus X$. By (19) we can write

$$\theta = -2\,\partial_w F_{\mathbb{C}}|_{M^*_{\mathbb{C}}\setminus X} = -2\, \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j \Bigg|_{M^*_{\mathbb{C}}\setminus X} \ .$$

In particular, there exists $j \in \{1,...,n\}$ such that $\frac{\partial F_{\mathbb{C}}}{\partial w_j}(p_o) \neq 0$. Assume that $\frac{\partial F_{\mathbb{C}}}{\partial w_n}(p_o) \neq 0$, for instance. In this case, we can parametrize $M^*_{\mathbb{C}}$ in a neighborhood of p_o as a graph $w_n = \varphi(z,w_1,...,w_{n-1})$, where $\varphi\colon V \to \mathbb{C}$ is holomorphic and V is an open neighborhood of $q_o = (z_1^o,...,z_n^o,w_1^o,...,w_{n-1}^o)$. It follows from (19) that in this parametrization we have

$$\theta = 2\,\partial_z F_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = \sum_{j=1}^n A_j(z,w_1,...,w_{n-1})\, dz_j \ ,$$

where

$$A_j(z,w_1,...,w_{n-1}) = 2\,\frac{\partial F_{\mathbb{C}}}{\partial z_j}\,(z,w_1,...,w_{n-1},\varphi(z,w_1,...,w_{n-1})) \ .$$

Since $\theta(p_o) \neq 0$ there exists $j \in \{1,...,n\}$ such that $A_j(q_o) \neq 0$. Let us assume $A_1(q_o) \neq 0$, so that $A_1(q) \neq 0$ for q in some neighborhood V_1 of q_o. Therefore, we can write

$$\theta = A_1.\left(dz_1 + \sum_{j=2}^{n} B_j\, dz_j\right) := A_1\,\alpha\ ,$$

where $B_j = A_j/A_1$ is holomorphic in V_1. Since θ is integrable, α is also integrable, so that $\alpha \wedge d\alpha = 0$. The coeficient of $dz_1 \wedge dw_j \wedge dz_k$ in this relation gives $\frac{\partial B_k}{\partial w_j} \equiv 0$. This implies that $B_k = b_k(z)$, $2 \le k \le n$, and so $\alpha = dz_1 + \sum_{j=2} b_j(z)dz_j$.

Now, $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ can be written in this parametrization as

$$g_{\mathbb{C}}(z, w_1, ..., w_{n-1}, \varphi(z, w_1, ..., w_{n-1})) := g_1(z, w_1, ..., w_{n-1})\ .$$

Since $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ is a first integral of $\mathcal{L}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{C}}$ is defined by α in the neighborhood V_1, we get

$$dg_1 \wedge \alpha \equiv 0 \implies dg_1 = \phi.\alpha\ ,\ \phi \in \mathcal{O}(V_1) \implies \frac{\partial g_1}{\partial w_j} \equiv 0\ ,\ \forall\, j = 1, ..., n\ .$$

It follows that g_1 depends only on z and we can write $g_1(z, w_1, ..., w_{n-1}) = g_\varphi(z)$, where g_φ is holomorphic in some neighborhood V_φ of z_o.

In particular, in the open set of $M^*_{\mathbb{C}}$, say W_φ, given by $W_\varphi := (V_\varphi \times \mathbb{C}^n) \cap M^*_{\mathbb{C}}$, we have

$$g_{\mathbb{C}}|_{W_\varphi} = g_\varphi|_{W_\varphi}\ .$$

It follows that g_φ is an extension of $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ to a neighborhood U_φ of $W_{\mathbb{C}}$ in $\mathbb{C}^{2n}$ satisfying

$$\frac{\partial g_\varphi}{\partial w_j} = 0\ ,\ \forall\ j = 1, ..., n\ .$$

If $\tilde{g}$ is a holomorphic extension of $g_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ to some neighborhood $\tilde{U}$ of an open set $\tilde{W}$ of $M_{\mathbb{C}}$ satisfying (20) on $\tilde{U}$ and $\tilde{W} \cap W_\varphi \neq \emptyset$ then $\tilde{g} = g_\varphi$ on $\tilde{U} \cap U_\varphi$. We leave the proof of this fact to the reader. This implies that $g_{\mathbb{C}}|_{M^*_{\mathbb{C}} \setminus X}$ can be extended to a holomorphic function G, defined in some neighborhood of $M^*_{\mathbb{C}} \setminus X$, and satisfying (20). □

Let us finish the proof of lemma 4.2. It is enough to prove that $G \in \mathcal{O}(U)$ can be extended to an open set W of $\mathbb{C}^{2n}$ such that $W \supset M_{\mathbb{C}}$, so that $0 \in W$.

We will first extend G to a neighborhood of $X = sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$. This can be done as follows : given $p \in X$, fix a coordinate system $(W, \Psi = (u, z) \in \mathbb{C}^{2n-1} \times \mathbb{C})$ such that $\Psi(W) = \mathbb{D}^{n-1} \times \mathbb{D}$, $\Psi(p) = 0$ and $M^*_{\mathbb{C}} \cap W = (z = 0)$. Since $cod_{M^*_{\mathbb{C}}}(X) \ge 2$ there exists a 2-plane $E \subset \mathbb{C}^{2n-1} \times \{0\}$ such that $p = (0, 0)$ is an isolated point of $E \cap X$. We can assume that $E = \mathbb{C}^2 \times \{0\} \subset \mathbb{C}^{2n}$ and that $\Psi = (x, y, z) \in \mathbb{C}^2 \times \mathbb{C}^{2n-3} \times \mathbb{C}$. The function G is defined and holomorphic in a neighborhood H of some sphere $S = \{(x, 0, 0)\,|\,|x| = \delta\}$, say of the form $H = \{(x, y, z)\,|\,\delta/2 < |x| < 2\delta\,,\ |y| + |z| < \epsilon\}$. Note that H is a Hartogs domain with holomorphic closure $\hat{H} = \{(x, y, z)\,|\,|x| < 2\delta\,,\ |y| + |z| < \epsilon\}$ (cf. [Si]). Therefore, G can be extended to the neighborhood $\hat{H}$ of p and we can assume that G is defined and holomorphic in a neighborhood V of $M^*_{\mathbb{C}}$ in $\mathbb{C}^{2n}$. Let us extend it to a neighborhood of $0 \in \mathbb{C}^{2n}$. We will use the following result of [Ce] :

Theorem 4.1. *Let $f\colon B_\rho \to \mathbb{C}$ be a holomorphic function, where $B_\rho = \{z \in \mathbb{C}^m\,|\,|z| < \rho\}$. Assume that :*

(a). *$m \geq 3$.*
(b). *$0 \in B_\rho$ is the unique singularity of f in B_ρ.*
(c). *For any $0 < r < \rho$ the hypersurface $f^{-1}(0)$ is transverse to the sphere $S_r = \{z \in \mathbb{C}^n \,|\, |z| = r\}$.*

Then any holomorphic function defined in a neighborhood of a knot $S_r \cap f^{-1}(0)$, $0 < r < \rho$, can be extended to a holomorphic function defined in an open set U which contains $f^{-1}(0)$. In particular, $0 \in U$.

In fact, the result proved in [Ce] is more general, but the weaker version above is sufficient to us. Observe also that for any f with an isolated singularity at $0 \in \mathbb{C}^m$ there exists a ball B_ρ such that $f|_{B_\rho}$ satisfies hypothesis (b) and (c) of theorem 4.1 (cf. [M]).

