A STRUCTURAL THEOREM FOR CODIMENSION ONE FOLIATIONS ON $\mathbb{P}^n$, $n \geq 3$, WITH APPLICATION TO DEGREE THREE FOLIATIONS.

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Abstract. Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$: to each point $p \in \mathbb{P}^n$ we set $\mathcal{J}(\mathcal{F}, p)$ = the order of the first non-zero jet $j_p^k(\omega)$ of a holomorphic 1-form ω defining $\mathcal{F}$ at p. The singular set of $\mathcal{F}$ is $sing(\mathcal{F}) = \{p \in \mathbb{P}^n \mid \mathcal{J}(\mathcal{F}, p) \geq 1\}$. We prove (main theorem 2) that a foliation $\mathcal{F}$ satisfying $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ has a non-constant rational first integral. Using this fact, we are able to prove that any foliation of degree three on $\mathbb{P}^n$, $n \geq 3$, is either the pull-back of a foliation on $\mathbb{P}^2$, or has a transverse affine structure with poles. This extends previous results for foliations of degree ≤ 2.

Contents

Notations.

- 1. $\mathcal{O}_n$: the ring of germs at $0 \in \mathbb{C}^n$ of holomorphic functions. $\mathcal{O}_n^* = \{f \in \mathcal{O}_n \mid f(0) \neq 0\}$. $\mathfrak{m}_n = \{f \in \mathcal{O}_n \mid f(0) = 0\}$.
- 2. $f \mid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and f divides g.
- 3. $f \nmid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and f does not divide g.
- 4. $[f, g]_0$: the intersection number of $f, g \in \mathfrak{m}_2 \setminus \{0\}$, when f and g have no common factor.
- 5. $< f, g >$: the ideal generated by $f, g \in \mathcal{O}_p$.
- 6. $Diff(\mathbb{C}^n, p)$: the group of germs at $p \in \mathbb{C}^n$ of biholomorphisms f with $f(p) = p$.
- 7. $i_X(\omega)$: the interior product of the vector field X and the form ω.
- 8. L_X : the Lie derivativative in the direction of the vector field X.
- 9. j_p^k : the k^{th}-jet at the point p.

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1. Introduction.

In a previous paper [Ce-LN 1] we have proved that the space of holomorphic codimension one foliations and degree two on $\mathbb{P}^n$, $n \geq 3$, has six irreducible components. A consequence of this classification is that we have two possibilities for a degree two foliation $\mathcal{F}$ on $\mathbb{P}^n$, $n \geq 3$: either $\mathcal{F}$ is defined by a meromorphic closed 1-form on $\mathbb{P}^n$, or $\mathcal{F} = g^*(\mathcal{G})$, where $g \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a linear map and $\mathcal{G}$ a degree two foliation of $\mathbb{P}^2$. A foliation defined by a meromorphic closed 1-form admits a special projective tranverse structure with poles, namely a translation structure. On the other hand, a foliation of the form $g^*(\mathcal{G})$ admits such a structure if, and only if, $\mathcal{G}$ admits (cf. [C]). This is not always the case : a foliation of $\mathbb{P}^2$ which admits a projective or affine transverse structure has always algebraic leaves, whereas for any $d \geq 2$, there are degree d foliations on $\mathbb{P}^2$ without algebraic invariant curves.

The following conjecture is attributed to different authors (Brunella, Lins Neto, ...) :

Main Conjecture. *Any codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$, $n \geq 3$, is either a pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ by a rational map $\Phi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, or admits a transverse projective structure with poles on some invariant hypersurface.*

In the first case, the leaves of $\mathcal{F}$ are sub-foliated by the levels of Φ and the dynamic properties of $\mathcal{F}$ are essentialy given by $\mathcal{G}$. In the second, we can associate a triple of meromorphic 1-forms $(\omega_0, \omega_1, \omega_2)$ such that ω_0 defines $\mathcal{F}$ outside its set of poles $|\omega_0|_\infty$ and the triple satisfies the $s\ell(2,\mathbb{C})$ structural relations :

$$\begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{aligned}$$

inducing the projective structure.

For instance, when $\omega_1 = \omega_2 = 0$, that is ω_0 is closed, then the integration of ω_0 on simply connected open sets $U \subset \mathbb{P}^n \setminus |\omega_0|_\infty$ gives $\omega_0 = df_U$, and defines the transverse translation structure : when $U \cap V \neq \emptyset$ then $f_U = f_V + c_{UV}$, where $c_{UV} \in \mathbb{C}$. On the other hand, if $\omega_2 = 0$ and $\omega_1 \neq 0$ then the transverse structrure is affine.

The main conjecture seems to be reasonable (at least for foliations of small degree) by the following reasons : first of all, if $\mathbb{K}$ is a field of positive characteristic every foliation on a projective manifold over $\mathbb{K}$, in particular on $\mathbb{P}^n_{\mathbb{K}}$, is defined by a closed 1-form (cf. [Ce-L-L-P-T]). On the other hand, if $\mathcal{F}$ is a foliation on $\mathbb{P}^n$ and p is a prime number then it is possible to define $\mathcal{F}_p$, the reduction modulo p of $\mathcal{F}$. There is a conjecture of Grothendieck-Katz type which says that if for almost all p the foliation $\mathcal{F}_p$ has a non-constant rational first integral then $\mathcal{F}$ itself has a non-constant rational first integral. Recently F. Touzet has communicated to one of the authors the following result :

Theorem. (F. Touzet) *The Grothendieck-Katz conjecture implies that any foliation of degree $\leq n-1$ on $\mathbb{P}^n$, either admits a projective transverse structure, or is a pull-back of some foliation on $\mathbb{P}^k$, $k < n$, by some rational map.*

Concerning the main conjecture, note that the first interesting case which is not covered by the above conditional result are the foliations of degree three on $\mathbb{P}^3$. In fact, one of the goals of this paper is to prove that the conjecture is true for foliations of degree three.

Theorem 1. *Let $\mathcal{F}$ be a holomorphic codimension one foliation of degree three on $\mathbb{P}^n$, $n \geq 3$. Then :*

- *either $\mathcal{F}$ has a rational first integral,*
- *or $\mathcal{F}$ has an affine transverse structure with poles on an invariant hypersurface,*
- *or $\mathcal{F} = g^*(\mathcal{G})$, where $g\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ a foliation on $\mathbb{P}^2$.*

One of the tools of the proof will be a result of [Ce-L-L-P-T] concerning foliations which admit a finite Godbillon-Vey sequence. This result says essentially that such a foliation is either a pull-back of a foliation on a surface or has a transversely projective structure. Let us explain briefly how we can apply the result.

By definition, a degree d foliation $\mathcal{F}$ on $\mathbb{P}^n$ has d tangencies with a generic straight line of $\mathbb{P}^n$. This implies that $\mathcal{F}$ can be represented in an affine coordinate system $\mathbb{C}^n \simeq E \subset \mathbb{P}^n$ by a polynomial integrable 1-form $\omega_E = \sum_{j=0}^{d+1} \omega_j$, where the coeffitients of the 1-form ω_j are homogeneous polynomials of degree j, $0 \leq j \leq d+1$, and $i_R(\omega_{d+1}) = 0$, with $R = \sum_{j=1}^{n} z_j\, \partial_{z_j}$, the radial vector field. The form ω_E can be considered as a meromorphic 1-form on $\mathbb{P}^n$ with poles of order $d+2$ at the hyperplane of infinity of E. Given $p \in E$, we set

$$\mathcal{J}(\mathcal{F}, p) = min\,\{k \geq 0 \,|\, j_p^k(\omega_E) \neq 0\}\ .$$

It can be proved that $\mathcal{J}(\mathcal{F}, p)$ depends only of p and $\mathcal{F}$ and not of E and ω_E. The singular set of $\mathcal{F}$ is defined as

$$sing(\mathcal{F}) = \{p \in \mathbb{P}^n \,|\, \mathcal{J}(\mathcal{F}, p) \geq 1\}\,.$$

This set is algebraic and has always irreducible components of codimension two (cf. [LN]).

Given a degree three foliation $\mathcal{F}$ of $\mathbb{P}^n$, we will consider two cases :

1. There exists $p \in sing(\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$.
2. For all $p \in sing(\mathcal{F})$ we have $\mathcal{J}(\mathcal{F}, p) = 1$.

Case (1) will be studied in section 2. We will see that $\mathcal{F}$ admits a finite Godbillon-Vey sequence in this case and we can apply the result of [Ce-L-L-P-T]. In case (2) we will see in section 3 that $\mathcal{F}$ has a meromorphic first integral.

In section 3 we will introduce the *Baum-Bott index* of an irreducible component, say Γ, of codimension two of $sing(\mathcal{F})$, which we will denote $BB(\mathcal{F}, \Gamma)$. As a consequence of the Baum-Bott theorem we will see that $sing(\mathcal{F})$ has always a codimension two irreducible component Γ with $BB(\mathcal{F}, \Gamma) \neq 0$.

Theorem 2. *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Assume that $sing(\mathcal{F})$ has an irreducible component of codimension two Γ such that*

(a). *$BB(\mathcal{F}, \Gamma) \neq 0$.*
(b). *The algebraic set $\{p \in \Gamma \,|\, \mathcal{J}(\mathcal{F}, p) > 1\}$ has codimension ≥ 4 in $\mathbb{P}^n$.*

Then $\mathcal{F}$ has a rational first integral.

As a consequence, we will get the following :

Corollary 1. *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. If $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ then $\mathcal{F}$ has a rational first integral.*

Remark 1.1. Recall that $p \in sing(\mathcal{F})$ is of Kupka type if $\mathcal{F}$ is defined in a neighborhood of p by a holomorphic 1-form ω such that $d\omega(p) \neq 0$. We define $K(\mathcal{F}) = \{p \in sing(\mathcal{F}) \,|\, p \text{ is of Kupka type}\}$. If $p \in K(\mathcal{F})$ then $\mathcal{J}(\mathcal{F}, p) = 1$. We would like to observe that if $sing(\mathcal{F})$ has a smooth irreducible component, say Γ, with $\Gamma \subset K(\mathcal{F})$, then a theorem due to Calvo Andrade and M. Brunella says that $\mathcal{F}$ has a rational first integral (cf. [Ce-LN 2], [CA 1], [CA 2] and [B 2]). In this sense, theorem 2 is a generalization of Calvo and Brunella theorem.

Remark 1.2. We would like to observe that the conclusion of corollary 1 is not true when we consider codimension one foliations on more general complex manifolds. For instance, let $M = \mathbb{P}^2 \times \mathbb{P}^k$, $k \geq 1$, and $\mathcal{F} = \Pi_1^*(\mathcal{G})$, where $\Pi_1 \colon \mathbb{P}^2 \times \mathbb{P}^k \to \mathbb{P}^2$ is the first projection and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ of degree ≥ 2 without non-constant rational first integral and with $\mathcal{J}(\mathcal{G}, p) \leq 1$ for all $p \in \mathbb{P}^2$. Then $\mathcal{F}$ satisfies the hypothesis of corollary 1 but not its conclusion. A natural question which arises is the following :

Problem 1. *For which compact complex manifolds of dimension* ≥ 3 *the conclusion of corollary 1 is true ?*

Remark 1.3. We say that a foliation admits a *purely* projective transverse structure (briefly p.p.t.s.) if it has a projective transverse structure with poles, but no affine transverse structure with poles. There are examples of foliations on $\mathbb{P}^3$, the so called Hilbert modular foliations, which admit a p.p.t.s. and are not pull-back of foliations on $\mathbb{P}^2$ (cf. [Ce-L-L-P-T]). In fact, these examples have degree at least five.

On the other hand, as a consequence of the proof of theorem 1, any foliation of degree three on $\mathbb{P}^n$, $n \geq 3$, that admits a p.p.t.s., is the pull-back of a Riccati foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ (see the 3^{rd} case in the proof of lemma 2.1). For instance, there are p.p.t.s. Riccati equations on $\mathbb{C}^2$ of the form

$$x(x-1)\, dy - (a_0(x) + a_1(x)\, y + a_2(x)\, y^2)\, dx = 0 \;, \tag{1}$$

where a_0, a_1 and a_2 are degree one polynomials. If $\mathcal{G}$ is a p.p.t.s. foliation defined by (1) on $\mathbb{P}^2$ then it has degree three. In particular, if $\Pi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is linear then $\Pi^*(\mathcal{G})$ is a p.p.t.s. degree three foliation on $\mathbb{P}^n$.

It seems reasonable to hope that theorem 1 will give a clue to a classification of the irreducible components of the space of degree three foliations on $\mathbb{P}^n$ which are not pull-back by rational maps of foliations on $\mathbb{P}^2$. However, the analysis of the components of rational pull-back type seems to be more delicate, since we have no control on the degrees of the objects that appear in our proofs.

Problem 2. *Classify the irreducible components of the space of foliations of degree three on* $\mathbb{P}^n$*,* $n \geq 3$.

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2. Proof of theorem 1.

The aim of this section is to prove theorem 1 by admitting theorem 2. In section 2.1 we will analyse the case where there exists $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$ and in section 2.2 we will finish the proof.

2.1. **The case $\mathcal{J}(\mathcal{F},p) \geq 2$ for some p.** Let $\mathcal{F}$ be a codimension one holomorphic foliation on a complex manifold X. A Godbillon-Vey sequence (briefly G-V-S) associated to $\mathcal{F}$ is a sequence of meromorphic 1-forms on X, say $(\omega_j)_{j\geq 0}$, such that

- 1. $\mathcal{F}$ is defined by ω_0 outside its set of poles, $|\omega_0|_\infty$. In particular, ω_0 is integrable, that is $\omega_0 \wedge d\omega_0 = 0$.
- 2. The 1-form defined by the formal power series

$$\Omega = dz + \omega_0 + z\,\omega_1 + \frac{z^2}{2}\,\omega_2 + \sum_{j\geq 3} \frac{z^j}{j!}\,\omega_j \tag{2}$$

is integrable.

When there exists N such that $\omega_N \neq 0$ but $\omega_j = 0$ for all $j > N$ then we say that $\mathcal{F}$ admits a finite G-V-S of lenght N. In this case, the form in (2) is meromorphic and can be extended meromorphically to $X \times \mathbb{P}^1$. Since it is integrable, it defines a codimension one foliation $\mathcal{H}$ on $X \times \mathbb{P}^1$ such that $\mathcal{H}|_{X\times\{0\}} = \mathcal{F}$.

Remark 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be foliations on complex manifolds X and Y, respectively. Assume that $\mathcal{G}$ admits a finite G-V-S of lenght N and that $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi\colon X \dashrightarrow Y$ is a rational map. Then $\mathcal{F}$ also admits a G-V-S of lenght N (cf. [Ce-L-L-P-T]).

Remark 2.2. When $\mathcal{F}$ admits a G-V-S of lenght $N \leq 2$ then $\mathcal{F}$ has a transverse projective structure with poles in a hypersurface. When $N = 1$ then the structure is in fact affine (cf. [Go] and [Sc]).

On the other hand, if it admits a finite G-V-S of lenght $N \geq 3$ then we have the following

Theorem 2.1. *([Ce-L-L-P-T]) Let $\mathcal{F}$ be a foliation on a compact holomorphic manifold X admiting a finite G-V-S of lenght $N \geq 3$. Then*

- *either $\mathcal{F}$ is transversely affine,*
- *or there exist a compact Riemann surface S, meromorphic 1-forms $\alpha_0, ..., \alpha_N$ on S and a rational map $\phi\colon X \dashrightarrow S \times \mathbb{P}^1$ such that $\mathcal{F}$ is defined by the meromorphic 1-form $\omega = \phi^*(\eta)$, where*

$$\eta = dz + \alpha_0 + z\,\alpha_1 + ... + z^N\,\alpha_N\ . \tag{3}$$

When $X = \mathbb{P}^n$, $n \geq 3$, necessarily $S = \mathbb{P}^1$ and (3) can be written as

$$\eta = dz - P(t,z)\,dt\ ,$$

where $P \in \mathbb{C}(t)[z]$ and $\mathcal{F} = \phi^*(\mathcal{G})$, where $\mathcal{G}$ is defined on $\mathbb{P}^1 \times \mathbb{P}^1$ by the differential equation $\frac{dz}{dt} = P(t,z)$. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is birrational to $\mathbb{P}^2$ we get the following consequence :

Corollary 2.1. *If $\mathcal{F}$ is a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, which admits a finite G-V-S then, either it has a tranverse projective structure, or it is a pull-back of foliation on $\mathbb{P}^2$ by a rational map.*

Now, let us consider a degree three codimension one foliation $\mathcal{F}$ on $\mathbb{P}^n$ and assume that there exists $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F},p) \geq 2$. In this case, if we take an affine coordinate system $\mathbb{C}^n \subset \mathbb{P}^n$ such that $p = 0 \in \mathbb{C}^n$ then $\mathcal{F}|_{\mathbb{C}^n}$ is defined by a polynomial 1-form $\omega = \alpha_2 + \alpha_3 + \alpha_4$, where the coefficients of α_j are homogeneous polynomials of degree j, $2 \leq j \leq 4$, and $i_R(\alpha_4) = 0$, R the radial vector field.

Lemma 2.1. *In the above situation we have three possibilities :*

(a) $\mathcal{F}$ *has an affine transverse structure with poles in a hypersurface.*
(b) $\mathcal{F}$ *is the pull-back by a rational map of a foliation on* $\mathbb{P}^2$*.*
(c) $\mathcal{F}$ *is the pull-back by a linear map* $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ *of a foliation of degree three on* $\mathbb{P}^{n-1}$*.*

In particular, if $n = 3$ *then* $\mathcal{F}$ *satisfies* (a) *or* (b).

Proof. With the previous notations, set $\alpha_j = \sum_{i=1}^n P_{ji}(z)\, dz_i$ and $F_j(z) = i_R(\alpha_{j-1}) = \sum_{i=1}^n z_i.P_{ji}(z)$, $j = 3, 4$. We will divide the proof in three cases :

1^{st}. $i_R(\omega) \equiv 0$, which is equivalent to $F_3 \equiv F_4 \equiv 0$. In this case, we will prove that $\omega = \alpha_4$ and we will get (c).

2^{nd}. $F_3 \not\equiv 0$. In this case, we will prove that there exists a birrational map $\Phi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, an affine coordinate system $(x, z) \in \mathbb{C}^{n-1} \times \mathbb{C} \simeq \mathbb{C}^n \subset \mathbb{P}^n$ and meromorphic 1-forms $\beta_1, \beta_2\, \beta_3$, with $i_{\partial_z}(\beta_j) = 0$ and $L_{\partial_z}(\beta_j) = 0$, $1 \le j \le 3$, such that $\Phi^*(\omega) = g.\, \eta$, where

$$\eta = dz + z\, \beta_1 + z^2\, \beta_2 + z^3\, \beta_3 \ . \tag{4}$$

We will show that we can apply theorem 2.1 to prove that $\mathcal{F}$ satisfies (a) or (b).

