Classical mechanics . . .

. . . viewed as a classical probability theory with a dynamical law

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Reminder of the Kolmogorov definition (1)

- **Abstract measurable space** $(\Omega, \mathcal{F})$, $\mathcal{F} \subseteq \mathcal{P}(\Omega)$.
  - $\Omega \in \mathcal{F}$,
  - $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
  - $(A_n)_{n\in\mathbb{N}} \in \mathcal{F} \Rightarrow \cup_{n\in\mathbb{N}} A_n \mathcal{F}$.
- **Probability measure** on $\Omega \in \mathcal{F}$, i.e. $\mathbb{P} : \mathcal{F} \to [0, 1]$
  - $\mathbb{P}(\Omega) = 1$,
  - $\mathbb{P}(\sqcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n)$.
- **Concrete measurable space** $(\mathbb{X}, \mathcal{X})$.
- $\mathbb{X}$**-valued random variable:** any $(\mathcal{F}, \mathcal{X})$-measurable map $X : \Omega \to \mathbb{X}$.

Reminder of the Kolmogorov definition (2)

Remark

Probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ does not intervene directly in the definition of X. It induces nevertheless a probability $\mathbb{P}_X$ on $(\mathbb{X}, \mathcal{X})$, **the law** of X, by

$$\mathcal{X} \ni A \mapsto \mathbb{P}_X(A) := \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}).$$

Remark

Important in definition of X the concrete space $\mathbb{X}$. The abstract space Ω is irrelevant.

Blackboard 1: 3 ways to toss a coin ...

Reducibility of classical randomness

Figure: From: Diaconis, Holmes, Montgomery, Dynamical bias in the coin toss, SIAM Review 2007.

Reminder on stochastic kernels

Definition

Let $(\Omega, \mathcal{F})$ and $(\mathbb{X}, \mathcal{X})$ measurable spaces. Map

$$K : \Omega \times \mathcal{X} \to [0, 1]$$

is **stochastic kernel** from $(\Omega, \mathcal{F})$ to $(\mathbb{X}, \mathcal{X})$ if

- $\forall \omega \in \Omega, K(\omega, \cdot)$ probability on $\mathcal{X}$, and
- $\forall A \in \mathcal{X}, K(\cdot, A)$ measurable function.

Blackboard 2: example of 2 coins

Action of K

- $K(\omega, \cdot)$ probability. Hence

$$\begin{aligned} b\mathcal{X} \ni f &\mapsto Kf \in b\mathcal{F} \\ Kf(\omega) &:= \int_{\mathbb{X}} K(\omega, dx) f(x). \end{aligned}$$

- $K(\cdot, A)$ (bounded) measurable function. Hence

$$\begin{aligned} \mathcal{M}_1(\mathcal{F}) \ni \mu &\mapsto \mu K \in \mathcal{M}_1(\mathcal{X}) \\ \mu K(A) &:= \int_{\Omega} \mu(d\omega) K(\omega, A). \end{aligned}$$

Blackboard 3: contravariant and covariant functors.

Deterministic kernel K

Definition

Stochastic kernel K is **deterministic** if

$$\forall \omega \in \Omega, \exists! x := x_K(\omega) \in \mathbb{X} : K(\omega, A) = \epsilon_x(A) = \mathbb{1}_A(x).$$

Blackboard 4: stochastic matrices and extremal stochastic matrices.
Blackboard 5: equivalence $X \leftrightarrow K$ for discrete r.v.
Blackboard 6: equivalence $X \leftrightarrow K$ for continuous r.v.

Random variables and detrerministic kernels

For X r.v. on $(\Omega, \mathcal{F}, \mu)$ and values in $(\mathbb{X}, \mathcal{X})$, kernel $K := K_X$

$$K(\omega, A) = \mathbb{1}_{X^{-1}(A)}(\omega) = \epsilon_{X(\omega)}(A)$$

conveys exactly same information as X.

$$\begin{aligned}
\forall \omega \in \Omega, X(\omega) &= \int_{\mathbb{X}} \epsilon_{X(\omega)}(dx)x = \int_{\mathbb{X}} K(\omega, dx)x = (K\mathrm{id}_{\mathbb{X}})(\omega),\\
\forall A \in \mathcal{X}, \mathbb{P}_X(A) &= \mu(X^{-1}(A)) = \int_{\Omega} \mu(d\omega)\mathbb{1}_{X^{-1}(A)}(\omega)\\
&= \int_{\Omega} \mu(d\omega)K(\omega, A) = (\mu K)(A).
\end{aligned}$$