Since $dim(sing(M_{\mathbb{C}})) \leq 2n - 4$ there exists a non-empty open set $\mathcal{U}$ of the grassmanian $Gr(4, 2n)$, of 4-planes in $\mathbb{C}^{2n}$ through 0, such that if $E \in \mathcal{U}$ then $0 \in E$ is an isolated singularity of $F_{\mathbb{C}}|_E$ and the function $E \in \mathcal{U} \mapsto \mu(F_{\mathbb{C}}|_E, 0) \in \mathbb{N}$ is constant, where μ denotes the Milnor's number. The invariance of the Milnor's number implies that we can find $0 < r_o < \rho$ and a non empty open set $\mathcal{B} \subset \mathcal{U}$ with the following property (see [L -R]) :

(i). For any $E \in \overline{\mathcal{B}}$ then $F_{\mathbb{C}}|_E$ satisfies the hypothesis of theorem 4.1 in the ball $B_{r_o} \cap E$ of E.

By theorem 4.1, there exists $\epsilon > 0$ such that $G|_{E \cap V}$ can be extend holomorphically to the ball $E \cap B_\epsilon$, for all $E \in \overline{\mathcal{B}}$. This implies that G can be extended to the open set $W := \bigcup_{E \in \mathcal{B}} (E \cap B_\epsilon)$. On the other hand, the open set W contains a Hartogs domain H_1 such that its holomorphic closure $\hat{H}_1$ contains $0 \in \mathbb{C}^{2n}$. In fact, if we fix $E_o \in \mathcal{B}$ then the sphere $S := \{z \in E_o \,|\, |z| = \epsilon/2\}$ is contained in the open set W of $\mathbb{C}^{2n}$. Consider coordinates $z = (x, y) \in \mathbb{C}^4 \times \mathbb{C}^{2n-4}$ such that $E_o = (y = 0)$ and $S = \{(x, 0) \,|\, |x| = \epsilon/2\}$. By compacteness of S, there exists $0 < \delta < \epsilon/4$ such that $H_1 := \{(x, y) \,|\, \epsilon/2 - \delta < |x| < \epsilon/2 + \delta \text{ and } |y| < \delta\} \subset W$. Since $dim(E_o) = 4 \geq 2$, H_1 is a Hartogs domain with holomorphic closure $\hat{H}_1 = \{(x, y) \,|\, |x| < \epsilon/2 + \delta$ and $|y| < \delta\}$. Hence, G can be extended to $\hat{H}_1$ and we are done. Call the extension $f_{\mathbb{C}}$. It follows from (20) that $\frac{\partial f_{\mathbb{C}}}{\partial w_j} \equiv 0$ for all $j = 1, ..., n$. This finishes the proof of lemma 4.2. □

Let us finish the proof of theorem 2. Since $\frac{\partial f_{\mathbb{C}}}{\partial w_j} \equiv 0$, $1 \leq j \leq n$, we can consider $f_{\mathbb{C}}$ as a germ in $\mathcal{O}_n$: $f_{\mathbb{C}}$ is the complexification of $f(z) := f_{\mathbb{C}}(z, w)$. The germ of f at 0 is an extension of the germ of $g|_{M^*}$. In particular, the foliation defined by $df = 0$ in $(\mathbb{C}^n, 0)$ is tangent to the Levi-flat M. Moreover, $M \subset (\mathcal{I}m(f) = 0)$, because $f|_M = g|_M$ is real valued. Now, $\mathcal{I}m(f)$ is irreducible in $\mathcal{A}_{n\mathbb{R}}$ by lemma 2.2, which implies $F = U.\mathcal{I}m(f)$, where U is an unit. Hence, $M = (\mathcal{I}m(f) = 0)$ and this finishes the proof of theorem 2.

4.2. **Proof of corollary 1.** Let $M = F^{-1}(0) \subset (\mathbb{C}^n, 0)$ be a Levi-flat, where $n \geq 2$ and $F(z) = \mathcal{R}e(z_1^2 + ... + z_n^2) + h.o.t.$. We want to prove that there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(x_1^2 + ... + x_n^2) = 0)$.

The idea is to use theorem 2 to prove that there exists a germ $f \in \mathcal{O}_n$ such that the foliation $\mathcal{F}$ defined by $df = 0$ is tangent to M and $M = (\mathcal{R}e(f) = 0)$. The foliation $\mathcal{F}$ can be viewed as an extension to a neighborhood of $0 \in \mathbb{C}^n$ of the Levi foliation $\mathcal{L}$ on M^*.

Let us assume for a moment that $M = (\mathcal{R}e(f) = 0)$ and conclude the proof. Without loss of generality, we can suppose that f is not a power in $\mathcal{O}_n$, so that $\mathcal{R}e(f)$ is irreducible by lemma 2.2. This implies that $\mathcal{R}e(f) = U.F$, where $U \in \mathcal{A}_{\mathbb{R}\,n}$ and $U(0) \neq 0$. If the Taylor series of f is $f = \sum_{j \geq 2} f_j$, where f_j is a homogeneous of degree j, $j \geq 2$, then

$$\mathcal{R}e(f_2) = j_0^2(\mathcal{R}e(f)) = j_0^2(U.F) = U(0).\mathcal{R}e(z_1^2 + ... + z_n^2) \implies$$

$f_2 = U(0).(z_1^2 + ... + z_n^2)$. Therefore, by Morse lemma there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $f \circ \phi^{-1}(x) = x_1^2 + ... + x_n^2$, so that $M = (\mathcal{R}e(x_1^2 + ... + x_n^2) = 0)$ and we are done.

Write $F(z) = \mathcal{R}e\left(\sum_{j=1}^n z_j^2\right) + \sum_{|\mu|+|\nu| \geq 3} F_{\mu\,\nu}\, z^\mu\, \overline{z}^\nu$, so that

$$F_{\mathbb{C}}(z, w) = \frac{1}{2}\left(\sum_{j=1}^n z_j^2 + \sum_{j=1}^n w_j^2\right) + \sum_{|\mu|+|\nu| \geq 3} F_{\mu\,\nu}\, z^\mu\, w^\nu \ .$$

Let us compute $sing(M_{\mathbb{C}})$ and $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with $\alpha = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j := \sum_{j=1}^n (z_j + A_j(z, w))\, dz_j$ and $\beta = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j := \sum_{j=1}^n (w_j + B_j(z, w))\, dw_j$, where A_j and B_j have order ≥ 2 at 0, $1 \leq j \leq n$. This implies that the sets

$$X_1 := \bigcap_{j=1}^n \left(\frac{\partial F_{\mathbb{C}}}{\partial z_j} = 0\right) \text{ and } X_2 := \bigcap_{j=1}^n \left(\frac{\partial F_{\mathbb{C}}}{\partial w_j} = 0\right)$$

have codimension n in $(\mathbb{C}^{2n}, 0)$ and that $sing(M_{\mathbb{C}}) = X_1 \cap X_2 = \{0\}$.

On the other hand, $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so

$$\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*} = (\eta_{\mathbb{C}} + i\, dF_{\mathbb{C}})|_{M_{\mathbb{C}}^*} = 2\, i\, \alpha|_{M_{\mathbb{C}}^*} = -2\, i\, \beta|_{M_{\mathbb{C}}^*} \ . \tag{21}$$

In particular, $\alpha|_{M_{\mathbb{C}}^*}$ and $\beta|_{M_{\mathbb{C}}^*}$ define $\mathcal{L}_{\mathbb{C}}$.