3^{rd}. $F_3 \equiv 0$ and $F_4 \not\equiv 0$. In this case, we will prove that, either $\mathcal{F}$ is the pull-back of a Ricatti equation on $\mathbb{P}^1 \times \mathbb{P}^1$, or it admits an affine transverse structure, or ω has an integrating factor, that is there exists a meromorphic function $h \ne 0$ such that $d(h\, \omega) = 0$.

Analysis of the 1^{st} case. First of all, note that $\alpha_4 \not\equiv 0$, for otherwise $\mathcal{F}$ would have degree ≤ 2. The integrability condition gives

$$\omega \wedge d\omega = 0 \implies \omega \wedge i_R(d\omega) = 0 \ .$$

On the other hand,

$$i_R(d\omega) = L_R(\omega) - d(i_R(\omega)) = L_R(\omega) = 3\, \alpha_2 + 4\, \alpha_3 + 5\, \alpha_4 \implies$$

$$0 \equiv (\alpha_2 + \alpha_3 + \alpha_4) \wedge (3\alpha_2 + 4\alpha_3 + 5\alpha_4) = \alpha_2 \wedge \alpha_3 + 2\, \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_4 \ .$$

Since the coefficients of α_j are homogeneous of degree j, $2 \le j \le 4$, we get

$$\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \alpha_4 = \alpha_3 \wedge \alpha_4 = 0 \ . \tag{5}$$

Since $\alpha_4 \not\equiv 0$, (5) implies that there are meromorphic functions f_j, $j = 2, 3$, such that $\alpha_j = f_j.\, \alpha_4$. If $f_j \not\equiv 0$ for some $j \in \{2, 3\}$ then the foliation $\mathcal{F}$ would have degree less than three. Therefore, $\alpha_2 = \alpha_3 = 0$.

In particular, we get $\omega = \alpha_4$. Since α_4 is integrable, it defines a foliation of degree three, say $\mathcal{F}_{n-1}$, on $\mathbb{P}^{n-1}$. If we consider $\mathbb{P}^{n-1}$ as the set of lines through $0 \in \mathbb{C}^n \subset \mathbb{P}^n$ and $\pi\colon \mathbb{P}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the natural projection then $\mathcal{F} = \pi^*(\mathcal{F}_{n-1})$. This finishes the analysis of the 1^{st} case.

In the analysis of the two other cases, we consider first a blowing-up $\pi\colon \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ at $0 \in \mathbb{C}^n \subset \mathbb{P}^n$. Let us compute the foliation $\pi^*(\mathcal{F})$. In the chart

$$(\tau_1, ..., \tau_{n-1}, x) = (\tau, x) \in \mathbb{C}^{n-1} \times \mathbb{C} \overset{\pi}{\mapsto} (x.\tau, x) = (z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$$

we get

$$\pi^*(\omega) = x^2\, \left[x\, \theta_2 + x^2\, \theta_3 + x^3\, \theta_4 + (F_3(\tau, 1) + x\, F_4(\tau, 1))dx\right] \ , \tag{6}$$

where

$$\theta_j = \sum_{i=1}^{n-1} P_{ji}(\tau,1)\, d\tau_i \ , \ 2 \leq j \leq 4 \ ,$$

depends only of τ.

Analysis of the 2nd case. Since $F_3 \not\equiv 0$ we have two possibilities :

2.a. $F_4 \equiv 0$. In this case, if we set $\beta_j = \frac{1}{F_3(\tau,1)}\, \theta_{j+2}$ then we can write $\pi^*(\omega) = x^2 .\, F_3(\tau,1).\, \eta$, where $i_{\partial_x}(\beta_j) = 0$, $L_{\partial_x}(\beta_j) = 0$, $1 \leq j \leq 3$, and

$$\eta = dx + x\,\beta_1 + x^2\,\beta_2 + x^3\,\beta_3 \ .$$

Therefore, we get (4).

2.b. $F_4 \not\equiv 0$. In this case, $\pi^*(\mathcal{F})$ is defined in this chart by

$$\nu := x^{-2}.\pi^*(\omega) = x\,\theta_2 + x^2\,\theta_3 + x^3\,\theta_4 + (F_3(\tau,1) + x\,F_4(\tau,1))dx$$

and we need one more birrational transformation. Consider the birrational map

$$\psi(\tau,z) = \left(\tau, \frac{\frac{F_3(\tau,1)}{F_4(\tau,1)} z}{1-z}\right) = (\tau,x) \text{ with inverse } \psi^{-1}(\tau,x) = \left(\tau, \frac{x}{x + \frac{F_3(\tau,1)}{F_4(\tau,1)}}\right) \ .$$

A straightforward computation gives $\psi^*(\nu) = \frac{F_3^2(\tau,1)}{F_4(\tau,1)}\,\eta$, where

$$\eta = dz + z\,\beta_1 + z^2\,\beta_2 + z^3\,\beta_3$$

with

$$\beta_1 = \tfrac{1}{F_3(\tau,1)}\,\theta_2 + \tfrac{dF_3(\tau,1)}{F_3(\tau,1)} - \tfrac{dF_4(\tau,1)}{F_4(\tau,1)}$$

$$\beta_2 = \tfrac{1}{F_4(\tau,1)}\,\theta_3 - 2\tfrac{1}{F_3(\tau,1)}\,\theta_2 + \tfrac{dF_4(\tau,1)}{F_4(\tau,1)} - \tfrac{dF_3(\tau,1)}{F_3(\tau,1)} \ .$$

$$\beta_3 = \tfrac{F_3(\tau,1)}{F_4^2(\tau,1)}\,\theta_4 - \tfrac{1}{F_4(\tau,1)}\,\theta_3 + \tfrac{1}{F_3(\tau,1)}\,\theta_2$$

Therefore, in both sub-cases we obtain (4).

At this point we should mention that, in order to use theorem 2.1, we have to assure that the G-V-S is of lenght ≥ 3. If η is like in (4) then the integrability condition of the form η implies that it admits the G-V-S $(\eta = \eta_0, \eta_1, \eta_2, \eta_3)$, where $\eta_j = L_{\partial_z}^{(j)}(\eta)$ (cf. [Ce-L-L-P-T]). The lenght is three if $\eta_3 = 6\,\beta_3 \not\equiv 0$. On the other hand, if $\beta_3 \equiv 0$ then $\mathcal{F}$ has an affine transverse structure.

In fact, if we set $z = 1/w$ in (4) with $\beta_3 = 0$ then we get $\eta = -w^{-2}\Omega$, where

$$\Omega = dw - \beta_2 - w\,\beta_1 \ .$$

Therefore, Ω admits the G-V-S of lenght one $(\Omega, -\beta_1)$. Note that $d\Omega = \beta_1 \wedge \Omega$ and $d\beta_1 = 0$. Hence, $\mathcal{F}$ has an affine transverse structure with poles in some hypersurface.

Analysis of the 3rd case. In this case, after the blowing-up π, we get $\pi^*(\omega) = x^3 .\, F_4(\tau,1).\, \eta$, where

$$\eta = dx + \beta_0 + x\,\beta_1 + x^2\,\beta_2 \ , \ \beta_j = \frac{1}{F_4(\tau,1)}\,\theta_{j+2} \ , \ 0 \leq j \leq 2 \ . \tag{7}$$

In particular, η admits a G-V-S of lenght ≤ 2, $(\eta = \eta_0, \eta_1, \eta_2)$, where $\eta_j = L^{(j)}_{\partial_x}(\eta)$, $j = 1, 2$. A foliation defined by a meromorphic form like in (7) has always a projective transverse structure, but in general has no affine transverse structure. Therefore, we have to work more to conclude the proof in this case.

We begin by recalling that $\pi^*(\alpha_2) = x^2(x\,\theta_2 + F_3(\tau,1)\,dx) = x^3\,\theta_2$. When $\alpha_2 \equiv 0$ we get $\beta_0 \equiv 0$ and (7) becomes similar to (4) with $\beta_3 \equiv 0$, which we have already considered; in that case we have an affine transverse structure. Let us assume $\alpha_2 \not\equiv 0$. The integrability condition $\omega \wedge d\omega = 0$ implies that $\alpha_2 \wedge d\alpha_2 = 0$. In particular, either $cod(sing(\alpha_2)) \geq 2$ and α_2 defines a foliation of degree one on $\mathbb{P}^{n-1}$, or $\alpha_2 = h.\,\alpha_1$, where α_1 defines a foliation of degree zero on $\mathbb{P}^{n-1}$. In both cases, it is known that α_2 has an integrating factor. In other words, there exists a homogeneous polynomial $f \not\equiv 0$ of degree three such that $d(f^{-1}\,\alpha_2) = 0$ (cf. [Ce-LN 1]). This implies

$$(8) \qquad \pi^*\left(\frac{\alpha_2}{f}\right) = \frac{\theta_2}{f(\tau,1)} \implies d\left(\frac{\theta_2}{f(\tau,1)}\right) = 0 \implies d\left(\frac{F_4(\tau,1)}{f(\tau,1)}\,\beta_0\right) = 0\ .$$

Set $F(\tau) := f(\tau,1)/F_4(\tau,1)$ and consider the birrational map $\Phi(\tau,z) = (\tau, F(\tau).\,z) = (\tau,x)$. If η is like in (7) then a straightforward computation gives $\Phi^*(\eta) = F.\,\tilde{\eta}$, where

$$\tilde{\eta} = dz + \tilde{\beta}_0 + z\,\tilde{\beta}_1 + z^2\,\tilde{\beta}_2\ ,\ \tilde{\beta}_0 := F^{-1}.\,\beta_0\ ,\ \tilde{\beta}_1 = \beta_1 + \frac{dF}{F}\ ,\ \tilde{\beta}_2 = F.\,\beta_2\ .$$

In this situation it is convenient to consider the birrational map $\Psi(\tau,w) = (\tau, 1/w) = (\tau,z)$. We have $\Psi^*(\tilde{\eta}) = -w^{-2}.\,\hat{\eta}$, where

$$\hat{\eta} = dw - \tilde{\beta}_2 - w\,\tilde{\beta}_1 - w^2\,\tilde{\beta}_0\ .$$

Since $i_{\partial_w}(\tilde{\beta}_j) = 0$ and $L_{\partial_w}(\tilde{\beta}_j) = 0$, $0 \leq j \leq 2$, the integrability of $\hat{\eta}$ implies

$$(9) \qquad \begin{cases} d\tilde{\beta}_0 = \tilde{\beta}_0 \wedge \tilde{\beta}_1 \\ d\tilde{\beta}_1 = 2\,\tilde{\beta}_0 \wedge \tilde{\beta}_2 \\ d\tilde{\beta}_2 = \tilde{\beta}_1 \wedge \tilde{\beta}_2 \end{cases} .$$

From (8) we get $d\tilde{\beta}_0 = 0$, and so the first relation in (9) gives $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$.

Let us denote by $\preceq \mathfrak{m}_k$ the set of meromorphic functions on $\mathbb{P}^k$. Since $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$ there exists $g \in \preceq \mathfrak{m}_{n-1}$ such that $\tilde{\beta}_1 = g.\,\tilde{\beta}_0$. The second relation in (9) gives

$$d\tilde{\beta}_1 = dg \wedge \tilde{\beta}_0 = 2\,\tilde{\beta}_0 \wedge \tilde{\beta}_2 \implies (dg + 2\,\tilde{\beta}_2) \wedge \tilde{\beta}_0 = 0 \implies$$

$$\exists\ h \in \preceq \mathfrak{m}_{n-1} \text{ such that } \tilde{\beta}_2 = h.\,\tilde{\beta}_0 - \frac{1}{2}\,dg\ .$$

The third relation in (9) implies

$$dh \wedge \tilde{\beta}_0 = d\tilde{\beta}_2 = \tilde{\beta}_1 \wedge \tilde{\beta}_2 = -g\,\tilde{\beta}_0 \wedge \left(h\,\tilde{\beta}_0 - \frac{1}{2}\,dg\right) = \frac{1}{2}\,g\,dg \wedge \tilde{\beta}_0 \implies$$

$$(10) \qquad d\left(h - \frac{1}{4}\,g^2\right) \wedge \tilde{\beta}_0 = 0\ .$$

Let us denote by $\mathcal{G}$ the foliation defined by $\tilde{\beta}_0$ on $\mathbb{P}^{n-1}$. We have two possibilities :

3.a. *$\mathcal{G}$ has no non-constant meromorphic first integral.* We assert that ω has an integrating factor.

In fact, relation (10) implies that

$$d\left(h - \frac{1}{4}\,g^2\right) = 0 \implies h = \frac{1}{4}\,g^2 + c\ ,\ c \in \mathbb{C}\ ,$$

for otherwise $h - \frac{1}{4}\,g^2$ would be a non-constant first integral of $\mathcal{G}$. From the above relations we get

$$\hat{\eta} = dw + \frac{1}{2}\,dg - (g^2/4 + c + g.\,w + w^2)\,\tilde{\beta}_0 = d(w + g/2) - ([w + g/2]^2 + c)\,\tilde{\beta}_0\ .$$

In particular, if we set $\mu := ((w + g/2)^2 + c)^{-1}.\,\hat{\eta}$ then

$$\mu = \frac{d(w + g/2)}{(w + g/2)^2 + c} - \tilde{\beta}_0 \implies d\mu = 0\ .$$

This implies that ω has an integrating factor, as asserted.

3.b. *$\mathcal{G}$ has a non-constant meromorphic first integral.* We assert that $\mathcal{F}$ is the pull-back of a Riccati equation on $\mathbb{P}^1 \times \mathbb{P}^1$ by a birrational map.

In fact, by Stein's fatorization theorem $\mathcal{G}$ has a meromorphic first integral, say f, with connected fibers : if $\phi \in\preceq \mathfrak{m}_{n-1}$ and $d\phi \wedge df = 0$ then there exists $\psi \in\preceq \mathfrak{m}_1$ such that $\phi = \psi(f)$, where $\psi(f) := \psi \circ f$. Since f is a first integral of $\mathcal{G}$ we have $\tilde{\beta}_0 = \phi_1.\,df$, for some $\phi_1 \in\preceq \mathfrak{m}_{n-1}$. This implies $d\phi_1 \wedge df = 0$, and so $\phi_1 = \psi_1(f)$, where $\psi_1 \in\preceq \mathfrak{m}_1 \setminus \{0\}$. On the other hand, relation (10) implies that there exists $\psi_2 \in\preceq \mathfrak{m}_1$ such that $h = \frac{1}{4}\,g^2 + \psi_2(f)$. In particular,

$$\hat{\eta} = d(w + g/2) - (g^2/4 + g\,w + w^2 + \psi_2(f))\,\psi_1(f)\,df =$$

$$= d(w + g/2) - ([w + g/2]^2 + \psi_2(f))\,\psi_1(f)\,df\ .$$

Consider the rational map $\Phi_1 \colon \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $\Phi_1(\tau, w) = (f(\tau), w - g(\tau)) := (x, y)$. Then $\hat{\eta} = \Phi^*(\theta)$, where

$$\theta = dy - (y^2 + \psi_2(x))\,\psi_1(x)\,dx\ .$$

Since $\theta = 0$ defines a Riccati equation on $\mathbb{P}^1 \times \mathbb{P}^1$ this finishes the proof of 3.b and of lemma 2.1. □

2.2. **End of the proof of theorem 1.** The proof is by induction on $n \geq 3$. If $n = 3$ then theorem 1 follows from lemma 2.1. Let us assume that theorem 1 is true for $n - 1 \geq 3$ and prove that it is true for n.

Let $\mathcal{F}$ be a codimension one foliation of degree three on $\mathbb{P}^n$, $n \geq 4$. It follows from lemma 2.1 and theorem 2 that, either $\mathcal{F}$ satisfies one of the conclusions of theorem 1, or $\mathcal{F}$ is the pull-back by a linear map $\pi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ of a foliation of degree three on $\mathbb{P}^{n-1}$. In this last case, since theorem 1 is true for $n - 1$, then one the three possilities bellow is true :

(i). $\mathcal{F}_{n-1}$ has a rational first integral, say $F \colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^1$. In this case, $F \circ \pi$ is a rational first integral of $\mathcal{F}$ and we are done.

(ii). $\mathcal{F}_{n-1} = \Phi^*(\mathcal{G})$, where $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ and $\Phi \colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^2$ a rational map. In this case, we get $\mathcal{F} = (\Phi \circ \pi)^*(\mathcal{G})$ and we are done.

(iii). $\mathcal{F}_{n-1}$ has an affine transverse structure. In this case, $\mathcal{F}_{n-1}$ admits a G-V-S of lenght one. Therefore, $\mathcal{F}$ also admits a G-V-S of lenght one by remark 2.1.

This finishes the proof of theorem 1.

3. Proof of theorem 2.

In § 3.1 we state some general facts about the Baum-Bott index that will be used in the proof. After that we give the proof of theorem 2 in dimension three and at the end we will see the proof in dimension $n \geq 4$.

3.1. The Baum-Bott index. We begin by recalling briefly the proof of [LN] that $sing(\mathcal{F})$ has components of codimension two. This proof is based in the Baum-Bott theorem for foliations on compact holomorphic surfaces.

Theorem 3.1. *Let $\mathcal{G}$ be a foliation on a compact surface with isolated singularites. Then*

$$\sum_{p \in sing(\mathcal{G})} BB(\mathcal{G}, p) = N_{\mathcal{G}}^2 \ , \tag{11}$$

where $N_{\mathcal{G}}$ is the normal bundle of the foliation $\mathcal{G}$ and $BB(\mathcal{G}, p)$ the Baum-Bott index of $\mathcal{G}$ at the point p.

A proof of theorem 3.1 and the definition of $N_{\mathcal{G}}$ can be found in [B 1]. The Baum-Bott index $BB(\mathcal{G}, p)$ is defined as follows : let $(U, (x, y))$ be a holomorphic chart around p such that $x(p) = y(p) = 0$ and $sing(\mathcal{G}) \cap U = \{0\}$ and $\omega = P(x, y)\, dy - Q(x, y)\, dx$ be a holomorphic 1-form representing $\mathcal{G}|_U$. Let η be a C^∞ $(1, 0)$-form on $U \setminus \{0\}$ such that $d\omega = \eta \wedge \omega$. For instance, one can take

$$\eta = \frac{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}{|P|^2 + |Q|^2} \left(\overline{P}\, dx + \overline{Q}\, dy\right) \ .$$

Then the 3-form $\eta \wedge d\eta$ is closed and

$$BB(\mathcal{G}, p) = \frac{1}{(2\pi i)^2} \int_{\partial B} \eta \wedge d\eta \tag{12}$$

where B is a closed ball containing $p = 0$ in its interior with $sing(\mathcal{G}) \cap B = \{p\}$ (cf. [B 1]). In particular, the integral in (12) does not depend on the form η chosen. We will also use the notation $BB(\omega, p) := BB(\mathcal{G}, p)$.