Archetypal example of a physical sharp measurement

- $X \leftrightarrow K_X$ with K_X deterministic kernel.
- For fixed X (hence K_X) define **sharp elementary observable:**

$$\mathcal{X} \ni A \mapsto M(A) := K(\cdot, A) = \mathbb{1}_{X^{-1}(A)} \in b\mathcal{F}.$$

- Random variable recovery: $X(\omega) = \int_{\mathbb{X}} M(dx)(\omega)x$.
- **Precise preparation** of the system $\mu \in \mathcal{M}_1(\mathcal{F})$.
- **Measurement:** $\mathbf{S} \times \mathbf{O} \ni (\mu, M) \mapsto \pi_M^\mu \in \mathcal{M}_1(\mathcal{X})$, where

$$\pi_M^\mu(A) = \int_\Omega \mu(d\omega) M(A)(\omega) = \int_\Omega \mu(d\omega) K(\omega, A) = (\mu K)(A).$$

Gambling with classical dice I

- Dice shows up a face $\omega \in \Omega := \{1, 2, \ldots, 6\}$.
- Gambler's net gain determined by the random variable X:

$$X(\omega) = [(\omega - 1 \mod 3) - 1] \in \mathbb{X} := \{-1, 0, 1\}.$$

- Two ways to represent information conveyed by X: either as a 6-dimensional vector V or as a 6×3 **stochastic deterministic matrix** K:

$$V := \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} ; K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Gambling with classical dice II

Observable M is the family $M = (M_x)_{x\in\mathbb{X}}$ of elementary observables , where

$$M_x(\omega) := K(\omega, x) = \mathbb{1}_x(X(\omega)) = \mathbb{1}_{X^{-1}(\{x\})}(\omega) = \delta_{X(\omega)}(\{x\}).$$

$$M_{-1} := \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}; M_0 := \begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}; M_1 := \begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix}.$$

Remark

- $\forall x \in \mathbb{X},\ \omega \in \Omega,\ M_x(\omega) \geq 0$ and $M_x^2(\omega) = M_x(\omega)$. (i.e. M_x projections).
- $\sum_{x\in\mathbb{X}} M_x = 1$. ($(M_x)_{x\in\mathbb{X}}$ resolution of identity).
- $X = \sum_{x\in\mathbb{X}} M_x x$. ("Spectral decomposition" of X).

Gambling with classical dice III

- For preparation of dice in state $\mu \in \mathcal{M}_1(\mathcal{F})$, **measurement** determines a probability $\pi^\mu_M \in \mathcal{M}_1(\mathbb{X})$ by

$$\begin{aligned}
\pi^\mu_M(x) &= \mu(\{\omega \in \Omega : X(\omega) = x\}) \\
&= \sum_{\omega \in X^{-1}(\{x\})} \mu(\omega) \\
&= \langle \mu, M_x \rangle \\
&= \sum_{\omega \in \Omega} \mu(\omega) M_x(\omega).
\end{aligned}$$

Definition

Observable decomposable into family of projections (M_x) called **sharp**.

Gambling with classical dice IV

- Example of 2 different **preparations** of the system "dice":

$$\mu_1 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6});\ \mu_2 = (\frac{1}{32}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$$

- Corresponding probability measures in $\mathcal{M}_1(\mathbb{X})$:

$$\pi_M^{\mu_1} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3});\ \pi_M^{\mu_2} = (\frac{5}{32}, \frac{9}{32}, \frac{18}{32}).$$

- Average value $\mathbb{E}_\mu(X) = \sum_{x\in\mathbb{X}} \pi_M^\mu(x)x$:

$$\mathbb{E}_{\mu_1}(X) = 0;\ \mathbb{E}_{\mu_2}(X) = -\frac{5}{32} + \frac{18}{32} = \frac{13}{32}.$$

Randomised gambling with classical dice I

Again gambler's net gain (observable) $\leftrightarrow K$ but now K **genuine stochastic matrix**, e.g.

$$K = \begin{pmatrix} \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \end{pmatrix} \Rightarrow M_{-1} = \begin{pmatrix} \frac{4}{5} \\ 0 \\ \frac{1}{5} \\ \frac{4}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} ; M_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; M_1 = \begin{pmatrix} \frac{1}{5} \\ 0 \\ \frac{4}{5} \\ \frac{1}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} \Rightarrow V = \begin{pmatrix} -\frac{3}{5} \\ 0 \\ \frac{3}{5} \\ -\frac{3}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix}$$

Exercise

What is the significance of the vector V?