Set $M_1 = \{(z, w) \in M_{\mathbb{C}} \mid \frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0$ for some $j = 1, ..., n\}$ and $M_2 = \{(z, w) \mid \frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0$ for some $j = 1, ..., n\}$. Note that $M_{\mathbb{C}}^* = M_1 \cup M_2$. We assert that

$$sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) = (X_1 \cap M_1) \cup (X_2 \cap M_2) \implies cod_{M_{\mathbb{C}}^*}(sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) = n \ .$$

Let us prove that $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) \cap M_1 = X_1 \cap M_1$. Since $\alpha|_{M_1}$ defines $\mathcal{L}_{\mathbb{C}}|_{M_1}$, we get $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) \cap M_1 \supset X_1 \cap M_1$. As the reader can check, the relation $dF_{\mathbb{C}} \wedge dz_1 \wedge ... \wedge dz_n = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial w_j}\, dw_j \wedge dz_1 \wedge ... \wedge dz_n$ implies that $(dz_1 \wedge ... \wedge dz_n)|_{M_1} \neq 0$. This fact, (21) and the relation

$$\left.\left(\frac{\partial F_{\mathbb{C}}}{\partial z_1}.\, dz_1 \wedge ... \wedge dz_n\right)\right|_{M_1} = (\alpha \wedge dz_2 \wedge ... \wedge dz_n)\,|_{M_1}$$

imply that if $p \in M_1$ and $\alpha(p)|_{T_p M_1} = 0$ then $\frac{\partial F_{\mathbb{C}}}{\partial z_1}(p) = 0$. Similarly, $\frac{\partial F_{\mathbb{C}}}{\partial z_j}(p) = 0$ for all $2 \leq j \leq n$. Hence, $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) \cap M_1 = X_1 \cap M_1$. Similarly, $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) \cap M_2 = X_2 \cap M_2$.

As a consequence, if $n \geq 3$ we can use part (a) of theorem 2 to guarantee the existence of f such that $M = (\mathcal{R}e(f) = 0)$. However, if $n = 2$ we have to work more to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral, in order to use (b) of theorem 2.

Proof in the case $n = 2$. We are going to prove directly that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral. The idea is to blow-up once the origin of $\mathbb{C}^4$, $\pi\colon (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$, and prove that the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} := \pi^*(\mathcal{L}_{\mathbb{C}})$ on the strict transform $\tilde{M}_{\mathbb{C}}$ of $M_{\mathbb{C}}$ by π, has a non-constant holomorphic first integral. Let us state the main result that we will use.

Consider the more general situation of a germ at $0 \in \mathbb{C}^2$ of analytic Levi-flat $M = F^{-1}(0)$, not necessarilly with first integral, where F is irreducible in $\mathcal{A}_{2\mathbb{R}}$. Let $F_{\mathbb{C}}$, $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$ and $M_{\mathbb{C}}^*$ be as before.

We will assume that the power series that defines $F_{\mathbb{C}}$ converges in a neighborhood of $\overline{\Delta} = \{((z,w) \in \mathbb{C}^4 \,|\, |z|\,,\,|w| \leq 1\} := \overline{B} \times \overline{B}$, so that $F(z) = F_{\mathbb{C}}(z, \overline{z})$ for all $|z| \leq 1$. Set $V := M_{\mathbb{C}}^* \backslash sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ and denote by L_p the leaf of $\mathcal{L}_{\mathbb{C}}$ through p, where $p \in V$.

Lemma 4.3. *In the above situation, for any $p = (z_o, w_o) \in V$ the leaf L_p is closed in $M_{\mathbb{C}}^*$.*

Proof. Consider the holomorphic vector fields X and Y on B defined by

$$X = \frac{\partial F_{\mathbb{C}}}{\partial z_2}\,\partial_{z_1} - \frac{\partial F_{\mathbb{C}}}{\partial z_1}\,\partial_{z_2} \text{ and } Y = \frac{\partial F_{\mathbb{C}}}{\partial w_2}\,\partial_{w_1} - \frac{\partial F_{\mathbb{C}}}{\partial w_1}\,\partial_{w_2}\ .$$

Since $X(F_{\mathbb{C}}) = Y(F_{\mathbb{C}}) \equiv 0$, they are tangent to the levels of $F_{\mathbb{C}}$ and in particular to $M_{\mathbb{C}}$. Denote by $\mathcal{G}$ and $\mathcal{H}$ the foliations by curves defined by X and Y on $M_{\mathbb{C}}^*$, respectively. We need some facts.

I. $X|_{M_{\mathbb{C}}^*}$ and $Y|_{M_{\mathbb{C}}^*}$ are tangent to $\mathcal{L}_{\mathbb{C}}$. In particular, if we denote by L_q^g and L_q^h the leaves of $\mathcal{G}$ and $\mathcal{H}$ through $q \in V$ then $L_q^g \subset L_q$ and $L_q^h \subset L_q$.

This follows from the fact that $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M_{\mathbb{C}}^*} = -\beta|_{M_{\mathbb{C}}^*}$ and

$$\begin{array}{lll} i_X(\alpha) \equiv 0 & \Longrightarrow & \text{if } p \in V \text{ then } X(p) \in T_p\mathcal{L}_{\mathbb{C}} \\ i_Y(\beta) \equiv 0 & \Longrightarrow & \text{if } p \in V \text{ then } Y(p) \in T_p\mathcal{L}_{\mathbb{C}} \end{array} \tag{22}$$

where $T_p\mathcal{L}_{\mathbb{C}} := T_p\,L_p$.

II. X and Y are linearly independent along V. In particular, for any $p \in V$ the leaves L_p^g and L_p^h intersect transversely in L_p at p : $T_p\,L_p = T_p\,L_p^g \oplus T_p\,L_p^h$.

We have seen in the proof of theorem 2 that for any $p \in V = M_{\mathbb{C}}^* \setminus sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ then, either $\frac{\partial F_{\mathbb{C}}}{\partial z_1}(p) \neq 0$, or $\frac{\partial F_{\mathbb{C}}}{\partial z_2}(p) \neq 0$, which implies that $X(p) \neq 0$. Similarly, $Y(p) \neq 0$. Therefore, $X(p) \wedge Y(p) \neq 0$, and the vector fields X and Y are linearly independent along V.

III. For any $p = (z_o, w_o) \in V$ the leaf L_p^g (resp. L_p^h) is contained in $A_p^g := \{(z, w_o) \in V \,|\, F_{\mathbb{C}}(z, w_o) = 0\}$ (resp. $A_p^h := \{(z_o, w) \in V \,|\, F_{\mathbb{C}}(z_o, w) = 0\}$). In particular, L_p^g and L_p^h are closed in $V = M_{\mathbb{C}}^* \setminus sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$.

It follows from $X(F) = X(w_1 - w_{o1}) = X(w_2 - w_{o2}) = 0$ that $L_p^g \subset A_p^g$. Let S_p^g be the irreducible component of A_p^g which contains p. For any $q \in V$ we have that, either $\frac{\partial F_{\mathbb{C}}}{\partial z_1}(q) \neq 0$, or $\frac{\partial F_{\mathbb{C}}}{\partial z_2}(q) \neq 0$. This implies that $dim_{\mathbb{C}}(S_p^g) = 1$. On the other hand, $L_p^g \subset S_p^g$ and $dim_{\mathbb{C}}(L_p^g) = 1$, so that $L_p^g = S_p^g$, by the definition of leaf. Hence, L_p^g is closed in V. Similarly, L_p^h is closed in V.

Fix $p = (z_o, w_o) \in V$ and set $N_p := \bigcup_{q \in L_p^h} L_q^g \subset V$. We assert that N_p is closed in V and $N_p = L_p$.