Remark 3.1. We would like to point out some consequences of (12).

- 1. $BB(\mathcal{G}, p)$ is invariant by local analytic equivalences.
- 2. If the foliation $\mathcal{G}$ has a holomorphic first integral in a neighborhood of the singular point p with an isolated singularity at p then $BB(\mathcal{G}, p) = 0$.
- 3. If the foliation $\mathcal{G}$ is represented in a neighborhood of p by a vector field X such that $DX(p)$ is non-singular then
$$BB(\mathcal{G}, p) = \frac{tr(DX(p))^2}{det(DX(p))} \ ,$$
where tr denotes trace and det determinant.
- 4. If $(\mathcal{G}_t)_{t \in (\mathbb{C}^k, 0)}$ is a germ of holomorphic deformation of $\mathcal{G}$ such that $\mathcal{G}_0 = \mathcal{G}$ and $sing(\mathcal{G}_t) \cap int(B) = \{p_1(t), ..., p_k(t)\}$ then
$$\lim_{t \to 0} \left(\sum_{j=1}^{k} BB(\mathcal{G}_t, p_j(t)) \right) = BB(\mathcal{G}, p) \ .$$

Let us prove that the singular set of the codimension one foliation $\mathcal{F}$ with $dg(\mathcal{F}) = d$ on $\mathbb{P}^n$ has at least one irreducible component of codimension two. If not, then there exists a linear embedding $i\colon \mathbb{P}^2 \to \mathbb{P}^n$ such that :

(i). $E \cap sing(\mathcal{F}) = \emptyset$.

(ii). The tangencies of $\mathcal{F}$ with $E := i(\mathbb{P}^2)$ are generic (of Morse type, see [LN]).

Let $\mathcal{G} = i^*(\mathcal{F})$. Note that (ii) implies that $dg(\mathcal{G}) = d$. Moreover, (i) and (ii) imply that if $p \in sing(\mathcal{G})$ then $\mathcal{G}$ has a local holomorphic first integral of Morse type in a neighborhood of p. In particular, we get from (2) in remark 3.1 that

$$\sum_{p \in sing(\mathcal{G})} BB(\mathcal{G}, p) = 0$$

On the other hand, $N_{\mathcal{G}} = (d+2)H$, where H is the class of a hyperplane, so that the Baum-Bott theorem gives (cf. [B 1]) :

$$\sum_{p \in sing(\mathcal{G})} BB(\mathcal{G}, p) = (d+2)^2 > 0 \ ,$$

a contradiction.

We will denote $sing_2(\mathcal{F})$ the union of the codimension two irreducible components of $sing(\mathcal{F})$. Let Γ be an irreducible component of $sing_2(\mathcal{F})$. Given a smooth point $p \in \Gamma$ and a germ of embedding $i\colon (\mathbb{C}^2, 0) \to (\mathbb{P}^n, p)$, transverse to Γ, define $BB(\mathcal{F}, \gamma, i, p) := BB(i^*(\mathcal{F}), 0)$. The following result can be proved :

Theorem 3.2. *There exists a proper analytic subset $\Gamma_1 \subset \Gamma$ such that :*

(a). If $p \in \Gamma \setminus \Gamma_1$ then $BB(\mathcal{F}, \Gamma, i, p)$ does not depend on the embedding $i\colon (\mathbb{D}^2, 0) \to (\mathbb{P}^n, p)$, transverse to Γ. We then denote $BB(\mathcal{F}, \Gamma, p) := BB(\mathcal{F}, \Gamma, i, p)$.

(b). The map $p \in \Gamma \setminus \Gamma_1 \mapsto BB(\mathcal{F}, \Gamma, p) \in \mathbb{C}$ is constant.

We then denote $BB(\mathcal{F}, \Gamma) := BB(\mathcal{F}, \Gamma, p)$, where $p \in \Gamma \setminus \Gamma_1$.

The proof in the general case can be done by using the results of J. F. Mattei about the equiresolution of integrable families of foliations of $(\mathbb{C}^2, 0)$ (cf. [Ma]) and also the fact that the Baum-Bott indexes of two germs of foliations on $(\mathbb{C}^2, 0)$ are the same if their Seidenberg resolutions of singularities are C^∞ isomorphic with corresponding singularities with the same Baum-Bott index (cf. [B 1]). We give the proof of theorem 3.2 in the case we are interested.

Lemma 3.1. *Let ω be a holomorphic integrable 1-form in a neighborhood of $0 \in U \subset \mathbb{C}^n$, $n \geq 3$, such that $\omega(0) = 0$ and $j^1_p(\omega) \neq 0$ for all $p \in U$. Assume that $sing(\omega)$ is connected and smooth of codimension two. Then for any $p, q \in sing(\omega)$ and any two transverse sections to $sing(\omega)$ through p and q, say Σ_p and Σ_q, then $BB(\omega, sing(\omega), \Sigma_p, p) = BB(\omega, sing(\omega), \Sigma_q, q)$.*

Proof. Denote by $\mathcal{F}$ the foliation defined by ω on U. We will prove that for any $p \in sing(\omega)$ there is a neighborhood V of p in $sing(\omega)$ such that for any two transverse sections Σ_p and Σ_q through p and $q \in V$, respectively, then $BB(\omega, sing(\omega), \Sigma_p, p) = BB(\omega, sing(\omega), \Sigma_q, q)$.

Fix $p \in sing(\omega)$. Assume first that p is a Kupka singularity, that is $d\omega(p) \neq 0$. In this case, the distribution defined by $E_q = \{v \,|\, i_v(d\omega(q)) = 0\}$ has codimension two and is integrable in some neighborhood W of p, defining a codimension two foliation $\mathcal{E}$ on W. Moreover, $sing(\omega) \cap W$ is a leaf of $\mathcal{E}$. If Σ is a germ of embedded two plane transverse to $sing(\omega)$ at p, we can define a germ of submersion $g\colon (\mathbb{C}^n,p)\to(\Sigma,p)$ by following the leaves of $\mathcal{E}$. It can be proved that $\omega = g^*(\omega|_\Sigma)$, so that $\mathcal{F}|_W$ is product of a singular foliation on Σ by the regular foliation of codimension two $\mathcal{E}$ (cf. [K]). This implies that if Σ' is another transverse section through a point $p'\in sing(\omega)$ near p then $BB(\omega, sing(\omega),\Sigma,p) = BB(\omega, sing(\omega),\Sigma',p')$.

When $d\omega(p) = 0$ the 1-jet $\omega_1 = j^1_p(\omega)$ is exact ($d\omega_1 = 0$). Since $\omega_1 \neq 0$ and $codim(sing(\omega)) = 2$ we must have $1 \le codim(sing(\omega_1)) \le 2$. Hence, after a change of variables we can suppose that $p = 0$ and, either $\omega_1 = x\,dy + y\,dx$, or $\omega_1 = x\,dx$. In the case $\omega_1 = x\,dy + y\,dx$ with $cod(sing(\omega)) = 2$, the situation is similar to the Kupka case. It is proved in [Ce-M] that $\mathcal{F}$ is equivalent in a neighborhood of p to the product of a dimension one foliation in a transversal section by regular foliation of codimension two. Hence, if Σ and Σ' are transverse sections to $sing(\omega)$ we have again $BB(\mathcal{F}, sing(\omega),\Sigma,p) = BB(\mathcal{F}, sing(\omega),\Sigma',p')$.

In the case $\omega_1 = x\,dx$ we use a result due to F. Loray. Since this case will appear before in the proofs, we give a formal definition.

Definition 1. We say that a singularity p of the foliation $\mathcal{F}$ is *nilpotent* if there exists an integrable holomorphic 1-form ω defining $\mathcal{F}$ in a neighborhood of p such that $j^1_p(\omega) = x\,dx$, in some coordinate chart $(x,y)\in\mathbb{C}\times\mathbb{C}^{n-1}$ around p such that $x(p) = 0$.

The next result is a consequence of corollary 3 in [Lo] page 710.

Theorem 3.3. ([Lo]) *Let θ be a germ at $(0,0)\in\mathbb{C}\times\mathbb{C}^m$ of holomorphic integrable 1-form, where*

$$\theta = g(w,z)\,dw + \sum_{j=1}^{m} f_j(w,z)\,dz_j \ , \ (w,z) = (w,z_1,...,z_m)\in\mathbb{C}\times\mathbb{C}^m \ .$$

Denote by $\mathcal{F}$ the germ of foliation defined by θ. Assume that $j^1_0(\theta) = w\,dw$. Then there exist local analytic coordinates $(x,\zeta)\in\mathbb{C}\times\mathbb{C}^m$, a germ $f\in\mathcal{O}_m$, with $f(0) = 0$, and germs $g,h\in\mathcal{O}_1$ such that $\mathcal{F}$ is defined in the chart (x,ζ) by the 1-form

$$\omega = x\,dx + [g(f(\zeta)) + x\,h(f(\zeta))]\,df(\zeta) \ . \tag{13}$$

In particular, $\mathcal{F} = \varphi^*(\mathcal{G})$ where $\varphi\colon(\mathbb{C}\times\mathbb{C}^m,0)\to(\mathbb{C}^2,0)$ is given by $\varphi(x,\zeta) = (x,f(\zeta))$ and $\mathcal{G}$ is the germ at $(\mathbb{C}^2,0)$ of foliation defined by

$$\eta := x\,dx + [g(t) + x\,h(t)]\,dt \ .$$

Let us finish the proof of lemma 3.1. Note that if ω is like in (13) then $sing(\omega)\subset(x=0)$. Since we are assuming that $sing(\omega)$ is smooth and has codimension two, after a holomorphic change of variables involving only ζ, we can assume that $sing(\omega)_p = (x=\zeta_1=0)$, where $sing(\omega)_p$ is the germ of $sing(\omega)$ at $p=0$. Therefore,

$$sing(\omega)_p = (x=\zeta_1=0) = (x = g(f(\zeta)) = 0)\cup(x = \partial f/\partial\zeta_1 = ... = \partial f/\partial\zeta_{n-1} = 0) \ .$$

Hence, either $g(0) = 0$ and $\zeta_1\,|\,f$, or $g(0)\neq 0$ and $\zeta_1\,|\,\partial f/\partial\zeta_j$ for all $j = 1,...,n-1$. In any case, we get $\zeta_1\,|\,f$ and so $f(\zeta) = \zeta_1^k.G(\zeta)$, where $G\in\mathcal{O}_n$, $k\in\mathbb{N}$ and $\zeta_1\nmid G$. We have two possibilities :

1^{st}. $G(0)\neq 0$. In this case, after the holomorphic change of variables

$$\Phi(x,\zeta) = (x,\zeta_1.G^{1/k}(\zeta),\zeta_2,...,\zeta_n) = (x,y,\zeta_2,...,\zeta_n) \ ,$$

where $G^{1/k}$ is a branch of the k^{th} rooth of G, we get $f \circ \Phi^{-1} = y^k$ and

$$\Phi_*(\omega) = x\,dx + [g(y^k) + x\,h(y^k)]\,k\,y^{k-1}\,dy := x\,dx + [g_1(y) + x\,h_1(y)]\,dy \ . \tag{14}$$

Hence, in this case $\mathcal{F}$ is locally the product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of codimension two and the argument is similar to the preceding cases.

2^{nd}. $G(0) = 0$. Since $sing(\omega)_p = (x = \zeta_1 = 0)$ and

$$\omega = x\,dx + (g(\zeta_1^k.\,G) + x\,h(\zeta_1^k.\,G))\,\zeta_1^{k-1}\,(\zeta_1.\,dG + k\,G.\,d\zeta_1)$$

we get

2.1. $g(0) \neq 0$, for otherwise $sing(\omega)_p \supset (x = \zeta_1.\,G(\zeta) = 0) \supsetneq (x = \zeta_1 = 0)$.
2.2. $k \geq 2$, for otherwise $\zeta_1 \,|\, G$.

Recall that $\omega = \varphi^*(\eta)$, where $\eta = x\,dx + (g(t) + x\,h(t))\,dt$ and $\varphi(x, \zeta) = (x, f(\zeta))$. Since $g(0) \neq 0$ we have $\eta(0,0) \neq 0$ and the foliation defined by η has a non-constant holomorphic first integral, say $H(x,t)$, in a neighborhood of $0 \in \mathbb{C}^2$, with $H(0,0) = 0$, $\frac{\partial H}{\partial t}(0,0) \neq 0$, $\frac{\partial H}{\partial x}(0,0) = 0$ and $\frac{\partial^2 H}{\partial x^2}(0,0) \neq 0$. This implies that $H_1(x, \zeta) := H(x, \zeta_1^k.\,G(\zeta))$ is a non-constant holomorphic first integral of ω in a neighborhood of $0 \in \mathbb{C}^n$. By using that $\frac{\partial H}{\partial t}(0,0) \neq 0$, $\frac{\partial H}{\partial x}(0,0) = 0$ and $\frac{\partial^2 H}{\partial x^2}(0,0) \neq 0$, it can be checked that for any $q \in sing(\omega)_p$ and any transverse section Σ_q through q then $H_1|_{\Sigma_q}$ has an isolated singularity at q. It follows from (2) in remark 3.1 that $BB(\omega, sing(\omega), \Sigma_q, q) = 0$. This finishes the proof of lemma 3.1. □

Remark 3.2. Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geq 3$. It follows from the argument of [LN] that $sing_2(\mathcal{F})$ has at least one irreducible component of codimension two, say Γ, such that $BB(\mathcal{F}, \Gamma) \neq 0$.

Assume that $\mathcal{J}(\mathcal{F}, p) = 1$ for all $p \in \Gamma$. Denote by $\tilde{\Gamma}$ the smooth part of $sing(\omega)$ contained in Γ. We would like to remark that for any $p \in \tilde{\Gamma}$ then the germ $\mathcal{F}_p$, of $\mathcal{F}$ at p, is equivalent to the product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of codimension two. In fact, we have seen in the proof of lemma 3.1 that the unique case in which perhaps this fact is not true is the 2^{nd}, where $BB(\mathcal{F}, \Gamma) = 0$.

Note that the irreducibility of Γ implies that $\tilde{\Gamma}$ is connected. In particular, there exists a germ of holomorphic 1-form η at $(\mathbb{C}^2, 0)$ such that for any $p \in \tilde{\Gamma}$ then there is a germ of submersion $\varphi\colon (\mathbb{P}^n, p) \to (\mathbb{C}^2, 0)$ such that $\mathcal{F}_p$ is defined by $\varphi^*(\eta)$.

Definition 2. The normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is, by definition, the equivalent class of the foliation defined by η on $(\mathbb{C}^2, 0)$.

Since in the proof of theorem 2 we will deal with nilpotent 1-forms, before closing this section we would like to state a result in which we compute the Baum-Bott index for this type of form.

Lemma 3.2. *Let $U \subset \mathbb{C}$ be an open set and $g_1, h_1 \in \mathcal{O}(U)$, $g_1 \not\equiv 0$. Consider the foliation $\mathcal{G}$ of $\mathbb{C} \times U$ defined by $\eta = 0$, where*

$$\eta = x\,dx + (g_1(t) + x\,h_1(t))\,dt \ .$$

Then for any $t_o \in U$ such that $g_1(t_o) = 0$ we have

$$BB(\mathcal{G}, (0, t_o)) = Res\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t = t_o\right) \ . \tag{15}$$

Proof. The vector field $X = -(x\,h_1(t) + g_1(t))\,\partial_x + x\,\partial_t$ also defines $\mathcal{G}$. Let $\ell \geq 1$ be the multiplicity of g_1 at t_o, so that $g_1(t) = (t - t_o)^\ell.\phi(t)$ and $\phi(t_o) \neq 0$.

Assume first that $\ell = 1$. In this case $g_1'(t_o) = \phi(t_o) \neq 0$ and $(0, t_o)$ is a non-degenerate singularity of X. Therefore, by computing the jacobian matrix of $DX(0, t_o)$ we get from (3) in remark 3.1 that

$$BB(\mathcal{G}, (0, t_o)) = \frac{(tr(DX(0,t_o))^2}{det(DX(0,t_o)} = \frac{h_1(t_o)^2}{g_1'(t_o)} = Res\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t_o\right) .$$

Suppose now that $\ell > 1$. Consider the family $(\mathcal{G}_s)_{s\in\mathbb{C}}$ of foliations defined by $\eta_s = x\,dx + (g_1(t) - s^\ell + x\,h_1(t))\,dt$ and set $\theta_s = \frac{(h_1(t))^2}{g_1(t)-s^\ell}\,dt$. For small $|s| \neq 0$, the equation $g_1(t) = s^\ell$ has exactly ℓ roots near t_o, say $t_1(s), ..., t_\ell(s)$, such that $\lim\limits_{t\to 0} t_j(s) = t_o$ and $g_1'(t_j(s)) \neq 0$ for $s \neq 0$. Therefore, the first case implies that $BB(\mathcal{G}_s, (0, t_j(s))) = Res(\theta_s, t_j(s))$, $1 \leq j \leq \ell$, $|s| \neq 0$ and small. On the other hand, by (4) in remark 3.1 we have

$$BB(\mathcal{G}, (0, t_o)) = \lim_{s\to 0}\left(\sum_{j=1}^{\ell} BB(\mathcal{G}_s, (0, t_j(s)))\right) =$$

$$= \lim_{s\to 0}\left(\sum_{j=1}^{\ell} Res(\theta_s, t_j(s))\right) = Res\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t = t_o\right) . \quad \square$$

3.2. **Proof of theorem 2 in dimension three.** Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^3$ and assume that $sing_2(\mathcal{F})$ has an irreducible component Γ with $BB(\mathcal{F}, \Gamma) \neq 0$ (see remark 3.2) and $\mathcal{J}(\mathcal{F}, p) = 1$ for all $p \in \Gamma$ (which is equivalent to hypothesis (b) of theorem 2 in the case $n = 3$). Since we are working in dimension three, the irreducible components of $sing(\mathcal{F})$ are either curves, or points. As a consequence, the connected component Δ of $sing(\mathcal{F})$ which contains Γ is of pure dimension one, and so is a finite union of irreducible algebraic curves. We denote $sing(\Delta)$ the singular set of Δ and $\tilde{\Gamma} = \Gamma \setminus sing(\Delta)$. Note that any point of $\tilde{\Gamma}$ is a smooth point of Γ. Let η a germ at $0 \in \mathbb{C}^2$ of 1-form representing the normal type of $\mathcal{F}$ along $\tilde{\Gamma}$.