Randomised gambling with classical dice II

Remark

- $\forall x \in \mathbb{X}$, $\omega \in \Omega$, $M_x^2(\omega) \leq M_x(\omega)$. ($M_x$ not projections).
- $\sum_{x \in \mathbb{X}} M_x = 1$. ($(M_x)_{x \in \mathbb{X}}$ resolution of identity).
- $\pi_M^\mu(x) = \langle \mu, M_x \rangle = \sum_{\omega \in \Omega} \mu(\omega) M_x(\omega)$. (But (M_x) do not provide spectral decomposition of X).
- But still average gain in state μ given by $\mathbb{E}_\mu X = \sum_{x \in \mathbb{X}} \pi_M^\mu(x) x$.

Definition

Resolution of identity $M = (M_x)$ with M_x positive but not necessarily projections called **unsharp or randomised observable**.

Randomised gambling with classical dice III

With previous μ_1 and μ_2:

$$\pi_M^{\mu_1} = \mu_1 K = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \begin{pmatrix} \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \end{pmatrix} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3});$$

$$\pi_M^{\mu_2} = \mu_2 K = (\frac{1}{32}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}) \begin{pmatrix} \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \end{pmatrix} = (\frac{38}{160}, \frac{45}{160}, \frac{77}{160}).$$

$$\mathbb{E}_{\mu_1}(X) = 0; \mathbb{E}_{\mu_2}(X) = \frac{39}{160}.$$

Postulates of classical mechanics

Phase space and states

Postulate (Phase space and states)

Phase space: *a measurable space* $(\Omega, \mathcal{F})$.
States: *possible preparations of the system* $\mathbf{S} = \mathcal{M}_1(\mathcal{F})$.

Set $\mathbf{S}$ is **convex**

$$\mu_1, \mu_2 \in \mathbf{S}, \lambda \in [0,1] \Rightarrow \lambda\mu_1 + (1-\lambda)\mu_2 \in \mathbf{S}.$$

Extremal points, i.e. states without non-trivial convex decomposition, are the **pure states** $\mathbf{S}_p = \{\epsilon_\omega, \omega \in \Omega\} \simeq \Omega$.

Postulates of classical mechanics

Dynamical law

Postulate (Dynamical law)

Time evolution of isolated[a] system: *a measurable invertible function* $T:\Omega\to\Omega$.

[a]Not exchanging mass or energy with environment.

T is a r.v. hence $\leftrightarrow K_T$ deterministic stochastic kernel.

Definition

State $\mu\in\mathbf{S}$ **invariant** if $T_*\mu := \mu K_T = \mu$, i.e.

$$\forall A\in\mathcal{F}, \mu(T^{-1}A)=\mu(A).$$

Postulates of classical mechanics

Sharp general and elementary observables

Postulate (Observables)

- ***General sharp $\mathbb{X}$-valued observables:*** *random variables*[a] *on phase space* $(\Omega, \mathcal{F})$ *taking values in* $(\mathbb{X}, \mathcal{X})$.
- ***Elementary sharp observables:*** *The* $\{0,1\}$*-valued spectral components* M_x.
- ***General (unsharp)*** $\mathbb{X}$***-valued*** *observables described by decompositions of identity* (M_x) *into positive but not necessarily projective components.*

[a] Recall r.v. $X \leftrightarrow$ deterministic stochastic kernel $K \leftrightarrow (M_x) : X = \sum_x M_x x$.

Postulates of classical mechanics

Measurement

For any $X \in \mathbf{O}$ consider family of questions $(M(A))_{A\in\mathcal{X}}$.

$$M(A) = 1 \Leftrightarrow X \in A.$$

Make system interact with **measuring apparatus** that determines whether question gets positive answer.

Postulate (Physical measurement)

For system in state μ ask question $M(A)$ and determine probability of positive answer.

$$\mathbf{S} \times \mathbf{O} \ni (\mu, M(A)) \mapsto \pi^{\mu}_{M}(A) = \mu K(A).$$

Postulates of classical mechanics

Composite systems

System composed of N subsystems, each with its own phase space $(\Omega_i, \mathcal{F}_i)$ for $i = 1, \ldots, N$.