1. N_p is closed in V. Let $(p_n = (z_n, w_n))_{n \geq 1}$ be a sequence in N_p such that $\lim\limits_{n\to\infty} p_n = q = (z', w') \in V$. Recall that we are working in $\Delta = B \times B$. In particular, $z', w' \in B$. By remark III, we get $(z_o, w_n) \in L^h_p$, for all $n \geq 1$, and $(z_o, w_n) \to (z_o, w')$. Again, by remark III, we have $(z', w') \in L^g_{(z_o,w')}$. Hence, $q \in N_p$.

2. $L_p = N_p$. Remark I implies that $N_p \subset L_p$. Since N_p is closed in V, it is also closed in $L_p \subset V$. On the other hand, it follows from remark II that N_p is open in L_p. Hence, $N_p = L_p$, because L_p is connected, . □

Let us finish the proof of the case $n = 2$. Let $F \in \mathcal{A}_{2\,\mathbb{R}}$ be such that $F(z) = \mathcal{R}e(z_1^2 + z_2^2) + h.o.t.$ and $F^{-1}(0)$ is a Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(z, w) = \frac{1}{2}\,(z_1^2 + z_2^2 + w_1^2 + w_2^2) \,+\, h.o.t\ .$$

Since $z_1^2 + z_2^2 = (z_2 + i\,z_1)(z_2 - i\,z_1)$, after the linear change of variables $x = \sqrt{2}\,(z_2 + i\,z_1)$, $y = \sqrt{2}\,(z_2 - i\,z_1)$, we will assume that $F(x, y) = 2\,\mathcal{R}e(x.y) + h.o.t. = x.y + \overline{x}.\overline{y} + h.o.t.$, with complexification of the form

$$F_{\mathbb{C}}(x, y, z, w) = x.y + z.w + \sum_{j+k+\ell+m\geq 3} f_{j\,k\,\ell\,m}\, x^j.y^k.z^\ell.w^m := x.y + z.w + R(x, y, z, w)\ .$$

We take the divisor $\mathbb{P}^3$ of the blow-up $\pi\colon (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$ with homogeneous coordinates $[x : y : z : w]$, $(x, y, z, w) \in \mathbb{C}^4 \setminus \{0\}$. The intersection of $\tilde{M}_{\mathbb{C}}$ with the divisor $\mathbb{P}^3$ is the quadric $Q := \{[x : y : z : w] \,|\, xy + zw = 0\}$, which is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Consider for instance the chart $(W, (t, u, v, w))$ of $\tilde{\mathbb{C}}^4$ where $\pi(t, u, v, w) = (t.w, u.w, v.w, w) = (x, y, z, w)$. We have

$$F_{\mathbb{C}} \circ \pi(t, u, v, w) = w^2(t.u + v + w\,R_1(t, u, v, w))\ ,$$

where $R_1(t, u, v, w) = R(t.w, u.w, v.w, w)/w^2$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W = (t.u + v + w.R_1(t, u, v, w) = 0) \implies Q \cap W = (w = v + t.u = 0)\ .$$

On the other hand, as we have seen in (21), The foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = \frac{\partial F_{\mathbb{C}}}{\partial x}\,dx + \frac{\partial F_{\mathbb{C}}}{\partial y}\,dy = \left(y + \frac{\partial R}{\partial x}\right)\,dx + \left(x + \frac{\partial R}{\partial y}\right)\,dy\ .$$

In particular, we get

$$\pi^*(\alpha) = w\,(u.w\,dt + t.w\,du + 2\,t.u\,dw + w.\theta_1)\ ,$$

where $\theta_1 = w\,[A\,dt + B\,du] + C\,dw$, $A = \left(\frac{\partial R}{\partial x} \circ \pi\right)/w^2$, $B = \left(\frac{\partial R}{\partial y} \circ \pi\right)/w^2$ and $C = \left(t\frac{\partial R}{\partial x} \circ \pi + u\frac{\partial R}{\partial x} \circ \pi\right)/w^2$.

It follows that $\mathcal{L}_{\mathbb{C}}$ is defined in this chart by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\alpha_1 = u.w\,dt + t.w\,du + 2\,t.u\,dw + w.\theta_1\ . \tag{23}$$

Note that (23) implies that Q is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. In particular, $S := Q \setminus sing(\tilde{\mathcal{L}}_{\mathbb{C}})$ is a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$. We assert that S is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

In fact, $sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap Q$ is the union of four lines of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$: two "vertical" rules of the form $V_j := \{a_j\} \times \mathbb{P}^1$, $a_1 \neq a_2$, and two "horizontal" rules $H_j := \mathbb{P}^1 \times \{b_j\}$, $b_1 \neq b_2$.

In the chart $(W, (t, u, v, w))$ we can see two of these lines (see (23)), one horizontal and one vertical : $sing(\mathcal{L}_{\mathbb{C}}) \cap W \cap Q = H_1 \cup V_1$, where $H_1 = (w = t = v = 0)$ and $V_1 = (w = u = v = 0)$.

In the chart $(Z,(t_1,u_1,v_1,z))$, where $\pi(t_1,u_1,v_1,z) = (u_1.z, t_1.z, z, v_1.z) = (x,y,z,w)$, we can see the other two rules of $sing(\tilde{\mathcal{L}}_{\mathbb{C}})$. The reader can check that $Q \cap Z = (z = v_1 + t_1.u_1 = 0)$ and $\pi^*(\alpha) = z\,\alpha_2$, where

$$\alpha_2 = t_1.z\,du_1 + u_1.z\,dt_1 + 2\,t_1.u_1\,dz + z.\theta_2 \ ,$$

with θ_2 holomorphic. This implies that $sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap Q \cap Z = H_2 \cup V_2$, where $H_2 = (z = t_1 = v_1 = 0)$ is an "horizontal" rule of Q, $H_2 \neq H_1$, and $V_2 = (z = u_1 = v_1 = 0)$ is a "vertical" rule, $V_2 \neq V_1$.

Let us prove that $\tilde{\mathcal{L}}_{\mathbb{C}}$ has a non-constant holomorphic first integral. Fix a point $p_o \in S$ and a transversal Σ to S. For instance, in the chart $(W,(t,u,v,w))$ take $p_o = (1,1,-1,0)$ and the section $\Sigma = \{(1,1,-1,w)\,|\,w \in \mathbb{C}\}$, parametrized by w. Call G the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ. The fundamental group $\Pi_1(S,p_o)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is generated by the curves $\delta_1(\theta) = (e^{i\theta}, 1, -e^{i\theta}, 0)$ and $\delta_2(\theta) = (1, e^{i\theta}, -e^{i\theta}, 0)$, $\theta \in [0,2\pi]$, $\Pi_1(S,p_o) =< [\delta_1],[\delta_2] >$. Therefore $G =< f_1, f_2 >$, where f_1 corresponding to $[\delta_1]$ and f_2 to $[\delta_2]$. We get from (23) that $f_1'(0) = f_2'(0) = -1$, so that $f_1(w) = -w + w^2\,r_1(w)$ and $f_2(w) = -w + w^2\,r_2(w)$. Moreover, $[f_1,f_2] := f_1^{-1} \circ f_2^{-1} \circ f_1 \circ f_2 = id$, because $\Pi_1(S,p_o)$ is abelian. Since $f_1'(0) = f_2'(0) = -1$ we get $G' := \{g'(0)\,|\,g \in G\} = \{1,-1\}$. We assert that the homomorphism $\psi\colon G \to G'$ defined by $\psi(g) = g'(0)$ is an isomorphism.