Remark 3.3. Any point $p \in \Gamma \setminus \tilde{\Gamma}$ is a nilpotent singularity of $\mathcal{F}$. Moreover, the normal type η of $\mathcal{F}$ along $\tilde{\Gamma}$ is, either Kupka, or nilpotent. In other words, either $d\eta(0) \neq 0$, or η is nilpotent.

Proof of the remark. As we have seen in the proof of lemma 3.1, for any $q \in \Gamma$ we have two possibilities :

(i). The germ of $\mathcal{F}$ at q is equivalent to a product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of dimension one.

(ii). q is a nilpotent singularity of $\mathcal{F}$.

In case (i) the germ of $sing(\mathcal{F})$ at q is smooth of codimension two and so $q \in \tilde{\Gamma}$. This proves the first assertion.

On the other hand, if the normal type is not Kupka then $d\eta(0) = 0$ and $\eta_1 = j^1_0(\eta) \neq 0$ is exact. If η_1 is not nilpotent, then $\eta_1 = x\,dy + y\,dx$ in some chart. But, this implies that $BB(\mathcal{F}, \Gamma) = 0$, which contradicts $BB(\mathcal{F}, \Gamma) \neq 0$. $\square$

Definition 3. A separatrix of $\mathcal{F}$ along Γ is a germ of hypersurface Σ along Γ which is $\mathcal{F}$-invariant. In other words, given $p \in \Gamma$ there exists a germ $u_p \in \mathfrak{m}_p \setminus \{0\}$ such that :

(a). The ideal of the germ Σ_p of Σ at p is generated by u_p.
(b). The germ Γ_p of Γ at p is contained in Σ_p.
(c). If $\mathcal{F}$ is represented by a holomorphic 1-form ω in a neighborhood of p then $du_p \wedge \omega = u_p.\Theta$, where Θ is a germ of holomorphic 2-form. This condition is equivalent to the $\mathcal{F}$-invariance of Σ.
(d). If u is a representative of u_p in a small neighborhood U of p then, for any $q \in \Gamma \cap U$ there exists $g \in \mathcal{O}_q^*$ such that $u_q = g.(u)_q$, where $(u)_q$ denotes the germ of u at q.

We say that Σ is smooth if $du_p(p) \neq 0$ for all $p \in \Gamma$.

Consider now the normal type $\eta = P(x,y)\,dy - Q(x,y)\,dx$ of $\mathcal{F}$ along $\tilde{\Gamma}$. Assume that η has a smooth separatrix $\sigma = (u(x,y) = 0)$, $u \in \mathfrak{m}_2 \setminus \{0\}$, $du(0) \neq 0$. Let $\Pi\colon (M,D) \to (\mathbb{C}^2, 0)$ be the minimal Seidenberg's resolution of singularities of η, in the sense of [C-LN-S] or [B 1]. Denote by $\mathcal{G}$ be the foliation on (M,D) defined by the strict transform of $\Pi^*(\eta)$. We would like to recall that :

(A). $D = \bigcup_{j=1}^k D_j$, where each divisor D_j is biholomorphic to $\mathbb{P}^1$. Moreover, if $i \neq j$ and $D_i \cap D_j \neq \emptyset$ then $D_i \cap D_j = \{p\}$ and D_i cuts D_j tranversely at p. The divisor D_j is *dicritical* if it is not $\mathcal{G}$-invariant. Otherwise, it is *non-dicritical*.
(B). All singularities of $\mathcal{G}$ in D are simple, in the sense that if $p \in sing(\mathcal{G}) \cap D$ and $\mathcal{G}$ is represented by a holomorphic vector field X in a neighborhood of p then the eigenvalues λ_1, λ_2 of $DX(p)$ satisfy one the conditions bellow :
  (B.1). If one of the eigenvalues is zero then the other is non-zero. In this case, p is a saddle-node of $\mathcal{G}$.
  (B.2). $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$.

Definition 4. Let σ be a smooth separatrix of η and $\Pi\colon (M,D) \to (\mathbb{C}^2,0)$, $D = \bigcup_i D_i$, and $\mathcal{G}$ be as above. Let $\hat{\sigma}$ be the strict transform of σ by Π, where $\hat{\sigma} \cap D = \{p\}$. We say that σ is a *distinguished* separatrix of η if for any other smooth separatrix, say σ_1, of η, with strict transform $\hat{\sigma}_1$ and $\hat{\sigma}_1 \cap D = \{q\}$ $(p \neq q)$ then there is no local biholomorphism $\Phi\colon (M,p) \to (M,q)$ such that $\Phi^*(\mathcal{G}_q) = \mathcal{G}_p$, where $\mathcal{G}_x$ denotes the germ of $\mathcal{G}$ at $x \in D$.

Remark 3.4. We would like to remark the following facts :

(I). When 0 is already a simple singularity of η then η has at least one and at most two analytic separatrices through 0, all smooth (cf. [C-S]). In the case (B.2) it has exactly two, each one tangent to an eigendirection of $DX(0)$. In the case (B.1) it has always one, which is tangent to the non-zero eigenvalue of $DX(0)$. Sometimes it has also another tangent to the eigendirection of the eigenvalue 0. We would like to observe that all separatrices of η are distingueshed, except when $\lambda_1 = -\lambda_2 \neq 0$. However, in this last case we have $BB(\eta,0) = 0$.

(II). If $\Phi \in Diff(\mathbb{C}^2,0)$ preserves the foliation defined by η and σ is a distinguished separatrix of η then $\Phi(\sigma) = \sigma$. In other words, if $\Phi^*(\eta) = h.\eta$, where $h \in \mathcal{O}_2^*$ then $\Phi(\sigma) = \sigma$. This follows from the fact that there is a germ of biholomorphism $\hat{\Phi}\colon (M,D) \to (M,D)$ such that $\Pi \circ \hat{\Phi} = \Phi \circ \Pi$.

(III). When the strict transform $\hat{\sigma}$ of σ cuts transversely some dicritical divisor in a regular point q of $\mathcal{G}$ then it is not distinguished. This follows from the fact that there exists a chart $(W,(u,v))$ around q such that $W\cap D=(v=0)$, $W\cap\hat{\sigma}=(u=0)$ and $\mathcal{G}|_W$ is defined by $du=0$.

We will say that a separatrix Σ of $\mathcal{F}$ along Γ *extends* a separatrix σ of η, if for some transverse section Λ through a point $p\in\tilde{\Gamma}$, where $\mathcal{F}|_\Lambda$ is defined by η, then σ coincides with $\Sigma\cap\Lambda$. We will say also that σ can be extended to Σ.

Lemma 3.3. *If the normal type $\eta=P(x,y)\,dy-Q(x,y)\,dx$ has a dintinguished smooth separatrix σ then it can be extended to a smooth separatrix Σ of $\mathcal{F}$ along Γ.*

Proof. Let us prove first that σ extends to a germ of separatrix $\tilde{\Sigma}$ of $\mathcal{F}$ along $\tilde{\Gamma}$. It follows from the definition of the normal type that there exists a covering $(W_\alpha)_{\alpha\in A}$ of $\tilde{\Gamma}$ by polydiscs with the following properties :

(i). $W_\alpha\cap\tilde{\Gamma}$ is connected and non-empty for all $\alpha\in A$. If $W_{\alpha\beta}:=W_\alpha\cap W_\beta\neq\emptyset$ then $W_{\alpha\beta}\cap\tilde{\Gamma}$ is connected and non-empty.

(ii). For all $\alpha\in A$ there is a chart $(x_\alpha,y_\alpha,z_\alpha)\colon W_\alpha\to\mathbb{C}^3$ such that $\mathcal{F}|_W$ is represented by $\eta_\alpha=P(x_\alpha,y_\alpha)\,dy_\alpha-Q(x_\alpha,y_\alpha)\,dx_\alpha$ and $\tilde{\Gamma}\cap W_\alpha=(x_\alpha=y_\alpha=0)$.

Let $u(x,y)=0$ be an equation of σ and define $u_\alpha(x_\alpha,y_\alpha,z_\alpha):=u(x_\alpha,y_\alpha)\in\mathcal{O}(W_\alpha)$. Set $\Sigma_\alpha=(u_\alpha=0)$. Since σ is smooth we have $du(0)\neq0$, which implies $du_\alpha(0,0,z_\alpha)\neq0$, so that Σ_α is smooth along $\tilde{\Gamma}\cap W_\alpha$.

Fix $W_{\alpha\beta}\neq\emptyset$. Since $\mathcal{F}|_{W_{\alpha\beta}}$ is represented by $\eta_\alpha|_{W_{\alpha\beta}}$ and by $\eta_\beta|_{W_{\alpha\beta}}$ there exists $\varphi\in\mathcal{O}^*(W_{\alpha\beta})$ such that $\eta_\alpha=\varphi.\eta_\beta$. Let Λ be a transverse section through a point $q\in\tilde{\Gamma}\cap W_{\alpha\beta}$. Then $\Sigma_\alpha\cap\Lambda$ and $\Sigma_\beta\cap\Lambda$ are separatrices of $\eta_\alpha|_\Lambda$ and $\eta_\beta|_\Lambda$, respectively. Since they correspond to σ, which is distinguished, they must coincide, by (II) in remark 3.4. This implies $\Sigma_\alpha\cap W_{\alpha\beta}=\Sigma_\beta\cap W_{\alpha\beta}$. In particular, there exists a germ of hypersurface $\tilde{\Gamma}$, which extends σ, and such that $\tilde{\Gamma}\cap W_\alpha=\Gamma_\alpha$ for all $\alpha\in A$.

This finishes the proof when $\tilde{\Gamma}=\Gamma$. Assume that $\Gamma\setminus\tilde{\Gamma}\neq\emptyset$ and let us prove that $\tilde{\Sigma}$ extends to a smooth separatrix Σ of $\mathcal{F}$ along Γ.

Fix a point $p\in\Gamma\cap sing(\Delta)=\Gamma\setminus\tilde{\Gamma}$. Since p is a nilpotent singularity of $\mathcal{F}$, by Loray's normal form, we can find a chart $(x,s,t)\colon U\to\mathbb{C}^3$ such that $\mathcal{F}|_U$ is represented by

$$\omega=x\,dx+(g(f(s,t))+x\,h(f(s,t)))\,df(s,t)$$

and $\Gamma\cap U=(x=f(s,t)=0)$. As we have seen before, given $q\in\tilde{\Gamma}\cap U$ there is a local chart $(W,(x,y,z))$ with $W\subset U$, $x(q)=y(q)=z(q)=0$, $f|_W=y^k$, $k\geq1$, and

$$\omega|_W=x\,dx+(g(y^k)+x\,h(y^k))\,k\,y^{k-1}\,dy:=x\,dx+(\tilde{g}(y)+x\,\tilde{h}(y))\,dy\ .$$

Let Λ be the transverse section $(z=0)$ and set

$$\theta=\omega|_\Lambda=x\,dx+(\tilde{g}(y)+x\,\tilde{h}(y))\,dy\ .$$

Note that $\tilde{g}(0)=0$, because $(0,0)$ is a singularity of θ.

Let $\alpha\in A$ be such that $q\in W_\alpha$ and $x(q)=y(q)=z(q)=0$. If we cut $\Sigma_\alpha=\tilde{\Sigma}\cap W_\alpha$ by the transverse section $\Lambda=(z=0)$, then we find a smooth separatrix $\tilde{\sigma}:=\Sigma_\alpha\cap\Lambda$ of the differential equation $\eta_\alpha|_\Lambda=0$, which is also a separatrix of $\theta=0$ and corresponds to the separatrix σ of η.

The idea is to prove that $\tilde{\sigma}$ admits an equation in the chart (x,y) of the form $x = \psi(y^k)$, $\psi \in \mathfrak{m}_1$. Since $f|_W = y^k$, this will imply that the form ω has a smooth separatrix with equation $x = \psi(f(s,t))$ which extends $\tilde{\Sigma}$ to a neighborhood of p. This will finish the proof of lemma 3.3.

We assert that $\tilde{\sigma}$ is not tangent to the x-axis. This will imply that $\tilde{\sigma}$ admits an equation of the form $x = \phi(y)$, $\phi \in \mathfrak{m}_1$, because it is smooth.

In fact, assume by contradiction that $\tilde{\sigma}$ is tangent to the x-axis. In this case, it admits an equation of the form $y = \phi(x)$, where $\phi(0) = \phi'(0) = 0$. Since $\tilde{\sigma}$ is a solution of $\theta = 0$, we get

$$x + (\tilde{g}(\phi(x)) + x\,\tilde{h}(\phi(x)))\,\phi'(x) \equiv 0\ .$$

Since $\tilde{g}(0) = \phi(0) = \phi'(0) = 0$, the above relation implies

$$x = j^1_0\,[x + (\tilde{g}(\phi(x)) + x\,\tilde{h}(\phi(x)))\,\phi'(x)] = 0\ ,$$

a contradiction. Therefore, $\tilde{\sigma}$ admits an equation of the form $x = \phi(y)$ with $\phi \in \mathfrak{m}_1$. When $k = 1$ this already proves that $\tilde{\Sigma}$ can be extended to a smooth surface in a neighborhood of p. When $k > 1$ we consider the automorphism $\Phi\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ given by $\Phi(x,y) = (x, \zeta.\,y)$, where ζ is a primitive k^{th}-root of unity. Since $\theta = x\,dx + (g(y^k) + x\,h(y^k))\,d(y^k)$ we get $\Phi^*(\theta) = \theta$. This implies $\Phi(\tilde{\sigma}) = \tilde{\sigma}$, because $\Phi(\tilde{\sigma})$ is a separatrix of θ and $\tilde{\sigma}$ is distinguished. On the other hand,

$$\Phi(\tilde{\sigma}) = \Phi(x-\phi(y)=0) = (x-\phi(\zeta.\,y)=0) \implies (x-\phi(y)=0) = (x-\phi(\zeta.\,y)=0)$$

$$\implies \phi(\zeta.\,y) = \phi(y)\ ,\ \forall\, y \in (\mathbb{C},0) \implies \phi(y) = \psi(y^k)\ ,\ \psi \in \mathfrak{m}_1\ .$$

This finishes the proof of lemma 3.3. □

The next result will be used several times in the rest of the proof.

Proposition 3.1. *Let γ be an irreducible curve of $\mathbb{P}^3$. Assume that there exists a germ Σ of smooth surface along γ. If N_Σ denotes the normal bundle of Σ and $c_1(N_\Sigma)$ its the first Chern class then*

$$\int_\gamma c_1(N_\Sigma) > 0\ .$$

In particular, the above integral is a positive integer.

Proof of proposition 3.1. According to the definition, by taking a representative of Σ in a sufficiently small neighborhood W of γ and a covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of W by polydiscs, we can say that there exist

I. A collection $(u_\alpha)_{\alpha\in A}$, where $u_\alpha \in \mathcal{O}(U_\alpha)$, $\Sigma_\alpha := \Sigma \cap U_\alpha = (u_\alpha = 0)$ and $du_\alpha(q) \neq 0$ for all $q \in \Sigma_\alpha$,

II. A multiplicative cocycle $(A_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$, $U_{\alpha\beta} := U_\alpha \cap U_\beta$, such that $u_\alpha = A_{\alpha\beta}.u_\beta$ for any $U_{\alpha\beta} \neq \emptyset$.

The cocycle $(A_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ represents the normal bundle N_Σ of Σ in $H^1(\mathcal{U},\mathcal{O}^*)$. Let $c_1(N_\Sigma)$ be the first Chern class of N_Σ, considered as an element of $H^2_{DR}(W)$.

Denote by $\mathcal{X}_3$ the set of holomorphic vector fields on $\mathbb{P}^3$. It is well known that $dim(\mathcal{X}_3) = 15$. Given $X \in \mathcal{X}_3$ let $Tang(X,\Sigma) \subset \Sigma$ be the divisor of tangencies of X with Σ. This divisor can be expressed as follows in the covering $\mathcal{U}$: if $q \in U_\alpha \cap \Sigma$ then $q \in |Tang(X,\Sigma)| \cap U_\alpha$ if, and only if, $X(u_\alpha)(q) = u_\alpha(q) = 0$. Set $\Sigma_\alpha := U_\alpha \cap \Sigma$ and $g_\alpha := X(u_\alpha)|_{\Sigma_\alpha}$.

Let $B = \{\alpha \in A \,|\, \Sigma_\alpha \neq \emptyset\}$. If $\alpha, \beta \in B$ and $\Sigma_{\alpha\beta} \neq \emptyset$ then

$$X(u_\alpha) = X(A_{\alpha\beta}.u_\beta) = A_{\alpha\beta}.X(u_\beta) + u_\beta.X(A_{\alpha\beta}) \implies g_\alpha = a_{\alpha\beta}.g_\beta \ ,$$

where $a_{\alpha\beta} = A_{\alpha\beta}|_\Sigma$. Hence, $(a_{\alpha\beta})_{\Sigma_{\alpha\beta}\neq\emptyset}$ is a multiplicative cocycle and $(g_\alpha)_{\alpha\in B}$ defines the divisor $Tang(X, \Sigma)$ of Σ.

If X is not completely tangent to Σ then $(g_\alpha)_{\alpha\in B}$ is effective ($g_\alpha \not\equiv 0$ for all α), which implies that $Tang(X, \Sigma).[\gamma] \geq 0$.

A straightforward computation in affine coordinates shows that, given $p \neq q \in \gamma$ there exists $X \in \mathcal{X}_3$ such that $X(p) \in T_p\Sigma$ and $X(q) \notin T_q\Sigma$, where $T_x\Sigma$ denotes the tangent space of Σ at $x \in \Sigma$. Let us fix such vector field. Since $X(q) \notin T_q\Sigma$, $Tang(X, \Sigma)$ is effective. Since $X(p) \in T_p\Sigma$ we have $p \in |Tang(X, \Sigma)| \cap \gamma$, which implies $Tang(X, \Sigma).[\gamma] > 0$. On the other hand, it is known that

$$Tang(X, \Sigma).[\gamma] = \int_\gamma c_1(Tang(X, \Sigma)) \implies \int_\gamma c_1(Tang(X, \Sigma)) > 0 \ .$$

Since the cocycle associated to $Tang(X, \Sigma)$ in the covering $(\Sigma_\alpha)_{\alpha\in B}$ is $(a_{\alpha\beta} = A_{\alpha\beta}|_\Sigma)_{\alpha\beta}$, we get

$$c_1(Tang(X, \Sigma)) = c_1(N_\Sigma)|_\Sigma \implies \int_\gamma c_1(N_\Sigma) > 0 \ . \quad \square$$

Lemma 3.4. *The normal type η is not nilpotent.*

Proof. The proof will be by contradicition. Assuming that all points of Γ are nilpotent, we will prove that $\mathcal{F}$ has a smooth separatrix Σ along Γ and that

$$\int_\Gamma c_1(N_\Sigma) = 0 \ ,$$

which contradicts proposition 3.1.

