Postulates of quantum mechanics viewed as a non-commutative extension of probability theory with a dynamical law

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Phase space and states

Postulate

***Phase space** quantum system: complex separable Hilbert space $\mathbb{H}$.*
***Pure states** are unit vectors[a] of $\mathbb{H}$.*

[a] Strictly speaking, equivalence classes of unit vectors differing by a global phase, called **rays**, so that $\mathbf{S}_p \simeq \mathbb{H}_1/\sim$ and $h' \sim h \Leftrightarrow h' = \lambda h, \lambda \in \mathbb{C}, |\lambda| = 1$.

- Hilbert space: $\mathbb{C}$-Banach space, with sesquilinear form $\langle \cdot | \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$, s.t.
  - $\langle \alpha_1\phi_1 + \alpha_2\phi_2 | \psi \rangle = \overline{\alpha_1}\langle \phi_1 | \psi \rangle + \overline{\alpha_2}\langle \phi_2 | \psi \rangle$,
  - $\langle \phi | \psi \rangle = \overline{\langle \psi | \phi \rangle}$,
  - $\|\psi\|^2 = \langle \psi | \psi \rangle$.
- Ray: $\psi \in \mathbb{H}$, $\|\psi\| = 1$.
  - But $\mathbb{H}$ separable $\Rightarrow \|\psi\|^2 = \sum_n |\psi_n|^2 = 1$.
  - Hence $(|\psi_n|^2)_n$ probability vector on set indexing the basis of $\mathbb{H}$.

Time evolution

Postulate

Time evolution *of isolated quantum system described by a unitary operator U acting on $\mathbb{H}$. Conversely, any unitary operator acting on $\mathbb{H}$ corresponds to possible invariance[a] of the system.*

[a] Time evolution of isolated system leaves physical quantity "energy" invariant. Unitary operators are associated with conserved quantities.

- U unitary: $U^*U = UU^* = I \Rightarrow U^{-1} = U^*$.
- ϕ ray: $\|U\phi\|^2 = \langle U\phi \mid U\phi \rangle = \langle \phi \mid U^*U\phi \rangle = \|\phi\|^2 = 1$.
- Hence $U\phi$ ray.

Sharp and unsharp observables

Postulate

General real sharp observable: *self-adjoint X acting on $\mathbb{H}$.*
Elementary real sharp observables *associated with X:*

$$\mathcal{B}(\mathbb{R}) \ni B \mapsto M(B) \in \mathfrak{P}(\mathbb{H}); M \text{ spectral projector of } X.$$

Unsharp real observable: *resolution of identity into positive operators that are not necessarily projections.*

- $X^* = X \Rightarrow \operatorname{spec} X := \mathbb{X} \subseteq \mathbb{R}$.
- M spectral measure of X; $\operatorname{supp} M = \operatorname{spec} X$ but can trivially be extended on $\mathbb{R}$.
- $X = \int_{\operatorname{spec} X} M(dx)x$.

Measurements (innocent-looking but against intuition)

Postulate

*Let M spectral measure of sharp observable X. **Measuring** observable X in pure state ψ is asking whether X has spectral values in B and determining probability $\pi_M^\psi(B)$ of occurrence, by*

$$\mathbf{S}_p \ni \psi \mapsto \pi_M^\psi(B) := \langle \psi \mid M(B)\psi \rangle \in [0,1].$$

- $\pi_M^\psi \in \mathcal{M}_1(\mathcal{B}(\mathbb{R}))$ supported by $\mathbb{X} := \operatorname{spec} X$.
- $X = \int_{\operatorname{spec} X} M(dx)x$
  $\Rightarrow \mathbb{E}_\psi(X) = \int_{\mathbb{X}} \pi_M^\psi(dx)x = \int_{\mathbb{X}} \langle \psi \mid M(dx)\psi \rangle x = \langle \psi \mid X\psi \rangle.$
- M in CM indicator; in QM projection operator acting on $\mathbb{H}$.
- Quid if 2 observables with spectral measures M and N?

Composite systems

Postulate

System composed from two subsystems — described respectively by $\mathbb{H}_1$ and $\mathbb{H}_2$ — is described by $\mathbb{H}_1 \otimes \mathbb{H}_2$ (where $\otimes$ denotes tensor product).

Blackboard 1: Tensor product (a first look).

A very simple illustrative example
Phase space, states, and time evolution

- $\mathbb{H} = \mathbb{C}^2$; $\forall f \in \mathbb{H} : f = f_1\epsilon_1 + f_2\epsilon_2$, $\epsilon_1 = \begin{pmatrix}1\\0\end{pmatrix}; \epsilon_2 = \begin{pmatrix}0\\1\end{pmatrix}$.
- If $\|f\| \neq 0$, then $\phi = \frac{f}{\|f\|} \in \mathbf{S}_p$, i.e. $|\phi_1|^2 + |\phi_2|^2 = 1$, therefore ϕ_1, ϕ_2 amplitudes of probability on $\{1, 2\}$.
- $\epsilon_1, \epsilon_2 \in \mathbf{S}_p$.
- $|\phi_1|^2 = \mathbb{P}$(system in state ϕ is in state ϵ_1).
- If $\phi_1 \neq 0, \phi_2 \neq 0$, the pure state ϕ is in a linear superposition of other pure states.
- Time evolution of isolated system is invertible and preserves pure states.