It is sufficient to prove that ψ is injective. Let $g \in G$ be such that $g'(0) = 1$. Suppose by contradiction that $g \neq id$. This implies that $g(w) = w(1 + a.w^k) + o(w^{k+2})$, where $a \neq 0$. In this case, it follows from [L] that any pseudo-orbit of g accumulates at $0 \in (\Sigma,0)$. But this implies that $\tilde{\mathcal{L}}_{\mathbb{C}}$ has non-closed leaves, which contradicts lemma 4.3.

In particular, we get $G =< f_1 >$ and $f_1^2 = id$. It follows that G is linearizable : in some holomorphic coordinate system z of $(\Sigma,0)$ we have $f_1(z) = f_2(z) = -z$. The function $H(z) = z^2 \in \mathcal{O}_1$ satisfies $H \circ f_1 = H \circ f_2 = H$. By [M-M] it can be extended to a non-constant holomorphic first integral, say $\tilde{h}$, of $\tilde{\mathcal{L}}_{\mathbb{C}}$, defined in some neighborhood of Q in $\tilde{M}_{\mathbb{C}}$. This finishes the proof of corollary 1. □

5. Appendix : an example.

The aim of this section is to give a class of examples of germs of non-dicritical foliations $\mathcal{F}$ on $(\mathbb{C}^2,0)$ with the following properties :

(I). $\mathcal{F}$ has no non-constant holomorphic first integral.

(II). If $\pi\colon(\tilde{\mathbb{C}}^2,D) \to (\mathbb{C}^2,0)$ is a blow-up of $0 \in \mathbb{C}^2$ then the foliation $\tilde{\mathcal{F}} := \pi^*(\mathcal{F})$ is tangent to an analytic real Levi-flat hypersurface $\tilde{M}$ containing $\pi^{-1}(0)$.

As consequence, the projection $M := \pi(\tilde{M})$ is a real hypersurface invariant by $\mathcal{F}$, which is analytic outside the origin but not at the origin.

Denote by $\mathcal{P}_k \subset \mathbb{C}[x,y]$ the set of homogeneous polynomials of degree k. Let $P(x,y) = x.y.(y-x).(y-b.x)$, where $b \neq 0,1$, and $Q(x,y) = a_0\,y^4 + a_1\,x.y^3 + a_2\,x^2.y^2 + a_3\,x^3.y + a_4\,x^4$. We will consider $Q \in \mathcal{P}_4$ as a parameter. Let $\mathcal{F}_Q$ be the foliation defined by $\omega_Q = 0$, where

$$\omega_Q = dP(x,y) + 2.Q(x,y)(x\,dy - y\,dx)$$

Proposition 5.1. *There are $\mathcal{I} \subset \mathcal{I}_{\mathbb{R}} \subset \mathcal{P}_4$, where $\mathcal{I}$ is a $\mathbb{C}$-linear subspace with $dim_{\mathbb{C}}(\mathcal{I}) = 3$ and $\mathcal{I}_{\mathbb{R}}$ is a real analytic quadric of real codimension one, such that :*

(a). *$\mathcal{F}_Q$ has a non-constant holomorphic first integral if, and only if, $Q \in \mathcal{I}$.*
(b). *If $Q \in \mathcal{I}_{\mathbb{R}} \setminus \mathcal{I}$ then $\mathcal{F}_Q$ satisfies (I) and (II).*

Proof. The resolution of singularities of $\mathcal{F}_Q$, involves just one blowing-up $\pi\colon (\tilde{\mathbb{C}}^2, D) \to (\mathbb{C}^2, 0)$. Set $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_Q := \pi^*(\mathcal{F}_Q)$. For instance, in the chart $\pi(t,x) = (x, t.x) = (x,y)$, we have $\pi^*(\omega_Q) = d(x^4.p(t)) + 2.x^6.q(t).dt = x^3.\eta$, where

$$\eta = 4\,p(t)\,dx + (p'(t).x + 2.q(t).x^3)dt\ ,\ p(t) := P(1,t)\ ,\ q(t) := Q(1,t)\ . \tag{24}$$

The differential equation $\eta = 0$ is of Bernouilli type and defines $\tilde{\mathcal{F}}$ in this chart. It has four singularities in D. In the above chart we can see three of these singularities $p_0 := (0,0)$, $p_1 := (1,0)$ and $p_2 := (b,0)$. The other one, p_3, is the point of infinity of D. Set $sing(\tilde{\mathcal{F}}) := \{p_0, ..., p_3\}$ and $D^* = D \setminus sing(\tilde{\mathcal{F}})$. Let

$$X := 4p(t)\,\partial_t - (p'(t).x + 2.q(t).x^3)\partial_x$$

be the vector field dual of η. Note that the eigenvalues of $DX(p_j)$ are $4p'(b_j) \neq 0$ and $-p'(b_j)$. The Camacho-Sad index of $\tilde{\mathcal{F}}$ at p_j with respect to D is $CS(\tilde{\mathcal{F}}, D, p_j) = -1/4$, $0 \le j \le 3$ (cf. [C-S]).

The main fact about η is that it admits a multivalued integrating factor : if we set $g := x^{-2}.p^{-1/2}$ then

$$\frac{\pi^*(\omega)}{x^6.p(t)^{3/2}} = \frac{d(x^4.p(t))}{x^6.p^{3/2}} + \frac{2.q(t)}{p(t)^{3/2}}dt = \frac{d(g^{-2})}{g^{-3}} + \frac{2.q(t)}{p(t)^{3/2}}dt = -2\,dg + \frac{2.q(t)}{p(t)^{3/2}}dt\,.$$

In particular, $\eta = 0$ is equivalent to the multivalued equation

$$dg - \frac{q(t)}{p(t)^{3/2}}dt = d\left(\frac{1}{x^2.p(t)^{1/2}}\right) - \frac{q(t)}{p(t)^{3/2}}dt = 0\ . \tag{25}$$

In order to study the differential equation (25) we consider a ramified covering of topological degree two. Let E be the bundle of rank two over $\mathbb{P}^1$ covered by two charts $(E_1 \simeq \mathbb{C}^3, (t,w,x))$ and $(E_2 \simeq \mathbb{C}^3, (s,z,y))$, where $s = 1/t$, $z = w/t^2$ and $y = tx$. Let $S \subset E$ be the elliptic curve given in the chart E_1 by $(w^2 - P(1,t) = x = 0)$ and in the chart E_2 by $(z^2 - P(s,1) = y = 0)$. We can define a line bundle $\Pi\colon L \to S$ by

$$\begin{cases} L \cap E_1 = \{(t,w,x)\,|\,w^2 = P(1,t)\} & ,\quad \Pi(t,w,x) = (t,w) \quad \text{on} \quad L\cap E_1 \\ L \cap E_2 = \{(s,z,y)\,|\,z^2 = P(s,1)\} & ,\quad \Pi(s,z,y) = (s,z) \quad \text{on} \quad L\cap E_1 \end{cases}$$

a holomorphic map $\Phi\colon L \to \tilde{\mathbb{C}}^2$ by

$$\begin{cases} \Phi(t,w,x) = (t,x) \quad \text{on} \quad L \cap E_1 \\ \Phi(s,z,y) = (s,y) \quad \text{on} \quad L \cap E_2 \end{cases}$$

and a ramified covering $\phi\colon S \to D \simeq \mathbb{P}^1$ by $\phi = \Phi|_S$. Of course, the diagram below commutes

$$\begin{array}{ccccc} & L & \overset{\Phi}{\to} & \tilde{\mathbb{C}}^2 \\ \Pi & \downarrow & & \downarrow \\ & S & \underset{\phi}{\to} & D \end{array}$$

The map ϕ is a ramified covering of degree two with four ramification points. The ramification points are $q_0 = (0,0,0)$, $q_1 = (1,0,0)$ and $q_2 = (b,0,0)$ in the chart E_1 and $q_3 = (0,0,0)$ in the chart E_2. Note that $\phi(q_j) = p_j$, $0 \le j \le 3$. The map Φ is also of degree two and ramifies along the fibers $\Pi^{-1}(q_j)$, $0 \le j \le 3$.