In the proof of the existence of the smooth separatrix we will need the resolution of singularities of a nilpotent 1-form η on $(\mathbb{C}^2, 0)$ with $BB(\eta, 0) \neq 0$. The following consequence of lemma 3.2 will be usefull.

Corollary 3.1. *Let*

$$\eta = x\,dx + (g(y) + x\,h(y))\,dy \ , \tag{16}$$

where $BB(\eta, 0) \neq 0$. Then $h \not\equiv 0$ and

$$\nu(g, 0) \geq 2\,\nu(h, 0) + 1 \ , \tag{17}$$

where $\nu(\,.\,, 0)$ denotes the multiplicity at $0 \in \mathbb{C}$.

Proof. It follows from lemma 3.2 that

$$0 \neq BB(\eta, 0) = Res\left(\frac{h(y)^2}{g(y)}\,dy, y = 0\right) \ .$$

This implies $h \not\equiv 0$ and $\nu(g, 0) > 2\,\nu(h, 0)$, because otherwise $\frac{h(y)^2}{g(y)}\,dy$ would be holomorphic at $0 \in \mathbb{C}$ and the residue would vanish. $\square$

Existence of the smooth separatrix along Γ. We can assume that the normal type is given by η as in (16). Since $BB(\eta, 0) = BB(\mathcal{F}, \Gamma) \neq 0$ we get from corollary 3.1 that $h \not\equiv 0$ and $m \geq 2n + 1 \geq 3$, where $\nu(g, 0) := m$ and $\nu(h, 0) := n$. Note that $n \geq 1$, because otherwise η would not be nilpotent. According to lemma 3.3 it is sufficient to prove that η has a dintinguished smooth separatrix.

Let us give a brief description of the Seidenberg resolution of η (cf. [Me]). Write $g(y) = y^m . \zeta_1(y)$ and $h(y) = y^n . \zeta_2(y)$, where $\zeta_j(0) \neq 0$, $j = 1, 2$, so that

$$\eta = x\,dx + (y^m . \zeta_1(y) + x\,y^n . \zeta_2(y))\,dy \ .$$

After the $(n+1)^{th}$ step of this resolution we find a chain of divisors $D_1, ..., D_{n+1}$ and a blowing-up map $\Pi\colon (M, D) \to (\mathbb{C}^2, 0)$, $D = \bigcup_j D_j = \Pi^{-1}(0)$, where :

(I). $D_j . D_i = 0$ if $j < i - 1$ and $D_j . D_{j+1} = 1$, $1 \leq j < i \leq n + 1$.
(II). $D_j^2 = -2$ if $1 \leq j \leq n$ and $D_{n+1}^2 = -1$.

Let us denote by $\mathcal{G}$ the strict transform of the foliation defined by $\Pi^*(\eta)$. It can be proved that (cf. [Me]) :

(III). All the divisors $D_1, ..., D_{n+1}$ are $\mathcal{G}$-invariant.
(IV). If $j < n + 1$ then $sing(\mathcal{G}) \cap D_j = D_j \cap D_{j+1} := \{p_j\}$. Moreover, if X_j is a vector field representing $\mathcal{G}$ around p_j then $DX_j(p_j)$ has eigenvalues $\lambda_1^j, \lambda_2^j \neq 0$ with $\lambda_1^j / \lambda_2^j \in \mathbb{Q}_-$. In particular, p_j is a simple singularity of X_j and $\mathcal{G}$ has only two separatrices through p_j, which are contained in the divisors D_j and D_{j+1}.
(V). The divisor D_{n+1} appears after the $(n + 1)^{th}$ blowing-up. Moreover, there is a chart $(u, y) \in \mathbb{C}^2$ around $D_{n+1} \setminus \{p_n\}$, where $\Pi(u, y) = (y^{n+1}.u, y) = (x, y)$. In this chart, we get $D_{n+1} \setminus \{p_n\} = (y = 0)$ and $\Pi^*(\eta) = y^{2n+1} . \alpha$, with

$$\alpha = u\,y\,du + \Big((n + 1)\,u^2 + \zeta_2(y)\,u + y^{m-(2n+1)}\,\zeta_1(y)\Big)\,dy \ .$$

The idea is to prove that $\mathcal{G}$ has a distinguished smooth separatrix $\hat{\sigma}$ transverse to D_{n+1} with an equation of the form $u = \zeta(y)$, where $\zeta \in \mathcal{O}_1^*$ and $(\zeta(0), 0)$ is a singularity of $\mathcal{G}$. In this case, if $\sigma = \Pi(\hat{\sigma})$ then σ admits an equation of the form $x = y^{n+1} . u = y^{n+1} . \zeta(y)$. In particular, σ will be a smooth distinguished separatrix of η.

The foliation $\mathcal{G}$ is defined around $D_{n+1} \setminus \{p_n\}$, in the chart (u, y), by the vector field

$$Z = \Big((n + 1)\,u^2 + \zeta_2(y)\,u + y^{m-(2n+1)}\,\zeta_1(y)\Big)\,\partial_u - u\,y\,\partial_y \ .$$

If we set $a = \zeta_2(0) \neq 0$, $b = 0$ if $m > 2n+1$ and $b = \zeta_1(0)$ if $m = 2n+1$, then, in this chart, the singularities of Z along $(y = 0) \subset D_{n+1}$ are $q_1 = (u_1, 0)$ and $q_2 = (u_2, 0)$, where u_1, u_2 are the roots of $(n+1)\,u^2 + a\,u + b = 0$. The eigenvalues of $DZ(q_i)$ are $\lambda_t^i = 2(n + 1)\,u_i + a$ and $\lambda_n^i = -u_i$, where λ_t^i corresponds the eigendirection of the separatrix $(y = 0)$, $i = 1, 2$. Since Z is not nilpotent at q_i, $i = 1, 2$, we can apply the classification of non-nilpotent singularities. According to the values of a and b, we have three possibilities :

1^{st}. $b \neq 0$ and $a^2/b = 4(n + 1)$. In this case, $q_1 = q_2 = (-a/2(n + 1), 0)$. The singularity is a saddle-node, $\lambda_t^1 = 0$ and $\lambda_n^1 = a/2(n + 1) \neq 0$. It follows that $\mathcal{G}$ has an unique separatrix $\hat{\sigma}$ through q_1, which is smooth and transverse to the divisor. The separatrix $\sigma = \Pi(\hat{\sigma})$ is the unique one of η and so it is distinguished.

2^{nd}. $b \neq 0$ and $a^2/b \neq 4(n + 1)$. In this case, $q_1 \neq q_2$ and $\lambda_t^i, \lambda_n^i \neq 0$, $i = 1, 2$. Since $\lambda_t^i = 2(n + 1)u_i + a$ and $\lambda_n^i = -u_i$, it follows that $\lambda_n^1/\lambda_t^1 \neq \lambda_n^2/\lambda_t^2$. On the other hand, a straightforward computation, using the values of λ_t^i and λ_n^i, $i = 1, 2$ (or the Camacho-Sad theorem [C-S]), shows that :

$$\frac{\lambda_n^1}{\lambda_t^1} + \frac{\lambda_n^2}{\lambda_t^2} = -\frac{1}{n + 1} \ .$$

Since $\lambda_n^1/\lambda_t^1 \neq \lambda_n^2/\lambda_t^2$, either $\lambda_n^1/\lambda_t^1 \notin \mathbb{Q}_+$, or $\lambda_n^2/\lambda_t^2 \notin \mathbb{Q}_+$. If, for instance, $\lambda_n^1/\lambda_t^1 \notin \mathbb{Q}_+$ then $\mathcal{G}$ has an unique smooth separatrix $\hat{\sigma}$ through q_1, transverse to the divisor $(y=0)$, with

$$CS(\mathcal{G},\hat{\sigma}) = \lambda_t^1/\lambda_n^1 \notin \mathbb{Q}_+ \ ,$$

where $CS(\mathcal{G},\hat{\sigma})$ denotes the Camacho-Sad index of the separatrix $\hat{\sigma}$ with respect to $\mathcal{G}$ (cf. [C-S]). If η has another smooth separatrix, say σ', then its strict transform $\hat{\sigma}'$ must satisfy $\hat{\sigma}' \cap (y=0) = \{q_2\}$ and

$$CS(\mathcal{G},\sigma') = \lambda_t^2/\lambda_n^2 \neq CS(\mathcal{G},\hat{\sigma}) \ .$$

Since the Camacho-Sad index is an analytic invariant of the pair ($\mathcal{G}$, separatrix), it follows that $\sigma = \Pi(\hat{\sigma})$ is a smooth dinguished separatrix of η.

3^{rd}. $b=0$. In this case, we can take $u_2 = 0$ and $u_1 = -a/(n+1) \neq 0$, which give $\lambda_n^2 = 0$, $\lambda_t^2 = a \neq 0$ and $\lambda_t^1/\lambda_n^1 = -(n+1) \notin \mathbb{Q}_+$. In particular, q_2 is a saddle-node and $\mathcal{G}$ has an unique separatrix $\hat{\sigma}$ through q_1. In this case, $\sigma = \Pi(\hat{\sigma})$ is a distinguished separatrix of η because q_2 is saddle-node and q_1 is not.

This proves the existence of the smooth separatrix Σ of $\mathcal{F}$ along Γ.

Proof of $\int_{\gamma_j} c_1(N_\Sigma) = 0$. We have seen that given $p \in \Gamma$ then :

(i). There exists a germ of local chart $\psi = (x,y,z)\colon (\mathbb{P}^3,p) \to (\mathbb{C}^3,0)$ such that the germ Γ_p, of Γ at p, satisfies $\Gamma_p \subset (x=0)$.

(ii). There exist $\zeta_1,\zeta_2 \in \mathcal{O}_1^*$ and $f \in \mathfrak{m}_2$, depending only on (y,z), such that $\mathcal{F}_p$ is represented by

$$\omega = x\,dx + (f^m.\,\zeta_1(f) + x\,f^n.\,\zeta_2(f))\,df \ .$$

In particular Γ_p is defined by $(x=f=0)$.

(iii). The germ Σ_p, of Σ at p, is defined by $x - f^{n+1}(y,z)\,\zeta(f(y,z)) = 0$, where $\zeta \in \mathcal{O}_1^*$. Set $\phi(t) = t^{n+1}\,\zeta(t)$.

(iv). When $p \in \tilde{\Gamma}$ then we can choose the chart in such a way that $\Gamma_p = (x = y = 0)$ and $f(y,z) = y^k$, $k \geq 1$.

When we consider the change of variables $\Psi(u,y,z) = (u + \phi(f(y,z)),y,z)$ then a straightforward computation shows that :

$$\Psi^*(\omega) = (u + \phi(f(y,z)))\,du + u\,\left[f^n(y,z).\,\zeta_2(f(y,z)) + \phi'(f(y,z))\right]\,df(y,z) \ .$$

In particular, in the new chart we have $\Sigma_p = (u=0)$. This implies that, if we choose a small neighborhood U of Γ, where the germ Σ has a representative, then we can find a finite covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of U by polydiscs with the following properties

(v). $U_\alpha \cap \Gamma \neq \emptyset$ for all α, and $U_{\alpha\beta} \cap \Gamma \neq \emptyset$ for all $U_{\alpha\beta} \neq \emptyset$.

(vi). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$ then $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma})$ contains just one point. Moreover, if $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then $p \notin U_\beta$ for all $\beta \in A$ with $\beta \neq \alpha$.

(vii). For all $\alpha \in A$ there exists a holomorphic chart $\Psi_\alpha = (u_\alpha,y_\alpha,z_\alpha)\colon U_\alpha \to \mathbb{C}^3$, such that $\Psi_\alpha(U_\alpha) = \mathbb{D}^3$ and $\Sigma \cap U_\alpha = (u_\alpha = 0)$, so that $\Psi_a(U_\alpha \cap \Sigma) = \{0\} \times \mathbb{D}^2$.

(viii). For all $\alpha \in A$ there exist $f_\alpha \in \mathcal{O}(\mathbb{D}^2)$ and $\phi_\alpha \in \mathcal{O}(f_\alpha(\mathbb{D}^2))$, with $\phi_\alpha(t) = t^{n+1}\,\zeta_\alpha(t)$, $\zeta_\alpha \in \mathcal{O}^*(f_\alpha(\mathbb{D}^2))$, such that $\mathcal{F}|_{U_\alpha}$ is represented in the chart (U_α,Ψ_α) by

$$\omega_\alpha = (u_\alpha + \phi_\alpha(f_\alpha))\,du_\alpha + u_\alpha\,\left(f_\alpha^n.\,\zeta_2(f_\alpha) + \phi_\alpha'(f_\alpha)\right)\,df_\alpha \ .$$

(ix). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$ then $f_\alpha(y_\alpha, z_\alpha) = y_\alpha^k$.

(x). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then $\Psi_a(p) = 0$. Moreover, if $q \in \Gamma \setminus \{p\}$ then there exists a chart $(W, (u_\alpha, v_\alpha, w_\alpha))$ around q such that $f_\alpha|_W = v_\alpha^k$.

It follows from (vii) that there exists a multiplicative cocycle $G = (g_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ such that $u_\alpha = g_{\alpha\beta}.u_\beta$ on $U_{\alpha\beta} \neq \emptyset$. The cocycle G represents N_Σ in $H^1(\mathcal{U}, \mathcal{O}^*)$. The idea is to prove that $g_{\alpha\beta}|_{\Gamma\cap U_{\alpha\beta}}$ is locally constant for all $U_{\alpha\beta} \neq \emptyset$. This will imply that $\int_\Gamma c_1(N_\Sigma) = 0$.

Since ω_α represents $\mathcal{F}|_{U_\alpha}$, there exists a multiplicative cocycle $(\varphi_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ such that $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$. Fix $U_{\alpha\beta} \neq \emptyset$ and $q \in \Gamma \cap U_{\alpha\beta}$. Let us prove that $g_{\alpha\beta}|_{\Gamma\cap U_{\alpha\beta}}$ is constant in a neighborhood of q.

Note that $q \in \tilde{\Gamma}$, because $(\Gamma \setminus \tilde{\Gamma}) \cap U_{\alpha\beta} = \emptyset$ by (vi). On the other hand, (ix) and (x) imply that we can find germs of charts $(u_\alpha, v_\alpha, w_\alpha)\colon (\mathbb{P}^3, q) \to (\mathbb{C}^3, 0)$ and $(u_\beta, v_\beta, w_\beta)\colon (\mathbb{P}^3, q) \to (\mathbb{C}^3, 0)$ such that

(xi). $\Gamma_q = (u_\alpha = v_\alpha = 0) = (u_\beta = v_\beta = 0)$ and $\Sigma_q = (u_\alpha = 0) = (u_\beta = 0)$.

(xii). If $i \in \{\alpha, \beta\}$ then $\mathcal{F}_q$ is represented by the germ at $0 \in \mathbb{C}^3$ of

$$\omega_i = (u_i + \phi_i(v_i^k))\, du_i + u_i\, (v_i^n.\, \zeta_2(v_i^k) + \phi_i'(v_i^k))\, d(v_i^k)\ .$$

In particular, we get from (xi) that $u_\alpha = g_{\alpha\beta}.u_\beta$ and $v_\alpha = h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta$, where $h_{\alpha\beta}, k_{\alpha\beta} \in \mathcal{O}_q$ and $g_{\alpha\beta}.k_{\alpha\beta} \in \mathcal{O}_q^*$. If we substitute these relations in ω_α then we get the expression of ω_α in the other coordinate system

$$\omega_\alpha = (g_{\alpha\beta}.u_\beta + \phi_\alpha((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k))\,(g_{\alpha\beta}\, du_\beta + u_\beta\, dg_{\alpha\beta}) + \tag{18}$$

$$+ g_{\alpha\beta}\, u_\beta\, (((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^s.\, \zeta_2(v_\alpha^k) + \phi_i'((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k))\, d(h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k$$

$$:= A(u_\beta, v_\beta, w_\beta)\, du_\beta + B(u_\beta, v_\beta, w_\beta)\, dv_\beta + C(u_\beta, v_\beta, w_\beta)\, dw_\beta\ .$$

Since $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$ and ω_β has no term with dw_β, we get $C \equiv 0$. Write

$$C(u_\beta, v_\beta, w_\beta) = \sum_{i,j\geq 0} C_{ij}(w_\beta)\, u_\beta^i\, v_\beta^j\ .$$

It follows from (18) that $C_{00}(w_\beta) = C_{10}(w_\beta) = C_{01}(w_\beta) = 0$ and

$$C_{20}(w_\beta) = g_{\alpha\beta}(0,0,w_\beta).\frac{\partial g_{\alpha\beta}(0,0,w_\beta)}{\partial w_\beta} = 0 \implies \frac{\partial g_{\alpha\beta}(0,0,w_\beta)}{\partial w_\beta} = 0 \implies$$

$$\implies g_{\alpha\beta}|_{U_{\alpha\beta}\cap\Gamma} \text{ is locally constant.}$$

Recall that $c_1(N_\sigma)|_\Gamma$ can be obtained from the additive cocycle of $\left(\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}|_{U_{\alpha\beta}}\right)_{U_{\alpha\beta}\neq\emptyset}$ by taking a fine resolution. Since $\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}|_{\Gamma\cap U_{\alpha\beta}} = 0$ we get $\int_\Gamma c_1(N_\Sigma) = 0$. This finishes the proof of lemma 3.4. □

Remark 3.5. Lemma 3.4 implies that $\tilde{\Gamma} \subset K(\mathcal{F})$, the set of singularities of Kupka type of $\mathcal{F}$.