Postulate (Composite system)

The phase space $(\Omega, \mathcal{F})$ of the composite system is

$$\begin{aligned} \Omega &= \times_{i=1}^{N} \Omega_i, \\ \mathcal{F} &= \otimes_{i=1}^{N} \mathcal{F}_i = \sigma(\times_{i=1}^{N} \mathcal{F}_i). \end{aligned}$$

States are not necessarily product measures!

Blackboard 7: Two illustrative examples.

Hidden variables hypothesis

- In the next lecture, postulates of Quantum Mechanics.
- QM never been contradicted by experiment up to now.
- Nevertheless, "measurement postulate" so counter-intuitive that physicists searched ways of circumvention.
- One of the criticism on this postulate concerns the irreducibility[1] of quantum randomness it imposes.
- One attempt of circumvention was the **"hidden variables" hypothesis**[2].
- Personal view: Situation similar to aether hypothesis in EM.

[1] Einstein's aphorism: "God does not play dice with the world".

[2] Bohm, A suggested interpretation of the quantum theory in terms of "hidden" variables. I+II, Phys. Rev. 85:166–179, 180–193 (1952).

Experiments with polarisers

Experimental facts:

- When photon passes through first polariser — in direction α — emerges polarised in **that** direction.
- When second polariser encountered — in direction β — photon passes through with probability $\cos^2(\alpha - \beta)$.
- If photon initially already polarised in direction α, nothing changes if the first polariser is removed.

Bell's inequalities

If hidden variables $\Rightarrow$ Kolmogorov theory holds.

Proposition (Four-variable Bell's inequality)

X_1, X_2, Y_1, Y_2 arbitrary quadruple of $\{0,1\}$-valued random variables. Then

$$\mathbb{P}(X_1 = Y_1) \leq \mathbb{P}(X_1 = Y_2) + \mathbb{P}(X_2 = Y_2) + \mathbb{P}(X_2 = Y_1).$$

Proof.

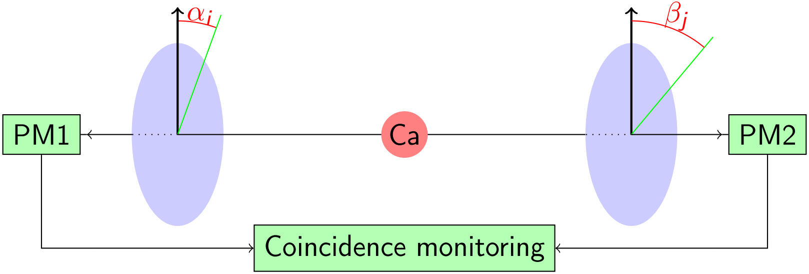
R.v. being $\{0,1\}$-valued, enough to check on all 16 possible realisations of quadruple $(X_1(\omega), X_2(\omega), Y_1(\omega), Y_2(\omega))$ that

$$\{X_1 = Y_1\} \subseteq \{[X_1 = Y_2] \vee [X_2 = Y_2] \vee [X_2 = Y_1]\}.$$

$\square$

The Orsay experiment

Aspect, Dalibard, Roger. Experimental test of Bell's inequalities using time-varying analyzers, Phys. Rev. Lett., 49: 1804–1807 (1982).



Experimental refutation of hidden variables hypothesis I

- $X_\alpha := 1 \Leftrightarrow$ {left photon passes if polariser oriented in α}.
- $Y_\beta := 1 \Leftrightarrow$ {right photon passes if polariser oriented in β}.
- Experimental fact: $\mathbb{P}(X_\alpha = Y_\beta) = \sin^2(\alpha - \beta)$.
- Bell's inequalities:

$$\mathbb{P}(X_{\alpha_1} = Y_{\beta_1}) \leq \mathbb{P}(X_{\alpha_1} = Y_{\beta_2}) + \mathbb{P}(X_{\alpha_2} = Y_{\beta_1}) + \mathbb{P}(X_{\alpha_2} = Y_{\beta_2}).$$

- With choice $\alpha_1 = 0$, $\alpha_2 = \pi/3$, $\beta_1 = \pi/2$, and $\beta_2 = \pi/6$:

$$\sin^2(\pi/2) \leq \sin^2(-\pi/6) + \sin^2(-\pi/6) + \sin^2(\pi/6)$$

or else $\Rightarrow 1 \leq 1/4 + 1/4 + 1/4$.

Experimental refutation of hidden variables hypothesis II

- Orsay experiment can be seen as game you play against nature and you always loose!
- Exist other experiments[3] refuting hidden variables, e.g. by use of Kochen-Specker theorem.

[3] Including by Chilean groups, e.g. Saavedra, Concepción.