Let $\mathcal{G} = \Phi^*(\tilde{\mathcal{F}})$. It follows from (25) that the foliation $\mathcal{G}$ can be defined in the chart E_1 by the differential equation $\Theta|_{L\cap E_1} = 0$, where

$$\Theta = d\left(\frac{1}{x^2.w}\right) - \frac{Q(1,t)}{w^3}dt \ .$$

As the reader can check, the form Θ extends to $L \cap E_2$ as $d\left(\frac{1}{y^2.z}\right) + \frac{Q(s,1)}{z^3}ds$. Note that we can write $\Theta = dG - \alpha$, where $G|_{E_1} = 1/x^2.w$, $G|_{E_2} = 1/y^2.z$ and $\alpha = \alpha_Q$ is the meromorphic 1-form on S given by $\alpha_Q|_{S\cap E_1} = \frac{Q(1,t)}{w^3}dt$ and $\alpha_Q|_{S\cap E_2} = -\frac{Q(s,1)}{z^3}ds$. Let us state some remarks.

(i). S is $\mathcal{G}$ invariant and $sing(\mathcal{G}) \cap S = \{q_0, ..., q_3\}$. In particular, $S^* = S \setminus \{q_0, ..., q_3\}$ is a leaf of $\mathcal{G}$.
(ii). If $Q \not\equiv 0$ then the poles of α are $q_0, ..., q_3$.
(iii). The order of q_j as a pole of α is ≤ 2 and $Res(\alpha, q_j) = 0$ for all $j = 0, ..., 3$.
(iv). For any $j = 0, ..., 3$, $\mathcal{G}$ has a local holomorphic first integral in a neighborhood of q_j of the type $u.v^2$, where $(W,(u,v))$ is a local chart with $S \cap W = (v = 0)$ and $\Pi^{-1}(q_j) \cap W = (u = 0)$.

We leave the proof of (i) and (ii) to the reader. Let us prove (iii) and (iv).

Fix for instance the pole $q_0 = (0,0,0)$. Since $p'(0) \neq 0$, we can solve locally the equation $w^2 = p(t) = P(1,t)$ as $t = h(w^2) = w^2.u(w^2)$, where $u(0) \neq 0$. Therefore, we can write

$$\alpha = \frac{Q(1,h(w^2))}{w^3}d(h(w^2)) = 2\frac{Q(1,h(w^2)).h'(w^2)}{w^2}dw := \frac{g(w^2)}{w^2}dw \ .$$

By considering the Taylor series of g at 0, it can be proved that there exists $\psi \in \mathcal{O}_1$ such that

$$\frac{g(w^2)}{w^2}dw = d\left(\frac{\psi(w^2)}{w}\right)$$

and this implies (iii). In particular, we get

$$\Theta = d\left(\frac{1}{x^2.w} - \frac{\psi(w^2)}{w}\right) \implies x^2\left(\frac{w}{1 - x^2.\psi(w^2)}\right) \tag{26}$$

is a local holomorphic first integral of $\mathcal{G}$. Since $(w,x) \mapsto (w/(1 - x^2.\psi(w^2),x) = (u,x)$ is a local biholomorphism, we get (iv).

Given $\gamma \in H_1(S^*,\mathbb{Z})$ define $per(\alpha_Q,\gamma) = \int_\gamma \alpha_Q$. We can write the first homology group of S^* as $H_1(S^*,\mathbb{Z}) =< [\delta_1], [\delta_2], [\gamma_0], ..., [\gamma_3] >$ where δ_1 and δ_2 are generators of $H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^2$ and γ_j is a simple cicle going around q_j once, $0 \leq j \leq 3$. It follows from (iii) that $per(\alpha_Q,\gamma_j) = 0$, $0 \leq j \leq 3$. Define the linear map $Per\colon \mathcal{P}_4 \to \mathbb{C}^2$ by

$$Per(Q) = (per(Q,\delta_1), per(Q,\delta_2)) \ .$$

Consider also the linear map $\sigma\colon \mathcal{P}_2 \to \mathcal{P}_4$ given by

$$\sigma(H) = \frac{1}{4}\left(\frac{\partial P}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial H}{\partial x}\right) \ .$$

Lemma 5.1. *The following sequence is exact :*

$$0 \to \mathcal{P}_2 \overset{\sigma}{\to} \mathcal{P}_4 \overset{Per}{\to} \mathbb{C}^2 \to 0 \ . \tag{27}$$

Proof. We will work in the chart E_1. Let us remark first that, Euler's identity for P and H implies that

$$\sigma(H) = \frac{1}{x}\left(P\frac{\partial H}{\partial y} - \frac{1}{2}\,H\frac{\partial P}{\partial y}\right) = \frac{1}{y}\left(\frac{1}{2}\,H\frac{\partial P}{\partial x} - P\frac{\partial H}{\partial x}\right) \ .$$

In particular, we have $\sigma(H)(1,t) = p(t).h'(t) - \frac{1}{2}\,h(t).p'(t)$, where $h(t) = H(1,t)$. Suppose by contradiction that $\sigma(H) = 0$, but $h(t) = H(1,t) \not\equiv 0$. This would imply that $p'/p = 2.h'/h$, which is equivalent to $p = c.h^2$, $c \in \mathbb{C}^*$. This is impossible because p is irreducible. Therefore, $H = 0$ and σ is injective. In particular, $dim_{\mathbb{C}}(Im(\sigma)) = dim_{\mathbb{C}}(\mathcal{P}_2) = 3$.

Consider the divisor Δ on S given by $\Delta = \sum_{j=0}^{2}(q_j)$ and set

$$\begin{cases} \mathcal{M}_\Delta = \{g \,|\, g \text{ is meromorphic on } S \text{ and } (g)_\infty \leq \Delta\} \cup \{0\} \\ \Omega_{2\Delta} = \{\omega \,|\, \omega \text{ is meromorphic form on } S \text{ and } (\omega)_\infty \leq 2\Delta\} \cup \{0\} \\ \Omega_1 = \{\omega \in \Omega_{2\Delta} \,|\, Res(\omega, q_j) = 0 \ , \ 0 \leq j \leq 3\} \end{cases} ,$$

where $(.)_\infty$ denotes the pole divisor. Some remarks :

(v). The Riemann-Roch theorem implies that $dim_{\mathbb{C}}(\mathcal{M}_\Delta) = 4$ and $dim_{\mathbb{C}}(\Omega_{2\Delta}) = 8$.

(vi). $d(\mathcal{M}_\Delta) \subset \Omega_{2\Delta}$, where $d\colon \mathcal{M}_\Delta \to \Omega_{2\Delta}$ is the differential. In particular, $dim_{\mathbb{C}}(d(\mathcal{M}_\Delta)) = 3$, because $dim_{\mathbb{C}}(ker(d)) = 1$.

(vii). Remark (iii) implies that $\alpha_Q \in \Omega_1$ for all $Q \in \mathcal{P}_4$. Therefore, we will consider α as a linear map $\alpha\colon \mathcal{P}_4 \to \Omega_1 \subset \Omega_{2\Delta}$. Note that α is injective.

Let us prove that $Im(\sigma) = ker(Per)$.