Corollary 3.2. *If* $\Gamma \setminus \tilde{\Gamma} = \emptyset$ *then theorem 2 is true in dimension three.*

Proof. If $\Gamma = \tilde{\Gamma}$ then it is smooth and $\Gamma \subset K(\mathcal{F})$. Therefore, $\mathcal{F}$ has a meromorphic first integral by [CA 2] and [B 2]. □

In view of corollary 3.2, from now on we will assume that $\Gamma \setminus \tilde{\Gamma} \neq \emptyset$. Another consequence of lemma 3.4 is the following :

Corollary 3.3. *Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and consider a germ of holomorphic chart $(x, y, z)\colon (\mathbb{P}^3, p) \to (\mathbb{C}^3, 0)$ such that $\mathcal{F}_p$ is represented in this chart by the form*

$$\omega = x\,dx + (f^m(y,z).\,\zeta_1(f(y,z)) + x\,f^n(y,z).\,\zeta_2(f(y,z)))\,df(y,z)\ ,$$

where $f \in \mathfrak{m}_2$ and $\zeta_1, \zeta_2 \in \mathcal{O}_1^*$. Then :

(a). $n = 0$ *and* $m \geq 1$.
(b). $0 \in \mathbb{C}^2$ *is a singularity of f and f is reduced in $\mathcal{O}_2$.*

Proof. Note that

$$d\omega = f^n(y,z).\,\zeta_2(f(y,z))\,dx \wedge df(y,z) =$$
$$= f^n(y,z).\,\zeta_2(f(y,z))\left[\frac{\partial f(y,z)}{\partial y}\,dx \wedge dy + \frac{\partial f(y,z)}{\partial z}\,dx \wedge dz\right]\ .$$

Since $\tilde{\Gamma} \subset K(\mathcal{F})$ we must have $(\omega = d\omega = 0) = \{0\}$. Therefore, $n = 0$, $m \geq 2n + 1 = 1$ and $\left(\frac{\partial f(y,z)}{\partial y} = \frac{\partial f(y,z)}{\partial z} = 0\right) = \{0\}$, which implies that f is reduced in $\mathcal{O}_2$. Finally, 0 must be a singular point of f because $p \in \Gamma \setminus \tilde{\Gamma}$ is a nilpotent singularity of $\mathcal{F}$. □

Lemma 3.5. *The normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is linearizable and can be defined by the germ of 1-form on $(\mathbb{C}^2, 0)$:*

$$\eta = m\,x\,dy - n\,y\,dx\ ,$$

where $m, n \in \mathbb{N}$, $gcd(m,n) = 1$ and $n > m \geq 1$.

Proof. Let $\eta = P(x,y)\,dy - Q(x,y)\,dx$ be a germ at $0 \in \mathbb{C}^2$ of holomorphic 1-form defining the normal type. Set $\eta_1 = j^1_0(\eta)$. Since $d\eta(0) \neq 0$ we get $d\eta_1 \neq 0$ and, after a linear change of variables, we have three possibilities :

(a). $\eta_1 = x\,dy$ (saddle-node).
(b). $\eta_1 = x\,dy - \lambda\,y\,dx$, where $\lambda \notin \{0, -1\}$.
(c). $\eta_1 = x\,dy - (x + y)\,dx$.

We will show first that $\lambda \in \mathbb{Q}_+$ in case (b). With the same type of argument we will show that case (a) is not possible in our situation. As a consequence, we will get that always $\lambda \neq 0$ and $\lambda \in \mathbb{Q}_+$. Concerning the linearization, we will use Poincaré-Dulac's normal form : when $\lambda, 1/\lambda \in \mathbb{Q}_+ \setminus \mathbb{N}$ then η is linearizable, whereas if $\lambda = n \in \mathbb{N}$, for instance, then η is equivalent to $\beta_n = x\,dy - (n\,y + a\,x^n)\,dx$. When $a = 0$ the form β_n is linear, whereas if $a \neq 0$ then it is not linearizable and we can assume that $a = 1$. However, in our situation, we will prove that $a = 0$. The same argument will imply that case (c) is not possible.

Let us examine the existence of distinguished separatrices. In case (a) the following normal form is known (cf. [M-R])

$$\eta = [x(1 + \mu\,y^n) + h.o.t.]\,dy - y^{n+1}\,dx\ ,$$

where $n \geq 1$. The separatrix $\sigma := (y = 0)$ is the unique one tangent to the direction of $y = 0$ and it is distinguished. Therefore, by lemma 3.3 it can be extended to a smooth separatrix Σ_1 of $\mathcal{F}$ along Γ. In this case, we will see that $\int_\Gamma c_1(N_{\Sigma_1}) = 0$, a contradicion with proposition 3.1.

On the other hand, in case (b), if X is the dual vector field of the normal type η then the eigenvalues of $DX(0)$ are 1 and λ. When $\lambda, 1/\lambda \notin \mathbb{N}$ then the vector field X has only two smooth separatrices through 0 : one, say σ_1, tangent to the eigenspace correspondent to 1, and the other, say σ_2, tangent to the eigenspace correspondent to λ. Both separatrices are distinguished because

$$CS(X,\sigma_1) = \lambda \neq 1/\lambda = CS(X,\sigma_2) \ .$$

Therefore, σ_j extends to a smooth separatrix Σ_j of $\mathcal{F}$ along Γ, $j = 1,2$. We will see that

$$\int_\Gamma c_1(N_{\Sigma_2}) = \lambda \int_\Gamma c_1(N_{\Sigma_1}) \ . \tag{19}$$

This will imply $\lambda \in \mathbb{Q}_+$, because $\int_\Gamma c_1(N_{\Sigma_i}) \in \mathbb{N}$, $i = 1,2$, by proposition 3.1.

In both cases, we will consider a covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of Γ satisfying (v) and (vi) of the proof of lemma 3.4. In particular, if $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. Let us analyse first case (b) with $\lambda, 1/\lambda \notin \mathbb{N}$.

We can assume that $\sigma_1 = (y = 0)$ and $\sigma_2 = (x = 0)$. Dividing η by some $\phi \in O_2^*$, the normal type becomes

$$\theta := \phi^{-1}.\,\eta = x\,dy - \lambda\, y(1 + R(x,y))\,dx \ , \ \nu(R,0) \geq 1 \ .$$

Therefore, we can suppose that :

(b.1) If $U_\alpha \cap \Gamma \setminus \tilde{\Gamma} = \emptyset$ then there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that

$$\omega_a = x_\alpha\, dy_\alpha - \lambda\, y_\alpha\,(1 + R(x_\alpha, y_\alpha))\,dx_\alpha \ . \tag{20}$$

In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and $\Sigma_2 \cap U_\alpha = (y_\alpha = 0)$. We take x_α and y_α as defining equations of $\Sigma_1 \cap U_\alpha$ and $\Sigma_2 \cap U_\alpha$, respectively.

Now, given $p \in \Gamma \setminus \tilde{\Gamma}$, by corollary 3.3 there is a chart $(U,(u,v,w))$, around p, where $\mathcal{F}|_U$ is defined by

$$\omega = u\,du + (f^r.\,\zeta_1(f) + u.\,\zeta_2(f))\,df \ , \ f = f(v,w) \ , \tag{21}$$

where f is reduced, $r \geq 1$ and $\zeta_i(0) \neq 0$, $i = 1,2$. If we fix some point $q \in U \cap \tilde{\Gamma}$ then there is a chart $(W,(u,y,z))$ such that $f|_W = y$, and so

$$\omega|_W = u\,du + (y^r\,\zeta_1(y) + u\,\zeta_2(y))\,dy \ .$$

Since $q \in \tilde{\Gamma}$, the form $\theta_1 := \omega|_W$ is analytically equivalent to θ. This implies $r = 1$ and

$$\frac{\zeta_2(0)^2}{\zeta_1(0)} = Res\left(\frac{\zeta_2^2(y)\,dy}{y\,\zeta_1(y)}, 0\right) = BB(\theta_1, 0) = BB(\theta, 0) = \frac{(\lambda+1)^2}{\lambda} \ .$$

In particular, we can assume that :

(b.2). If $\emptyset \neq U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then there is a chart $(u_\alpha, v_\alpha, w_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $p = (0,0,0)$ and $\mathcal{F}|_{U_\alpha}$ is defined by

$$\omega_\alpha = u_\alpha\, du_\alpha + (f_\alpha.\,\zeta_1(f_\alpha) + u_\alpha.\,\zeta_2(f_\alpha))\,df_\alpha \ , \ f_\alpha = f_\alpha(v_\alpha, w_\alpha) \ . \tag{22}$$

As we have seen before, in this chart we can set $\Sigma_i \cap U_\alpha = (u_\alpha - \phi_i(f_\alpha(v_\alpha, w_\alpha)) = 0)$, where $\phi_i(t) = t\,\psi_i(t)$, $i = 1,2$, and $\psi_1(0)$ and $\psi_2(0)$ are the roots of $z^2 + \zeta_2(0)\,z + \zeta_1(0) = 0$. We take $x_\alpha := u_\alpha - \phi_1(f_a(v_\alpha, w_\alpha))$ and $y_\alpha := u_\alpha - \phi_2(f_\alpha(v_\alpha, w_\alpha))$ as the defining equations of $\Sigma_1 \cap U_\alpha$ and $\Sigma_2 \cap U_\alpha$, respectively.

Let $(g_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$, $(k_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ and $(\varphi_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ be the multiplicative cocycles such that $x_\alpha = g_{\alpha\beta}.x_\beta$, $y_\alpha = k_{\alpha\beta}.y_\beta$ and $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$. We assert that

$$\frac{dk_{\alpha\beta}}{k_{\alpha\beta}} - \lambda \frac{dg_{\alpha\beta}}{g_{\alpha\beta}}\bigg|_{U_{\alpha\beta}\cap\Gamma} \equiv 0 \ , \ \forall\ U_{\alpha\beta}\cap\Gamma \neq \emptyset. \tag{23}$$

Note that (23) implies (19).

Proof of (23). Fix $\alpha, \beta \in A$ such that $U_{\alpha\beta}\cap\Gamma \neq \emptyset$. Since the covering satisfies (vi), we can assume that $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$, so that ω_α is like in (20). When we substitute $x_\alpha = g_{\alpha\beta}.x_\beta$ and $y_\alpha = k_{\alpha\beta}.y_\beta$ in ω_α, we get the expression of $\omega_\alpha|_{U_{\alpha\beta}}$ in the other chart :

$$\omega_\alpha = g_{\alpha\beta}.x_\beta\, d(k_{\alpha\beta}.y_\beta) - \lambda.k_{\alpha\beta}.y_\beta\,(1 + R(g_{\alpha\beta}.x_\beta, k_{\alpha\beta}.y_\beta))\, d(g_{\alpha\beta}.x_\beta) \ . \tag{24}$$

We have two possibilities :

1^{st}. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. In this case, ω_β is also like in (20) and we get :

$$g_{\alpha\beta}.x_\beta\, d(k_{\alpha\beta}.y_\beta) - \lambda.k_{\alpha\beta}.y_\beta\,(1 + R(g_{\alpha\beta}.x_\beta, k_{\alpha\beta}.y_\beta))\, d(g_{\alpha\beta}.x_\beta) =$$

$$:= A\, dx_\beta + B\, dy_\beta + C\, dz_\beta = \varphi_{\alpha\beta}\ (x_\beta\, dy_\beta - \lambda\, y_\beta\,(1 + R(x_\beta, y_\beta))\, dx_\beta) \ .$$

Since ω_β does not contain terms with dz_β, we get $C \equiv 0$. If we set $C(x_\beta, y_\beta, z_\beta) = \sum_{ij\geq 0} C_{ij}(z_\beta).x_\beta^i.y_\beta^j$ then

$$C_{11}(z_\beta) = g_{\alpha\beta}(0,0,z_\beta).\frac{\partial k_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} - \lambda\, k_{\alpha\beta}(0,0,z_\beta).\frac{\partial g_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} = 0 \ .$$

Since $U_{\alpha\beta}\cap\Gamma = (x_\beta = y_\beta = 0)$, the above relation implies (23).

2^{nd}. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$. In this case, ω_β is like in (22) and the substitution of $x_\alpha = g_{\alpha\beta}.x_\beta$ and $y_\alpha = k_{\alpha\beta}.y_\beta$ in (24) becomes more complicated. However, if we fix $q \in U_{\alpha\beta}\cap\tilde{\Gamma}$ then we can find a chart $(W,(u,s,t))$ around q such that $u_\beta = u$ and $f_\beta(v_\beta, w_\beta)|_W = s$. In this new chart, ω_β does not contain terms in dt. Since $x_\alpha = g_{\alpha\beta}.(u - \phi_1(s))$ and $y_\beta = k_{\alpha\beta}.(u - \phi_2(s))$ in this chart, a direct computation shows that the term in dt of ω_α, after the substitution, is $C(u,s,t) =$

$$= (u - \phi_1(s))(u - \phi_2(s)) \left\{ g_{\alpha\beta}\frac{\partial k_{\alpha\beta}}{\partial t} - \lambda[1 + R(x_\alpha, y_\alpha)]k_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t} \right\} \equiv 0 \implies$$

$$g_{\alpha\beta}(0,0,t)\frac{\partial k_{\alpha\beta}(0,0,t)}{\partial t} - \lambda\, k_{\alpha\beta}(0,0,t)\frac{\partial g_{\alpha\beta}(0,0,t)}{\partial t} \equiv 0 \implies (23) \ .$$

This proves that in case (b) we must have $\lambda \in \mathbb{Q}_+$. Moreover, Poincaré's linearization theorem implies that if $\lambda, 1/\lambda \notin \mathbb{N}$ then the normal type is equivalent to $m\, x\, dy - n\, y\, dx$, $m, n \in \mathbb{N}$.

Let us analyse case (a), in which, a priori, $\mathcal{F}$ has just the separatrix Σ_1. We can take the covering $\mathcal{U}$ in such a way that :

(a.1). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$ then there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by :

$$\omega_\alpha = [y_\alpha(1 + \mu\, x_\alpha^n) + R(x_\alpha, y_\alpha)]\, dx_\alpha - x_\alpha^{n+1}\, dy_\alpha \ , \tag{25}$$

where $\nu(R, 0) \geq n+2$. In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and we take x_α as the defining equation of $\Sigma \cap U_\alpha$.

(a.2). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$ and $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then there is a chart $(u_\alpha, v_\alpha, w_\alpha) \colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by a form like in (21) :

$$\omega_\alpha = u_\alpha \, du_\alpha + (f_\alpha^r . \zeta_1(f_\alpha) + u_\alpha . \zeta_2(f_\alpha)) \, df_\alpha \ , \ f_\alpha = f_\alpha(v_\alpha, w_\alpha) \ ,$$

where $r \geq 1$, f_α is reduced and $\zeta_j(0) \neq 0$, $j = 1, 2$.

Let $(\varphi_{\alpha\beta})_{U_{\alpha\beta} \neq \emptyset}$ be the multiplicative cocycle such that $\omega_\alpha = \varphi_{\alpha\beta} . \omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$.

We would like to observe that $r = n + 1 > 1$ in the situation (a.2). In fact, if we take $\beta \in A$ such that $\beta \neq \alpha$ with $\tilde{\Gamma} \cap U_{\alpha\beta} \neq \emptyset$ and $q = (0, v_o, w_o) \in \tilde{\Gamma} \cap U_{\alpha\beta}$, then $f_\alpha(v_o, w_o) = 0$ and there is a chart $(W, (u, y, z))$ around q such that $u = u_\alpha$ and $f_\alpha|_W = y$. In particular, in this chart

$$\omega_\alpha = u \, du + (y^r \, \zeta_1(y) + u . \, \zeta_2(y)) \, dy \ .$$

Therefore, the multiplicity (Milnor number) of the singularity 0 of ω_α (in a transverse section) is $\mu(\omega_\alpha, 0) = [u, y^r \, \zeta_{1\alpha}(y) + u \, \zeta_{2\alpha}(y)]_0 = r$. Since $\omega_\alpha = \varphi_{\alpha\beta} . \omega_\beta$, where ω_β is like in (25), we get $r = \mu(\omega_\beta, 0) = n + 1$.

In particular, a straightforward computation shows that the equation of Σ_1 in the chart (u, y, z) is of the form $u + \phi(y) = 0$, where $\phi(0) = 0$ and $\phi'(0) = \zeta_2(0) \neq 0$. Therefore, the equation of $\Sigma_1 \cap U_\alpha$ is $u_\alpha + \phi(f_\alpha(v_\alpha, w_\alpha)) = 0$. We take $x_\alpha := u_\alpha + \phi(f_\alpha(v_\alpha, w_\alpha))$ as the defining equation of $\Sigma_\alpha \cap U_\alpha$ in the situation (a.2).

Note $< u_\alpha, x_\alpha > = < u_\alpha, f_\alpha >$, because $\phi(0) = 0$ and $\phi'(0) \neq 0$. In particular, $\Gamma \cap U_\alpha$ is defined by the ideal $< u_\alpha, x_\alpha >$.

Let $G = (g_{\alpha\beta})_{U_{\alpha\beta}}$ be the multiplicative cocycle such that $x_\alpha = g_{\alpha\beta} . \, x_\beta$ on $U_{\alpha\beta} \neq \emptyset$. We will see that $g_{\alpha\beta}|_{\Gamma \cap U_{\alpha\beta}}$ is locally constant and this will imply $\int_\Gamma c_1(N_{\Sigma_1}) = 0$.

Fix $\alpha, \beta \in A$ such that $\Gamma \cap U_{\alpha\beta} \neq \emptyset$. By the construction of the covering $\mathcal{U}$, we can assume that $(\Gamma \setminus \tilde{\Gamma}) \cap U_\alpha = \emptyset$, so that ω_α is like in (25) and $\Gamma \cap U_\alpha = (x_\alpha = y_\alpha = 0)$. We have two possibilities :

- 1^{st}. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. In this case, ω_β is also like in (25) and $\Gamma \cap U_{\alpha\beta} = (x_\alpha = y_\alpha = 0) = (x_\beta = y_\beta = 0)$. This implies that $y_\alpha = h_{\alpha\beta} . x_\beta + k_{\alpha\beta} . y_\beta$ on $U_{\alpha\beta}$, where $g_{\alpha\beta} . k_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$.
- 2^{nd}. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$. In this case, ω_β is like in (21) and $\Gamma \cap U_{\alpha\beta} = (x_\alpha = y_\alpha = 0) = (x_\beta = f_\beta = 0)$. This implies that $y_\alpha = h_{\alpha\beta} . x_\beta + k_{\alpha\beta} . f_\beta$ on $U_{\alpha\beta}$, where $g_{\alpha\beta} . k_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$.

In the 2^{nd} case, if $q \in U_{\alpha\beta} \cap \Gamma$ then $dx_\beta \wedge df_\beta(q) \neq 0$. Hence, we can find a chart $(W, (x_\beta, y_\beta, z_\beta))$ around some point $q \in U_{\alpha\beta} \cap \Gamma$ such that $f_\beta|_W = y_\beta$. In both cases, we have $y_\alpha = h_{\alpha\beta} . x_\beta + k_{\alpha\beta} . y_\beta$ and ω_β do not contain terms with dz_β. On the other hand, if we write ω_α in the coordinates $(x_\beta, y_\beta, z_\beta)$, using the relations $x_\alpha = g_{\alpha\beta} . x_\beta$ and $y_\alpha = h_{\alpha\beta} . x_\beta + k_{\alpha\beta} . y_\beta$, we get $\omega_\alpha = A(x_\beta, y_\beta, z_\beta) \, dx_\beta + B(x_\beta, y_\beta, z_\beta) \, dy_\beta + C(x_\beta, y_\beta, z_\beta) \, dz_\beta$, where we can write

$$C(x_\beta, y_\beta, z_\beta) = \sum_{i,j \geq 0} C_{ij}(z_\beta) \, x_\beta^i \, y_\beta^j \equiv 0 \quad \Longrightarrow \quad C_{ij}(z_\beta) \equiv 0 \ , \ \forall i, j \geq 0 \ .$$

By substituting explicitly $x_\alpha = g_{\alpha\beta} . x_\beta$ and $y_\alpha = h_{\alpha\beta} . x_\beta + k_{\alpha\beta} . y_\beta$ in (25) we get

$$C_{11}(z_\beta) = k_{\alpha\beta}(0, 0, z_\beta) . \frac{\partial g_{\alpha\beta}(0, 0, z_\beta)}{\partial z_\beta} = 0 \ ,$$

which implies $\frac{\partial g_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} = 0$. Hence, $g_{\alpha\beta}|_{\Gamma \cap U_{\alpha\beta}}$ is locally constant and this finishes the analysis of case (a).