A very simple illustrative example
Observables I

- Take $X = \begin{pmatrix} 1 & 2i \\ -2i & -2 \end{pmatrix} = X^*$ as an example of sharp observable.
- $\mathbb{X} = \operatorname{spec} X = \{-3, 2\}$. Compute easily

Eigenvalues x	Eigenvectors $\mathbf{u}_x$	Orthoprojectors M_x
-3	$\frac{1}{\sqrt{5}} \begin{pmatrix} -i \\ 2 \end{pmatrix}$	$\frac{1}{5} \begin{pmatrix} 1 & -2i \\ 2i & 4 \end{pmatrix}$
2	$\frac{1}{\sqrt{5}} \begin{pmatrix} 2i \\ 1 \end{pmatrix}$	$\frac{1}{5} \begin{pmatrix} 4 & 2i \\ -2i & 1 \end{pmatrix}$

- $\sum_{x \in \operatorname{spec} X} M_x = I_{\mathbb{H}}$, $M_x \geq 0$ and $M_x^2 = M_x$. Thus (M_x) family of sharp elementary observables.
- For $x \in \operatorname{spec} X$, $M_x \mathbb{H} = \mathbb{C}\mathbf{u}_x$.
- $X^* = X \Rightarrow \langle \mathbf{u}_x \mid \mathbf{u}_y \rangle = \delta_{x,y}, x, y \in \operatorname{spec} X \Rightarrow M_x, M_y$ orthogonal orthoprojections.

A very simple illustrative example
Observables II

- Since $(\mathbf{u}_x)_x$ orhonormal basis, $\mathbf{S}_p \ni \psi = \sum_x \alpha_x \mathbf{u}_x$, with $\sum_x |\alpha_x|^2 = 1$, i.e. $(|\alpha_x|^2)_x$ probability vector on spec X.
- $\mathbb{E}_\psi X = \langle \psi \,|\, X\psi \rangle = \langle \sum_x \alpha_x \mathbf{u}_x \,|\, (\sum_{x'} M_x x) \sum_{x''} \alpha_{x''} \mathbf{u}_{x''} \rangle = \sum_x x |\alpha_x|^2$.
- If X was a classical $\mathbb{X}$-valued r.v., with probability $(p_x)_{x\in\mathbb{X}}$,

$$\mathbb{E}X = \sum_x x p_x = \sum_x \sqrt{p_x} x \sqrt{p_x} = \sum_x e^{-i\theta_x} \sqrt{p_x} x \sqrt{p_x} e^{i\theta_x} = \langle \psi \,|\, \hat{X}\psi \rangle,$$

where $\psi = \begin{pmatrix} e^{i\theta_{-3}} \sqrt{p_{-3}} \\ e^{i\theta_2} \sqrt{p_2} \end{pmatrix}$ and $\hat{X} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$.

- Classical r.v. = quantum r.v. with only diagonal entries.

A very simple illustrative example
Measurement

- System in $\psi \in \mathbf{S}_p$. Ask question M_x, i.e. "does X takes value x?" and determine its probability.
- Answer: yes, with probability $\pi_M^\psi(x) = \langle\, \psi \,|\, M_x \psi \,\rangle = \|M_x \psi\|^2$, where $M_x \psi = \begin{cases} \langle\, \mathbf{u}_x \,|\, \psi \,\rangle \mathbf{u}_x & \text{if } \ x \in \mathbb{X}, \\ 0 & \text{otherwise.} \end{cases}$
- BUT: Once the question has been asked, and got positive answer, the system now in new state $\phi_x = \frac{\langle\, \mathbf{u}_x \,|\, \psi \,\rangle \mathbf{u}_x}{|\langle\, \mathbf{u}_x \,|\, \psi \,\rangle|}$.
- Re-asking the question, gives answer "yes" with probability 1.
- Asking a question on a quantum system, IRREVERSIBLY alters its state.

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A very simple illustrative example
Composite systems

- $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$, $\mathbb{H}_i \simeq \mathbb{C}^2$. Each $\mathbb{H}_i$ with basis (ϵ_1, ϵ_2).
- Basis of $\mathbb{H}$ $(\epsilon_1 \otimes \epsilon_1, \epsilon_1 \otimes \epsilon_2, \epsilon_2 \otimes \epsilon_1, \epsilon_2 \otimes \epsilon_2)$.
- $\mathbb{H} \ni \psi = \psi_{11}\epsilon_1 \otimes \epsilon_1 + \psi_{12}\epsilon_1 \otimes \epsilon_2 + \psi_{21}\epsilon_2 \otimes \epsilon_1 + \psi_{22}\epsilon_2 \otimes \epsilon_2$.
- If $\psi_{11} = \psi_{12} = 0$, $\psi_{21} \neq 0; \psi_{22} \neq 0$, then

$$\psi = \psi_{21}\epsilon_2 \otimes \epsilon_1 + \psi_{22}\epsilon_2 \otimes \epsilon_2 = \epsilon_2 \otimes (\psi_{21}\epsilon_1 + \psi_{22}\epsilon_2),$$

  i.e. $\psi = \psi^{(1)} \otimes \psi^{(2)}$. Joint probability = product measure (independence).
- If both $\psi_{11} \neq 0; \psi_{22} \neq 0$ then $\psi \neq \psi^{(1)} \otimes \psi^{(2)}$ (non-independence). Exercise: is-it possible classically?
- "Quantenverschränkung", **"Quantum entanglement"**. "Intrication (enchevêtrement) quantique". Probably "Entrelazamiento (enredo) cuántico" in Spanish.

An artist's view of entanglement

Figure: Ruth Bloch. Entanglement (bronze, 71cm, 1995). (From the web page of the artist, reproduced here with her permission; the existence of this sculpture has been brought to my attention by Dénes Petz).