Define $\delta\colon \mathcal{P}_2 \to \Omega_{2\Delta}$ by $\delta(H) = d\left(\frac{H(1,t)}{w}|_S\right)$ (in the chart E_1). An easy computation shows that

$$\delta(H) = \frac{\sigma(H)(1,t)}{w^3}\,dt = \alpha_{\sigma(H)} \implies \alpha_{\sigma(H)} \text{ is exact} \implies \sigma(H) \in ker(Per) \ .$$

This implies that $Im(\sigma) \subset ker(Per)$ and that δ is injective. On the other hand, if $Q \in ker(Per)$ then α_Q is exact, and so $\alpha(ker(Per)) \subset d(\mathcal{M}_\Delta)$. In particular, we get $Im(\delta) \subset \alpha(ker(Per)) \subset d(\mathcal{M}_\Delta)$. This implies that $dim_{\mathbb{C}}(\alpha(ker(Per)) = 3$. Since α is injective, we get $dim_{\mathbb{C}}(ker(Per)) = dim_{\mathbb{C}}(\alpha(ker(Per)) = 3$. Hence, $Im(\sigma) = ker(Per)$.

Let us prove that Per is surjective.

For each $j = 0,...,3$ consider a local chart (D_j, z_j) of S around q_j such that $z_j(q_j) = 0$. Given $\omega \in \Omega_{2\Delta}$ we can write $\omega|_{D_j} = \frac{a_j}{z_j^2} + \frac{Res(\omega,q_j)}{z_j} + g_j(z_j)$, where $g_j \in \mathcal{O}(D_j)$, $0 \leq j \leq 3$. Define $T\colon \Omega_{2\Delta} \to \mathbb{C}^8$ and $T_1\colon \Omega_{2\Delta} \to \mathbb{C}^4$ by :

$T(\omega) = (Res(\omega, q_0), ..., Res(\omega, q_3), a_0, ..., a_3)$ and $T_1(\omega) = (Res(\omega, q_0), ..., Res(\omega, q_3))$.

By the residue theorem, we have $T(\Omega_{2\Delta}) \subset \Sigma$ and $T_1(\Omega_{2\Delta}) \subset \Sigma_1$, where $\Sigma = \{(x_1, ..., x_8) \in \mathbb{C}^8 \,|\, \sum_{j=1}^4 x_j = 0\}$ and $\Sigma_1 = \{(x_1, ..., x_4) \in \mathbb{C}^4 \,|\, \sum_{j=1}^4 x_j = 0\}$. Moreover, $ker(T_1) = \Omega_1$ and $ker(T) =$ the set of holomorphic 1-forms on S, which has dimension one. This implies

$$dim_{\mathbb{C}}(Im(T)) = dim_{\mathbb{C}}(\Omega_{2\Delta}) - 1 = 7 \implies T(\Omega_{2\Delta}) = \Sigma \implies T_1(\Omega_{2\Delta}) = \Sigma_1$$

and

$$ker(T_1) = \Omega_1 \implies dim_{\mathbb{C}}(\Omega_1) = dim_{\mathbb{C}}(\Omega_{2\Delta}) - dim_{\mathbb{C}}(\Sigma_1) = 8 - 3 = 5 \ .$$

Consider now, the map $Per_1\colon \Omega_1 \to \mathbb{C}^2$ given by $Per_1(\omega) = \left(\int_{\delta_1}\omega, \int_{\delta_2}\omega\right)$. Note that $dim_{\mathbb{C}}(Im(Per)) = dim_{\mathbb{C}}(Im(Per_1))$. Since $ker(Per_1) = d(\mathcal{M}_\Delta)$, we get $dim_{\mathbb{C}}(Im(Per_1)) = dim_{\mathbb{C}}(\Omega_1) - dim_{\mathbb{C}}(d(\mathcal{M}_\Delta)) = 5-3 = 2$. □

Let $\mathcal{I} = ker(Per) = Im(\sigma)$ and $\mathcal{I}_{\mathbb{R}} = \{Q \,|\, per(\alpha_Q,\delta_1), per(\alpha_Q,\delta_2)$ are $\mathbb{R}$-linearly dependent$\}$. As a consequence of lemma 5.1, we get :

(viii). $\mathcal{I}$ is a complex linear subspace of $\mathcal{P}_4$ with $dim_{\mathbb{C}}(\mathcal{I}) = 3$.

(ix). $\mathcal{I}_{\mathbb{R}}$ is defined by

$$\mathcal{R}e(per(Q,\delta_1)).\mathcal{I}m(per(Q,\delta_2)) - \mathcal{R}e(per(Q,\delta_2)).\mathcal{I}m(per(Q,\delta_1)) = 0\,.$$

In particular it is a homogeneous real quadric and $dim_{\mathbb{R}}(\mathcal{I}_{\mathbb{R}}) = 9$.

Lemma 5.2. *The following properties are true :*

(a). *$\mathcal{F}_Q$ has a non-constant holomorphic first integral in a neighborhood of $0 \in \mathbb{C}^2$ if, and only if, $Q \in \mathcal{I}$.*

(b). *If $Q \in \mathcal{I}_{\mathbb{R}} \setminus \mathcal{I}$ then $\tilde{\mathcal{F}}_Q$ is tangent to a real analytic hypersurface $\tilde{M} \subset (\tilde{\mathbb{C}}^2, D)$.*

Proof. Let us prove (a). If $Q \in \mathcal{I}$ then $Q = \sigma(H)$ where $H \in \mathcal{P}_2$. Set $f = P/(1-H)^2$. Then

$$\frac{df}{f} = \frac{dP}{P} + \frac{2\,dH}{1-H} = \frac{1}{P(1-H)}\left[dP + (2P\,dH - H\,dP)\right]\ .$$

It follows from Euler's identity that $2PdH - HdP = 2\sigma(H)(y\,dx - x\,dy)$. Therefore, f is a first integral of $\mathcal{F}_Q$.

Conversely, suppose that $\mathcal{F}_Q$ has a non-constant holomorphic first integral $f \in \mathcal{O}_2$ such that $f(0) = 0$. The idea is to prove that α_Q is exact, which will imply that $Q \in Im(\sigma)$. Note that $F := f\circ\Phi\circ\pi$ is a holomorphic first integral of $\mathcal{G}$. This implies that the holonomy group G of the leaf S^* with respect to the foliation $\mathcal{G}$ is finite, say $\#(G) = m$. In particular, if γ is a cycle in $S^*\cap E_1$ then γ^m can be lifted by Π to a neighbour leaf as a closed path, say $\tilde{\gamma}$, and we can assume that $\tilde{\gamma} \subset E_1 \setminus K$, where $K := \cup_{j=0}^4 \Pi^{-1}(q_j) \cup S$. Recall that $\mathcal{G}$ is defined outside K by Θ. Since $\tilde{\gamma}$ is contained in a leaf of $\mathcal{G}$, we get

$$0 = \int_{\tilde{\gamma}} \Theta = \int_{\tilde{\gamma}} \left(d\left(\frac{1}{x^2.w}\right) - \alpha_Q\right) \implies \int_{\tilde{\gamma}} \alpha_Q = \int_{\tilde{\gamma}} d\left(\frac{1}{x^2.w}\right) = 0\ .$$

On the other hand, $\Pi\circ\tilde{\gamma} = \gamma^m$, so that the cycles γ^m and $\tilde{\gamma}$ are equivalent in $H_1(L\setminus \cup_{j=0}^3\Pi^{-1}(q_j),\mathbb{Z})$. Hence, $m\int_\gamma \alpha_Q = \int_{\tilde{\gamma}}\alpha_Q = 0$, which implies that α_Q is exact and $Q \in \mathcal{I}$.