Next, we prove that in our situation the normal type cannot be equivalent to

$$\beta_n = x\,dy - (n\,y + x^n)\,dx \ , \ n \geq 1\,.$$

This will imply that case (c) is not possible in our situation : if $\eta_1 = x\,dy - (x + y)\,dx$ then by Poincaré's linearization theorem η is equivalent η_1 and so to β_1, a contradiction.

Let us suppose by contradiction that the normal type of $\mathcal{F}$ at Γ is equivalent to β_n.

Remark 3.6. We would like to observe the following facts about the foliation defined by β_n :

(a). It has no meromorphic non-constant first integral in a neighborhood of $0 \in \mathbb{C}^2$. This implies that $\mathcal{F}$ has no meromorphic non-constant meromorphic first integral in a neighborhood of any point $p \in \Gamma$.

(b). The separatrix $\sigma = (x = 0)$ is the unique analytic separatrix of η through $0 \in \mathbb{C}^2$. In particular, it is distinguished and can be extended to a smooth separatrix Σ of $\mathcal{F}$ along Γ.

(c). β_n has an integrating factor : if we set $\theta := x^{-(1+n)}.\,\beta_n$ then :

$$\theta = d\left(\frac{y}{x^n}\right) - \frac{dx}{x} \quad \Longrightarrow \quad d\theta = 0\ .$$

Let us sketch the proof. First of all, we will prove that the closed meromorphic 1-form θ, on some transverse section Λ, can be extended from the transverse section to a closed meromorphic 1-form Θ on some connected neighborhood U of Γ, in such a way that :

(i). The divisor $(\Theta)_\infty$, of poles of Θ, is $(\Theta)_\infty = (\Sigma)^{n+1}$.
(ii). Θ defines $\mathcal{F}$ on $U \setminus \Sigma$.
(iii). $Res(\Theta, \Sigma) = -1$.

By using an extension theorem of meromorphic functions on U (cf. [Ba] and [Ro]), the form Θ can be extended to a global closed meromorphic 1-form. The contradiction will be a consequence of (i) and (iii), as we will see.

Extension of θ to a neighborhood of $\tilde{\Gamma}$. Fix a covering $(U_\alpha)_{\alpha\in A}$ of $\tilde{\Gamma}$ such that, for all $\alpha \in A$, there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by $\eta_\alpha = x_\alpha\,dy_\alpha - (n\,y_\alpha + x_\alpha^n)\,dx_\alpha$. We can assume also that, if $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty. Note that $\Sigma \cap U_\alpha = (x_\alpha = 0)$ for all $\alpha \in A$.

Set $\Theta_\alpha = d\left(\frac{y_\alpha}{x_\alpha^n}\right) - \frac{dx_\alpha}{x_\alpha}$, $\alpha \in A$. We assert that, if $U_{\alpha\beta} \neq \emptyset$ then $\Theta_\alpha = \Theta_\beta$ on $U_{\alpha\beta}$.

In fact, fix $U_{\alpha\beta} \neq \emptyset$ and $g, \varphi \in \mathcal{O}^*(U_{\alpha\beta})$ such that $x_\alpha = g.\,x_\beta$ and $\eta_\alpha = \varphi.\,\eta_\beta$ on $U_{\alpha\beta}$. From $\Theta_\alpha = x_\alpha^{-(n+1)}.\,\eta_\alpha$ and $\Theta_\beta = x_\beta^{-(n+1)}.\,\eta_\beta$ we get $\Theta_\alpha = \phi.\,\Theta_\beta$, where $\phi = \varphi/g^{n+1} \in \mathcal{O}^*(U_{\alpha\beta})$. Since Θ_α and Θ_β are closed, we get

$$0 = d\Theta_\alpha = d\phi \wedge \Theta_\beta \quad \Longrightarrow \quad d\phi \wedge \eta_\beta = 0 \quad \Longrightarrow$$

ϕ is a holomorphic first integral of $\mathcal{F}|_{U_{\alpha\beta}}$. This implies that ϕ is a constant, because $U_{\alpha\beta} \cap \tilde{\Gamma} \neq \emptyset$ and $\mathcal{F}$ has no non-constant holomorphic first integral in a neighborhood of any point $q \in \tilde{\Gamma}$ (see remark 3.6). Now, observe that

$$Res(\Theta_\alpha, \Sigma) = Res\left(-\frac{dx_\alpha}{x_\alpha}, \Sigma\right) = Res\left(-\frac{dx_\alpha}{x_\alpha}, (x_a = 0)\right) = -1\ .$$

Similarly, $Res(\Theta_\beta, \Sigma) = -1$. Since ϕ is a constant, we get

$$-1 = Res(\Theta_\alpha, \Sigma) = \phi.Res(\Theta_\beta, \Sigma) = -\phi \implies \phi \equiv 1 \ ,$$

which proves the assertion.

It follows that there exists a meromorphic 1-form $\tilde{\Theta}$ on the neighborhood $\tilde{U} = \bigcup_\alpha U_\alpha$ of $\tilde{\Gamma}$, such that $\tilde{\Theta}|_{U_\alpha} = \Theta_\alpha$ for all $\alpha \in A$. Let us extend $\tilde{\Theta}$ to a neighborhood $U \supset \tilde{U}$ of Γ.

Extension of $\tilde{\Theta}$ *to a neighborhood of* Γ. Fix $p \in \Gamma \backslash \tilde{\Gamma}$ and a chart $(u, s, t) \colon V \to \mathbb{C}^3$, around p, such that $\mathcal{F}|_V$ is defined by

$$\omega = u\, du + (f.\, \zeta_1(f) + u\, \zeta_2(f))\, df \ , \ f = f(s,t) \ .$$

Choose a point $q \in \tilde{\Gamma} \cap V$ and a chart $(W, (u, v, w))$, around q, such that $v = f(s,t)$, $\tilde{\Gamma} \cap V = (u = v = 0)$ and $\omega = u\, du + (v\, \zeta_1(v) + u\, \zeta_2(v))\, dv$. Since $\tilde{\Theta}|_W$ defines $\mathcal{F}$ on $W \setminus \Sigma$, there exists a meromorphic function $h = h(u, v, w)$ on W such that $\tilde{\Theta} = h.\omega$. We assert that $\frac{\partial h}{\partial w} = 0$, so that $h = h(u, v)$.

In fact, since $\tilde{\Theta}$ is closed, we get

$$0 = d\tilde{\Theta} = dh \wedge \omega + h\, d\omega \implies \zeta_2(v)\, du \wedge dv = d\omega = -\frac{dh}{h} \wedge \omega \implies$$

$$-\zeta_2(v)\, du \wedge dv = \frac{dh}{h} \wedge [u\, du + (v\, \zeta_1(v) + u\, \zeta_2(v))\, dv] \implies \frac{\partial h}{\partial w} = 0 \ ,$$

which proves the assertion. It follows that the meromorphic 1-form $h(u, f(s,t)).\omega$ is closed and extends $\tilde{\Theta}$ to some neighborhood of p. Hence, $\tilde{\Theta}$ can be extended to a closed meromorphic 1-form Θ, on some connected neighborhood U of Γ, which satisfies (i), (ii) and (iii).

Now, we use the following result :

Theorem 3.4. ([Ba], [Ro]). *Let Y be a connected analytic subset of $\mathbb{P}^n$, $n \geq 2$, with $dim(Y) \geq 1$. Then any meromorphic function in a connected neighborhood of Y extends to a meromorphic function on all of $\mathbb{P}^n$.*

As a consequence of theorem 3.4, the form Θ can be extended to a global meromorphic 1-form on $\mathbb{P}^3$. In fact, consider an affine coordinate system $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{P}^3$, such that $\Gamma \not\subset L_\infty$, where L_∞ denotes the plane at infinity of $\mathbb{C}^3$. We can write

$$\Theta|_{\mathbb{C}^3 \cap U} = A\,.dx + B\, dy + C\, dz \ ,$$

where A, B and C are meromorphic functions on $\mathbb{C}^3 \cap U$. Since dx, dy and dz are global meromorphic forms on $\mathbb{P}^3$, the functions A, B and C can be extended to meromorphic functions on U, and, as a consequence, to meromorphic functions on $\mathbb{P}^3$, by theorem 3.4, which proves the assertion. Denote the extension again by Θ. Let $|\Theta|_\infty$ be the set of poles of Θ.

Note that $|\Theta|_\infty$ is an algebraic hypersurface of $\mathbb{P}^3$. Since $|\Theta|_\infty \cap U = \Sigma$, the separatrix Σ extends to an irreducible algebraic hypersurface of $\mathbb{P}^3$, which we still denote Σ. On the other hand, if S is an irreducible component of $|\Theta|_\infty$ then

$$S \cap \Gamma \neq \emptyset \implies S \cap U \neq \emptyset \implies S \cap U = \Sigma \cap U \implies S = \Sigma \implies |\Theta|_\infty = \Sigma \ .$$

Now, we arrive to a contradiction : we have seen that $Res(\Theta, \Sigma) = -1$. If we take a line $\mathbb{P}^1 \simeq \ell \subset \mathbb{P}^3$ cutting Σ transversely in $dg(\Sigma)$ points, then $\sum_{p \in \ell} Res(\Theta|_\ell, p) = -dg(\Sigma) \neq 0$, which is not possible by the residue theorem. Therefore, the normal type is equivalent to $m\,x\,dy - n\,y\,dx$, where $m,n \in \mathbb{N}$, where we can assume that $n \geq m \geq 1$ and $gcd(m,n) = 1$.

It remains to prove that in our situation we don't have $m = n = 1$. This is a consequence of the fact that we are assuming $\Gamma \setminus \tilde{\Gamma} \neq \emptyset$.

In fact, suppose by contradiction that the normal type was $\eta = x\,dy - y\,dx$. Fix a point $q \in \tilde{\Gamma}$ and a transverse section Λ to $\tilde{\Gamma}$ through q. Let X be a vector field on Λ defining $\mathcal{F}|_\Lambda$. Since the normal type is η, we have $\eta(X) = 0$, which implies that the linear part $DX(q)$ of X at q, must be of the form $\delta\, I$, where I is the identity and $\delta \neq 0$. On the other hand, if $p \in \Gamma \setminus \tilde{\Gamma}$ then there is a chart $(U, \psi = (x,y,z))$ around p such that $\psi(p) = 0$ and $\mathcal{F}|_U$ is represented by

$$\omega = x\,dx + (f^r\,\zeta_1(f) + x\,\zeta_2(f))\,df \ , \ f = f(y,z) \ ,$$

where f is reduced, $r \geq 1$ and $\zeta_1(0), \zeta_2(0) \neq 0$. If $q \in U \cap \tilde{\Gamma}$ then we can find a chart $(W,(u,v,w))$ around q such that $x = u$ and $f(y,z) = v$. In this chart $\mathcal{F}|_W$ is represented, in a normal section to $\tilde{\Gamma}$ through q by $u\,du + (v^r\,\zeta_1(v) + u\,\zeta_2(v))\,dv$. The dual vector field of this form is $X = (v^r\,\zeta_1(v) + u\,\zeta_2(v))\,\partial_u - u\,\partial_v$. As the reader can check, $DX(0) \neq \delta\, I$. This finishes the proof of lemma 3.5. □

End of the proof of theorem 2 in dimension three. We have proved that the normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is given by $\eta = m\,x\,dy - n\,y\,dx$, where $m,n \in \mathbb{N}$, $gcd(m,n) = 1$ and $n > m \geq 1$. Let us give an idea of the proof.

Consider a meromorphic integrable 1-form Ω on $\mathbb{P}^3$ representing $\mathcal{F}$ outside its set of poles. By using the normal type, we will see that there exists a closed meromorphic 1-form $\tilde{\Lambda}$, on some connected neighborhood U of Γ, such that $d\Omega = \tilde{\Lambda} \wedge \Omega$ on U. The extension theorem of [Ba] and [Ro] will imply that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form Λ on $\mathbb{P}^3$ with $d\Omega = \Lambda \wedge \Omega$. Next, working with the pole divisors and residues of Λ, we will see that $\Lambda = \frac{dF}{F}$, where F is meromorphic on $\mathbb{P}^3$. In particular, we will get $d\left(\frac{\Omega}{F}\right) = 0$, that is, F is an integrating factor of Ω. Finally, by studing $\frac{\Omega}{F}$ around Γ, we will show that $\mathcal{F}$ has a rational first integral of the form f_2^m/f_1^n, where $m.\,dg(f_2) = n.\,dg(f_1)$.

Remark 3.7. Since $n > m$, the separatrix $\sigma = (x = 0)$ is distinguished. In particular, it extends to a smooth separatrix Σ_1 of $\mathcal{F}$ along Γ. When $n > m > 1$ the other separatrix, $\sigma_2 = (y = 0)$, is also distinguished and can be extended to another separatrix, say Σ_2, of $\mathcal{F}$ along Γ.

Another fact that we would like to observe is that $f(x,y) := y^m/x^n$ is a mermorphic first integral of η. On the other hand, η has no non-constant holomorphic first integral in a neighborhood $0 \in \mathbb{C}^2$.

Fix an affine chart $(x,y,z) \in \mathbb{C}^3 \subset \mathbb{P}^3$ and a polynomial integrable 1-form Ω on $\mathbb{C}^3$ which represents $\mathcal{F}|_{\mathbb{C}^3}$. Without lost of generality, we can assume that Γ is transverse to the line at infinity $L_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3$.

Construction of $\tilde{\Lambda}$ in a neighborhood of $\tilde{\Gamma}$. Let $(U_\alpha)_{\alpha \in A}$ be a covering of $\tilde{\Gamma}$ with the following properties :

(a). $U_\alpha \cap \tilde{\Gamma}$ is connected and non-empty for all $\alpha \in A$.
(b). If $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty.
(c). For all $\alpha \in A$ there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\tilde{\Gamma} \cap U_\alpha = (x_\alpha = y_\alpha = 0)$ and $\mathcal{F}|_{U_\alpha}$ is represented by $\eta_a = m\,x_\alpha\,dy_\alpha - n\,y_\alpha\,dx_a$.

In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$, $f_\alpha := y_\alpha^m / x_\alpha^n$ is a meromorphic first integral of $\mathcal{F}|_{U_\alpha}$ and

$$df_\alpha = \frac{y_\alpha^{m-1}}{x_\alpha^{n+1}} . \, \eta_\alpha \ , \ \forall \, \alpha \in A \ . \tag{26}$$

Fix $U_{\alpha\beta} \neq \emptyset$ and let $\varphi_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ and $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ be such that $\eta_\alpha = \varphi_{\alpha\beta} . \, \eta_\beta$ and $x_\alpha = g_{\alpha\beta} . \, x_\beta$ on $U_{\alpha\beta}$. From (26) we get

$$df_\alpha = a_{\alpha\beta} . \, df_\beta \ , \ a_{\alpha\beta} = \frac{(y_\alpha / y_\beta)^{m-1}}{g_{\alpha\beta}^{n+1}} \, \varphi_{\alpha\beta} \ .$$

Note that $a_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$. In fact, if $m = 1$ this is clear. On the other hand, if $m > 1$ then by remark 3.7, there is a separatrix Σ_2 along Γ such that

$$\Sigma_2 \cap U_{\alpha\beta} = (y_\alpha = 0) \cap U_\beta = (y_\beta = 0) \cap U_\alpha \, .$$

As a consequence, there exists $h_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $y_\alpha = h_{\alpha\beta} . \, y_\beta$. Hence, $a_{\alpha\beta} = h_{\alpha\beta}^{m-1} . \varphi_{\alpha\beta} / g_{\alpha\beta}^{n+1} \in \mathcal{O}^*(U_{\alpha\beta})$.

From $df_\alpha = a_{\alpha\beta} . df_\beta$ we get

$$da_{\alpha\beta} \wedge df_\beta = 0 \implies da_{\alpha\beta} \wedge \eta_\beta = 0$$

and $a_{\alpha\beta}$ is a holomorphic first integral of $\mathcal{F}$ in a neighborhood of $U_{\alpha\beta} \cap \Gamma$. This implies that $a_{\alpha\beta} \in \mathbb{C}^*$, because the normal type has no non-constant holomorphic first integral.