Let us prove (b). Assume that $P_1 := per(Q,\delta_1)$ and $P_2 := per(Q,\delta_2)$ are $\mathbb{R}$-linearly dependent, but $(P_2,P_2) \neq (0,0)$. In this case, we get $P_1, P_2 \in \beta.\mathbb{R}$, for some β with $|\beta| = 1$. In particular, if $Q_1 = \beta^{-1}.Q$ then $per(Q_1,\delta_1), per(Q_1,\delta_2) \in \mathbb{R}$. As a consequence, the periods of the form α_{Q_1} are real and there exists a harmonic function f_1 on S^* such that $df_1 = \mathcal{I}m(\alpha_{Q_1}) = \frac{1}{2i}(\alpha_{Q_1} - \overline{\alpha_{Q_1}})$. Recall that $\Theta = dg - \alpha_Q$, where $g|_{E_1\cap L} = 1/x^2.w$ and $g|_{E_2\cap L} = 1/y^2.z$. If we define $g_1 := \beta^{-1}.g$ and $\Theta_1 = \beta^{-1}.\Theta$ then $\Theta_1 = dg_1 - \alpha_{Q_1}$. In particular, we get $\mathcal{I}m(\Theta_1) = d(\mathcal{I}m(g_1) - f_1)$. Therefore, the form $\mathcal{I}m(\Theta_1)$ is exact with primitive $\hat{F}_1 := \mathcal{I}m(g_1) - f_1$. For any $c \in \mathbb{R}$ the real analytic hypersurface $\hat{F}_1^{-1}(c)$ of $L\setminus K$ is a Levi-flat tangent to $\mathcal{G}$.

Denote by $\hat{M}_c^1$ the germ of $\hat{F}_1^{-1}(c)$ at S. Let us prove that $\hat{M}_c^1$ extends analytically to K. Without loss of generality, we will assume $\beta = 1$.

Extension to S^*. Let $q = (t,w,0) \in S^* \cap E_1$. Then $\hat{F}_1$ can be written in a neighborhood of q as

$$\hat{F}_1(t,w,x) = \mathcal{I}m\left(\frac{1}{x^2.w}\right) - f_1(t,w) = \frac{\mathcal{I}m(\overline{x}^2.\overline{w})}{|x|^4.|w|^2} - f_1(t,w) \implies$$

$\hat{M}_c^1$ is defined in a neighborhood of q by $\mathcal{I}m(\overline{x}^2.\overline{w}) - |x|^4.|w|^2(f_1(t,w) - c) = 0$.

Extension to a neighborhood of q_j, $0 \le j \le 3$. We have seen in (iv) that for any $j = 0,...,3$ there exists a holomorphic chart $(W,(u,v))$ such that $S \cap W = (v = 0)$, $\Pi^{-1}(q_j) \cap W = (v = 0)$ and $\Theta|_W = d\left(\frac{1}{u.v^2}\right)$. In this chart, we have

$$\mathcal{I}m(\Theta) = d\,\mathcal{I}m\left(\frac{1}{u.v^2}\right) \implies \hat{F}_1|_W = \mathcal{I}m\left(\frac{1}{u.v^2}\right) + c_1\ ,\ c_1 \in \mathbb{R}\ .$$

This implies that $\hat{M}_c^1$ can be defined in W by $\mathcal{I}m(\overline{u}.\overline{v}^2) + (c_1 - c)|u|^2.|v|^4 = 0$. Therefore, $\hat{M}_c^1$ at S is an analytic subset of (L,S).

We will construct now the real the real Levi-flat $\tilde{M}$ satisfying (II). Let $\mu\colon L \to L$ be the involution defined by $\mu|_{E_1}(t,w,x) = (t,-w,x)$ and $\mu|_{E_2}(s,z,y) = (s,-z,y)$. Note that the group of automorphisms of the ramified covering Φ is $\{id,\mu\}$.

We have seen in (26) that around q_0 we can write $\Theta = d\left(\frac{1}{x^2.w} - \frac{\psi(w^2)}{w}\right)$, where ψ is holomorphic in a neighborhood of 0. This implies that $\hat{F}_1 = \mathcal{I}m\left(\frac{1}{x^2.w} - \frac{\psi(w^2)}{w}\right) + c_1$, $c_1 \in \mathbb{R}$, in a neighborhood of q_0. In particular, if we set $\hat{F} := \hat{F}_1 - c_1$ then

$$\hat{F} \circ \mu = -\hat{F} \implies \hat{F}^2 \circ \mu = \hat{F}^2\ .$$

This implies that there exists a real analytic function $\tilde{F}$ on $\tilde{\mathbb{C}}^2 \setminus \pi^{-1}(Sep(\mathcal{F}_Q))$ such that $\tilde{F} \circ \Phi = \hat{F}^2$. We assert that, if $c > 0$ then the germ of $\tilde{F}^{-1}(c)$ at D extends to a real analytic Levi-flat $\tilde{M}_c$ tangent to $\tilde{\mathcal{F}}_Q$.

For instance, let us exhibit an analytic equation of $\tilde{M}_c$ in a neighborhood of $p_0 = \Phi(q_0)$. Near q_0 we have $\hat{F} = \mathcal{I}m(1/u.x^2)$, where $u = w/(1 - x^2.\psi(w^2))$. Note that $u^2 = w^2/(1 - x^2.\psi(w^2))^2$. Since $w^2 = p(t)$ along S, we get $u^2 = p(t)/(1 - x^2.\psi(p(t))^2$, so that u^2 is well defined and holomorphic near $p_0 = (t = x = 0)$. Since $p'(0) \neq 0$, the map $(t,x) \mapsto (p(t)/(1 - x^2.\psi(p(t))^2, x) := (s,x)$ is a local biholomorphism. In the coordinate system (s,x) we can write $u^2 = s$. On the other hand,

$$\hat{F}^2 = \mathcal{I}m^2(1/u.x^2) = \frac{\mathcal{I}m^2(\overline{u}.\overline{x}^2)}{|u^2|^2.|x|^8} = \frac{1}{4\,|u^2|^2.|x|^8}(2\,|u^2|.|x|^4 - u^2.x^4 - \overline{u}^2.\overline{x}^4) \implies$$

$$\hat{F}^2 - c = 0 \iff 2|u^2|.|x|^4 = 4c\,|u^2|^2.|x|^8 + u^2.x^4 + \overline{u}^2.\overline{x}^4 \implies$$

$4|u^2|^2.|x|^8 = (4c|u^2|^2.|x|^8 + u^2.x^4 + \overline{u}^2.\overline{x}^4)^2$. This implies that $\tilde{M}_c$ can be defined near p_0 by the real analytic equation

$$4|s|^2.|x|^8 - (s.x^4 + \overline{s}.\overline{x}^4 + 4c|s|^2.|x|^8)^2 = 0 \tag{28}$$

Similarly, it can be proved that $\tilde{M}_c$ has a local real analytic equation near all points of D. We leave the details to the reader. □

Final remarks : let Γ be the intersection of $\tilde{M}_c$ with the section $(x = 1)$. If we set $s = r\,e^{i\theta}$ in (28) then we get, after simplifications, the polar equation for Γ :

$$r = c^{-1}.sin^2(\theta/2)\ ,\ \theta \in [0, 2\pi]\ .$$

The above equation represents a cardioid in polar coordinates. In particular, $\tilde{M}_c$ is irreducible. This implies that $M_c := \pi(\tilde{M}_c)$ is sub-analytic and irreducible, but not real analytic at the origin, by theorem 1. □

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