Given $\alpha \in A$, $\Omega|_{U_\alpha}$ and df_α represent $\mathcal{F}$ in the complement of their poles. Hence, there is a meromorphic function g_α on U_α such that $\Omega = g_\alpha . \, df_\alpha$. Since $df_\alpha = a_{\alpha\beta} . df_\beta$ on $U_{\alpha\beta} \neq \emptyset$, we get

$$\Omega = g_\alpha . df_\alpha = g_\alpha . a_{\alpha\beta} . df_\beta = g_\beta . df_\beta \implies g_\beta = a_{\alpha\beta} . g_\alpha \text{ , on } U_{\alpha\beta} \ .$$

Since $a_{\alpha\beta} \in \mathbb{C}^*$, we get

$$\frac{dg_\alpha}{g_\alpha} = \frac{dg_\beta}{g_\beta} \text{ , on } U_{\alpha\beta}$$

and this implies that there exists a meromorphic 1-form $\tilde{\Lambda}$ on $\tilde{U} := \bigcup_\alpha U_\alpha$ such that $\tilde{\Lambda}|_{U_\alpha} = \frac{dg_\alpha}{g_\alpha}$ for all $\alpha \in A$. Finally, $d\Omega = \tilde{\Lambda} \wedge \Omega$ because

$$d\Omega|_{U_\alpha} = dg_\alpha \wedge df_\alpha = \frac{dg_\alpha}{g_\alpha} \wedge \Omega|_{U_\alpha} = \tilde{\Lambda} \wedge \Omega|_{U_\alpha} \ .$$

Extension of $\tilde{\Lambda}$ *to a neighborhood of* $\Gamma \setminus \tilde{\Gamma}$. Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and a local chart $(V, (u, s, t))$ around p such that $\mathcal{F}|_V$ is represented by

$$\omega = u \, du + (h. \, \zeta_1(h) + u. \, \zeta_2(h)) \, dh \ , \ h = h(s,t)$$

and $\Gamma \cap V = (u = h(s,t) = 0)$. Choose $q \in V \cap \tilde{\Gamma}$ and a chart $(W, (u, v, w))$ around q with $W \subset V$ and $h(s,t) = v$, so that

$$\omega|_W = u \, du + (v \, \zeta_1(v) + u \, \zeta_2(v)) \, dv \ .$$

Let $\alpha \in A$ be such that $q \in U_\alpha$. We can assume that $W \subset U_\alpha$. Since $\eta_\alpha|_W$ and $\omega|_W$ represent $\mathcal{F}|_W$ there is $\varphi = \varphi(u, v, w) \in \mathcal{O}^*(W)$ sucht that $\eta_\alpha = \varphi . \omega$ on W. This implies $df_\alpha|_W = h . \omega|_W$, where $h(u, v, w) = \varphi . y_\alpha^{m-1} / x_\alpha^{n+1}$ is meromorphic on

W. In particular, $d(h.\,\omega|_W)=0$, which implies $d\omega|_W=-\frac{dh}{h}\wedge\omega|_W$. Since $d\omega|_W$ do not contain terms with $du\wedge dw$ and $dv\wedge dw$, from the last relation we get

$$\frac{\partial h}{\partial w}\equiv 0 \implies h=h(u,v)\ .$$

Therefore, the closed 1-form $\theta := h(u,f(s,t)).\omega$ is meromorphic in some neighborhood U_p of p and extends df_α to this neighborhood. As before, we have $\Omega = g.\,\theta$, where g is meromorphic on U_p and is an extension of g_α to U_p. This implies that $\frac{dg}{g}$ extends $\tilde{\Lambda}$ to U_p. In particular, $\tilde{\Lambda}$ can be extended meromorphically to some connected neighborhood U of Γ. Finally, theorem 3.4 implies that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form Λ on $\mathbb{P}^3$ with $d\Omega=\Lambda\wedge\Omega$.

Poles and residues of Λ. Let $|\Lambda|_\infty$ be the set of poles of Λ. Fix $p\in\tilde{\Gamma}$ and $\alpha\in A$ such that $p\in U_\alpha$. Note that $L_\infty=\mathbb{P}^3\setminus\mathbb{C}^3$ is a pole of Ω of order $d+2$, where $d=dg(\mathcal{F})$ (cf. [B 1]). Let $(u_\alpha=0)$ be a reduced equation of $L_\infty\cap U_\alpha$. Since $\Omega|_{U_\alpha}$ and η_α represent $\mathcal{F}|_{U_\alpha}$ there is $\phi_\alpha\in\mathcal{O}^*(U_\alpha)$ such that

$$\Omega|_{U_\alpha}=\frac{\phi_\alpha}{u_\alpha^{d+2}}.\,\eta_\alpha=\frac{\phi_\alpha.x_\alpha^{n+1}}{u_\alpha^{d+2}.y_\alpha^{m-1}}.\,df_a \implies g_\alpha=\frac{\phi_\alpha.x_\alpha^{n+1}}{u_\alpha^{d+2}.y_\alpha^{m-1}}.\ .$$

From the above expression, we get

$$\Lambda|_{U_\alpha}=\frac{dg_\alpha}{g_\alpha}=(n+1)\,\frac{dx_\alpha}{x_\alpha}-(m-1)\,\frac{dy_\alpha}{y_\alpha}-(d+2)\,\frac{du_\alpha}{u_\alpha}+\frac{d\phi_\alpha}{\phi_\alpha}\ . \tag{27}$$

We have two possibilities :

1^{st}. $1<m<n$. In this case, $|\Lambda|_\infty\cap U_\alpha=(x_\alpha=0)\cup(y_\alpha=0)\cup(u_\alpha=0)$. Since $\Sigma_1\cap U_\alpha=(x_\alpha=0)$ and $\Sigma_2\cap U_\alpha=(y_\alpha=0)$, they extend to global algebraic irreducible surfaces, which we call again Σ_1 and Σ_2, respectively. Moreover, we get $|\Lambda|_\infty\supset\Sigma_1\cup\Sigma_2\cup L_\infty$. We assert that $|\Lambda|_\infty=\Sigma_1\cup\Sigma_2\cup L_\infty$.

Let S be an irreducible component of $|\Lambda|_\infty$, $S\neq L_\infty$, and let us prove that $S\subset\Sigma_1\cup\Sigma_2$. We assert that S is $\mathcal{F}$-invariant.

In fact, fix a smooth point $p\in S\setminus(L_\infty\cup sing(\mathcal{F}))$. Consider a local chart $\psi=(x_1,x_2,x_3)\colon W\to\mathbb{C}^3$ around p such that $\psi(p)=0$, $W\cap(L_\infty\cup sing(\mathcal{F}))=\emptyset$ and $S\cap W=|\Lambda|_\infty\cap W=(x_3=0)$. We can write

$$\Lambda|_W=\frac{\theta}{x_3^k}\ ,\ \theta=A_1\,dx_1+A_2\,dx_2+A_3\,dx_3\ ,$$

where $A_i\in\mathcal{O}(W)$, $i=1,2,3$, $x_3\nmid A_i$ for some $i=1,2,3$, and $k\geq 1$. From $d\Lambda=0$, we get

$$x_3^{-k}\,d\theta-k\,x_3^{-(k+1)}\,dx_3\wedge\theta=0 \implies d\theta=k\,\frac{dx_3}{x_3}\wedge\theta\ ,$$

which implies that $x_3\mid A_1,A_2$ and $x_3\nmid A_3$.Therefore, we can write $\theta=x_3\,\alpha+A_3\,dx_3$, where α is holomorphic on W. Since $d\Omega=\Lambda\wedge\Omega$, we get

$$x_3^k\,d\Omega|_W=\theta\wedge\Omega|_W \implies A_3\,dx_3\wedge\Omega|_W=x_3(x_3^{k-1}d\Omega|_W-\alpha\wedge\Omega|_W)\ .$$

From the last relation above, we obtain that $\frac{dx_3}{x_3}\wedge\Omega|_W:=\beta$ is holomorphic. Hence, S is $\mathcal{F}$-invariant, because $dx_3\wedge\Omega|_W=x_3\,\beta$, where β is holomorphic.

Since S is $\mathcal{F}$-invariant and $\Gamma\cap S\neq\emptyset$, S must contain some separatrix of $\mathcal{F}$ along Γ. In particular, $S\cap U_\alpha\neq\emptyset$, which implies that $S\cap U_\alpha\subset(x_\alpha=0)\cup(y_\alpha=0)$. Therefore, either $S=\Sigma_1$, or $S=\Sigma_2$.

Let f_1, f_2, f_3 be irreducible homogeneous polynomials on $\mathbb{C}^4$, f_3 of degree one, such that $f_i = 0$ is an equation of Σ_i, $i = 1, 2$, and $f_3 = 0$ is an equation of L_∞ (in homogeneous coordinates). By (27) the residues of Λ are $n+1$ (on Σ_1), $-(m-1)$ (on Σ_2) and $-(d+2)$ (on L_∞). Therefore, Λ can be written in homogeneous coordinates as dF/F, where

$$F = \frac{f_1^{n+1}}{f_2^{m-1}.f_3^{d+2}} \ .$$

2^{nd}. $n > m = 1$. In this case, $|\Lambda|_\infty \cap U_\alpha = (x_\alpha = 0) \cup (u_\alpha = 0)$. With the same argument of the 1^{st} case, we get $|\Lambda|_\infty = \Sigma_1 \cup L_\infty$. Let f_1, f_3 be irreducible homogeneous polynomials on $\mathbb{C}^4$, f_3 of degree one, such that $f_1 = 0$ is an equation of Σ_1 and $f_3 = 0$ is an equation of L_∞ (in homogeneous coordinates). By (27) the residues of Λ are $n+1$ (on Σ_1) and $-(d+2)$ (on L_∞). Therefore, Λ can be written in homogeneous coordinates as dF/F, where

$$F = \frac{f_1^{n+1}}{f_3^{d+2}} \ .$$

The first integral. Let $\Pi \colon \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ be the canonical projection and (x_0, x_1, x_2, x_3) be homogeneous coordinates such that $L_\infty = (f_3 = x_0 = 0)$ and the previous affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$ is $(x_0 = 1)$. In this chart,

$$\Pi(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \ .$$

Since $dg(\mathcal{F}) = d$ we can write $\Pi^*(\Omega) = \frac{1}{x_0^{d+2}}\,\omega$, where the coefficients of ω are homogenous of degree $d+1$ and $i_R(\omega) = 0$, $R = \sum_i x_i \partial_{x_i}$. If $m = 1$ we set $f_2^{m-1} := 1$. With this convention, we can write $F = \frac{f_1^{n+1}}{f_2^{m-1}.x_0^{d+2}}$. On the other hand, the relation $d\Omega = \frac{dF}{F} \wedge \Omega$ is equivalent to $d(F^{-1}\,\Omega) = 0$, and so the form

$$\mu := \frac{\Omega}{F} = \frac{f_2^{m-1}\,\omega}{f_1^{n+1}}$$

is closed. Since it is closed and its pole divisor is f_1^{n+1}, it can written as

$$\mu = \lambda\,\frac{df_1}{f_1} + d\left(\frac{h}{f_1^n}\right)$$

where $\lambda \in \mathbb{C}$, h is a homogeneous polynomial and $dg(h) = n\,dg(f_1)$.

Since $0 = i_R(\mu) = \lambda\,dg(f_1)$, we get $\lambda = 0$. It follows that h/f_1^n is a rational first integral of $\mathcal{F}$. If $m > 1$ then $\Sigma_2 = (f_2 = 0)$ is $\mathcal{F}$-invariant. Hence, there exists $b \in \mathbb{C}$ such that $(f_2 = 0) \subset (h + b\,f_1^n = 0)$. In particular, there exist $k \in \mathbb{N}$ and a homogeneous polynomial g such that $g.f_2^k = h + b\,f_1^n$, where $f_1, f_2 \nmid g$ and $dg(g) + k\,dg(f_2) = n\,dg(f_1)$. This implies

$$\frac{f_2^{m-1}}{f_1^{n+1}}\,\omega = d\left(\frac{h}{f_1^n}\right) = d\left(\frac{g.f_2^k}{f_1^n}\right) \implies$$

$$f_2^{m-1}\,\omega = f_2^{k-1}\,(f_1\,f_2\,dg + k\,f_1\,g\,df_2 - n\,g\,f_2\,df_1) \implies m = k$$

and g is a constant, because otherwise in a point $q \in (g = f_1 = f_2 = 0) \cap \Gamma$ we would have $j_q^1(\omega) > 1$. This implies that $\mathcal{F}$ has a first integral of the form f_2^m/f_1^n.

When $m = 1$, then h is irreducible and we take $f_2 = h$. This finishes the proof of theorem 2 in dimension three.

3.3. **Proof of theorem 2 in dimension** $n \geq 4$**.** The idea is to use the case of dimension three and the following known result (cf. [C-LN-S 1]) :

Theorem 3.5. *Let $\mathcal{G}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Assume that there is a k-plane $E \simeq \mathbb{P}^k$, $2 \leq k < n$ such that E is in general position with $\mathcal{G}$ and $\mathcal{G}|_E$ is represented by a closed meromorphic 1-form ω on E outside its poles. Then ω can be extended to a closed meromorphic 1-form Ω on $\mathbb{P}^n$ representing $\mathcal{G}$ outside its poles. In particular, if $\mathcal{G}|_E$ has a rational first integral then it can be extended to rational first integral of $\mathcal{G}$.*

Recall that E is in general position with $\mathcal{G}$ if :

(a). E is not $\mathcal{G}$-invariant.
(b). The divisor of tangencies between $\mathcal{G}$ and E has codimension ≥ 2 in E.

Moreover, the set of k-planes in general position with $\mathcal{G}$ is a Zariski open and dense subset of the respective grassmanian (cf. [C-LN-S 1]).

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geq 4$, such that $sing_2(\mathcal{F})$ has an irreducible component Γ with $BB(\mathcal{F},\Gamma) \neq 0$ and the set $X := \{p \in \Gamma \,|\, \mathcal{J}(\mathcal{F},p) > 1\}$ has codimension ≥ 4 in $\mathbb{P}^n$. Set $\mathcal{N}_\Gamma = \{p \in \Gamma \,|\, p$ is a nilpotent singularity of $\mathcal{F}\}$ and $K_\Gamma = \{p \in \Gamma \,|\, p$ is a singularity of Kupka type of $\mathcal{F}\}$. Since $cod_{\mathbb{P}^n}(X) \geq 4$ and $cod_{\mathbb{P}^n}(\Gamma) = 2$, we have $\Gamma = \mathcal{N}_\Gamma \cup K_\Gamma \cup X$ and

- Either $\Gamma = \mathcal{N}_\Gamma \cup X$, or K_Γ is a Zariski open and dense subset of Γ.

When $\mathcal{N}_\Gamma \cup X = \emptyset$ then $\Gamma \subset K(\mathcal{F})$ and so theorem 2 is true by [C-LN 2], [CA 2] and [B 2]. Therefore, from now on we will assume that $\mathcal{N}_\Gamma \cup X \neq \emptyset$. In view of theorem 3.5, the next result will reduce the problem to the case $n = 3$.

Lemma 3.6. *In the above situation, there is a $(n-1)$-plane $\mathbb{P}^{n-1} \simeq E \subset \mathbb{P}^n$ in general position with $\mathcal{F}$ and such that :*

(a). $\Gamma \cap E \subset sing_2(\mathcal{F}|_E)$.
(b). *The set $X_E := \{p \in \Gamma \cap E \,|\, \mathcal{J}(\mathcal{F}|_E, p) > 1\}$ has codimension ≥ 4 in E.*
(c). *If Γ' is an irreducible component of $\Gamma \cap E$ then $BB(\mathcal{F}|_E, \Gamma') \neq 0$.*

Proof. Fix an affine chart $(z_1,...,z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form Ω representing $\mathcal{F}$ in this chart. Given $p \in \mathbb{C}^n \cap \mathcal{N}_\Gamma$ there is $\ell_p \in \mathbb{C}[z_1,...,z_n]$, of degree one, such that $\ell_p(p) = 0$ and

$$j^1_p(\Omega) = \ell_p \, d\ell_p \ .$$

Note that the hyperplane $H_p = \overline{(\ell_p = 0)} \in \check{\mathbb{P}}^n$ does not depend on the affine chart containing p. As a consequence, the correspondence $p \mapsto H_p$ defines an analytic map $H \colon \mathcal{N}_\Gamma \to \check{\mathbb{P}}^n$. Since $dim(\mathcal{N}_\Gamma) \leq n-2$, we get $dim(H(\mathcal{N}_\Gamma)) \leq n-2$. In particular, the set

$$A := \check{\mathbb{P}}^n \setminus \overline{H(\mathcal{N}_\Gamma)}$$

is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Let $B = \{E \in A \,|\, E$ is in general position with $\mathcal{F}\}$.

Note that B is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Moreover, if $E \in B$ then all points of $\mathcal{N}_\Gamma \cap E$ are nilpotent singularities of $\mathcal{F}|_E$. In fact, fix $p \in \mathcal{N}_\Gamma \cap E$, an affine coordinate system $z = (z_1,...,z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form Ω representing $\mathcal{F}$ in this chart, such that $z(p) = 0$ and $E \cap \mathbb{C}^n = (z_n = 0)$. Let ℓ_p be a degree one polynomial with $\ell_p(p) = 0$, $H_p \cap \mathbb{C}^n = (\ell_p = 0)$ and $j^1_p(\Omega) = \ell_p \, d\ell_p$. Since $\ell_p(0) = 0$ and $E \neq H_p$, we can set $\ell_p(z) = \sum_{j=1}^n a_j \, z_j$, where $a_j \neq 0$ for some $j \in \{1, ..., n-1\}$. The polynomial $\tilde{\ell}_p := \ell_p|_{E \cap \mathbb{C}^n}$ is non-constant. In particular,

$$j^1_0(\Omega|_E) = \tilde{\ell}_p \, d\tilde{\ell}_p \neq 0 \ .$$

Therefore, p is a nilpotent singularity of $\mathcal{F}|_E$.

Now, consider an algebraic stratification $sing(\mathcal{F}) := S_0 \supset S_1 \supset ... \supset S_r = \emptyset$, where $dim(S_0) = n-2$, $dim(S_{j+1}) < dim(S_j)$ and $S_j \setminus S_{j+1}$ is a smooth manifold, for all $0 \leq j < r$. By transversality theory, there exists $E \in B$ transverse to all manifolds $S_j \setminus S_{j+1}$, $0 \leq j < r$. We assert that E satisfies properties (a), (b) and (c).

In fact, since $\Gamma \subset sing_2(\mathcal{F})$ we must have $\Gamma \setminus S_1 \neq \emptyset$, and so $cod(\Gamma \cap E) = 2$, which implies (a), because $\Gamma \cap E \subset sing(\mathcal{F}|_E)$. On the other hand, since K_Γ is smooth of codimension two, we get $K_\Gamma \subset S_0 \setminus S_1$. In particular, E is transverse to K_Γ and this implies that $K_\Gamma \cap E \subset K(\mathcal{F}|_E)$. Therefore, $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$ for all $p \in (\Gamma \setminus X) \cap E$. This implies also that $X_E = X \cap E$. Since $X \subset S_1$, by transversality we get $cod_E(X_E) \geq 4$.

Finally, if Γ' is an irreducible component of $\Gamma \cap E$ then $BB(\mathcal{F}|_E, \Gamma')$ can be computed in any dimension two transverse section, say Λ, through any point in the smooth part of $\Gamma \cap E$. If we take such a point in the smooth part of Γ then we see that Λ is also transverse to Γ at this point, which implies

$$BB(\mathcal{F}|_E, \Gamma') = BB(\mathcal{F}, \Gamma) \neq 0 \ . \quad \square$$

By using lemma 3.6 inductively $n-3$ times we get

Corollary 3.4. *In the situation of lemma 3.6 there is a 3-plane $\mathbb{P}^3 \simeq E \subset \mathbb{P}^n$, in general position with $\mathcal{F}$, with $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$, for all $p \in \Gamma \cap E$, and $BB(\mathcal{F}|_E, \Gamma') \neq 0$, for all irreducible components of Γ' of $\Gamma \cap E$.*

In particular, $\mathcal{F}|_E$ has a rational first integral of the form f_1^m / f_2^n, where $gcd(m,n) = 1$, $1 \leq m < n$, $m \, dg(f_1) = n \, dg(f_2)$ and f_1, f_2 are irreducible. By theorem 3.5 this first integral can be extended to a rational first integral of $\mathcal{F}$. This finshes the proof of theorem 2. $\square$

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