UNIVERSITÉ DE RENNES 1

Deformations of complex manifolds: Exam

1. On the course (40 minutes, without documents)

Exercise 1

1 Recall the definition of a holomorphic family of compact complex manifolds.
2 Recall the definition of completeness and versality for such a family.
3 Recall the theorem that gives a sufficient condition on the Kodaira-Spencer map to ensure completeness.
4 Recall the theorem that characterizes versality in terms of the Kodaira-Spencer map.
5 State the Kodaira-Spencer-Nirenberg theorem.

Exercise 2

1 Recall the definition of a sheaf.
2 Is the sheaf of germs of continuous functions on a complex manifold a fine sheaf ?
3 Recall the definition of the coboundary δ of the Čech cohomology.
4 Check that $\delta \circ \delta = 0$.

2. Problems (1h20, with documents)

Exercise 3

Let V be a 3-dimensional complex vector space and (e_1, e_2, e_3) a basis. Let (Z_1, Z_2, Z_3) be the corresponding homogeneous coordinates on $\mathbb{P}^2 := P(V)$ the space of lines in V. The tangent sheaf $T\mathbb{P}^2$ of $\mathbb{P}^2$ is computed through the Euler sequence derived from the differential dp of the quotient map $p : V - \{0\} \to \mathbb{P}^2$:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\mathbb{P}^2} & \to & \mathcal{O}_{\mathbb{P}^2}(1) \otimes V & \to & T\mathbb{P}^2 & \to & 0 \\ & & 1 & \mapsto & Z_1\frac{\partial}{\partial Z_1} + Z_2\frac{\partial}{\partial Z_2} + Z_3\frac{\partial}{\partial Z_3} & & & & \\ & & & & \ell(Z_1,Z_2,Z_3)\frac{\partial}{\partial Z} & \mapsto & \ell(Z_1,Z_2,Z_3)dp\left(\frac{\partial}{\partial Z}\right) & & \end{array}$$

where $(\frac{\partial}{\partial Z_1}, \frac{\partial}{\partial Z_2}, \frac{\partial}{\partial Z_3})$ is the basis (e_1, e_2, e_3) of V seen as a basis of TV and ℓ a linear map.

1 Compute the cohomology groups $\check{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ and $\check{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ for all integers m.

Solution : Let W be a 2-dimensional complex vector space and (ϵ_1, ϵ_2) a basis. Let (X_1, X_2) be the corresponding homogeneous coordinates on $\mathbb{P}^1 := P(W)$. On the open set $U_1 := \{X_1 \neq 0\}$ set $x_2 := \frac{X_2}{X_1}$ and on the open set $U_2 := \{X_2 \neq 0\}$ set $x_1 := \frac{X_1}{X_2}$. A frame for $\mathcal{O}_{\mathbb{P}^1}(m)$ on U_1 is $(\epsilon_1 + x_2\epsilon_2)^{\star m}$ and a frame for $\mathcal{O}_{\mathbb{P}^1}(m)$ on U_2 is $(x_1\epsilon_1 + \epsilon_2)^{\star m}$. On the overlap $U_1 \cap U_2$, they compare by

$$(x_1\epsilon_1 + \epsilon_2)^{\star m} = x_2^m(\epsilon_1 + x_2\epsilon_2)^{\star m}.$$

A data $(U_1, f_1(x_2)), (U_2, f_2(x_1))$ gives a global section of $\mathcal{O}_{\mathbb{P}^1}(m)$ if on the overlap

$$f_1(x_2)(\epsilon_1 + x_2\epsilon_2)^{\star m} = f_2(x_1)(x_1\epsilon_1 + \epsilon_2)^{\star m}$$

that is $f_1(x_2) = f_2(x_1)x_2^m = f_2(\frac{1}{x_2})x_2^m$. Expanding as a power series (in the analytic setting or polynomials in the algebraic setting), we infer that f_1 and f_2 has to be reciprocal polynomials of degree less or equal to m. We find that in terms of homogeneous polynomials $f_1(x_2)(\epsilon_1 + x_2\epsilon_2)^{\star m} = P(X_1, X_2)(X_1\epsilon_1 + X_2\epsilon_2)^{\star m}$, where $P(X_1, X_2) := X_1^m f_1(\frac{X_2}{X_1})$ that is $\check{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = Sym^m V^\star$.

A 1-Cech cocycle on $U_1 \cap U_2$ is given by a Laurent polynomials [1]

$$\left(f(x_2) + g(\frac{1}{x_2})\right)(\epsilon_1 + x_2\epsilon_2)^{\star m}.$$

From the previous computations, we know that 1-coboundaries are of the form

$$f_1(x_2) - x_2^m f_2(\frac{1}{x_2}).$$

Hence for $m \geq -1$ all cocycles are coboundaries. For $m \leq -2$, cohomology classes have a unique representative of the form

$$\left(\frac{a_1}{x_2} + \frac{a_2}{(x_2)^2} + \cdots + \frac{a_{-m-1}}{(x_2)^{-m-1}}\right)(\epsilon_1 + x_2\epsilon_2)^{\star m}$$
$$= \left(\frac{a_1}{X_1^{-m-1}X_2} + \frac{a_2}{X_1^{-m-2}X_2^2} + \cdots + \frac{a_{-m-1}}{X_1X_2^{-m-1}}\right)(X_1\epsilon_1 + X_2\epsilon_2)^{\star m}.$$

Then $\check{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ is of dimension $-m-1$.

2 Determine the sheaf cokernel of the multiplication map $\mathcal{O}_{\mathbb{P}^2}(m-1) \to \mathcal{O}_{\mathbb{P}^2}(m)$ by Z_1.

Solution : The maps

$$\begin{array}{ll} \text{On } U_1 & (e_1 + z_2^1 e_2 + z_3^1 e_3)^{\star m-1} \mapsto (e_1 + z_2^1 e_2 + z_3^1 e_3)^{\star m} \\ \text{On } U_2 & (z_1^2 e_1 + e_2 + z_3^2 e_3)^{\star m-1} \mapsto z_1^2(z_1^2 e_1 + e_2 + z_3^2 e_3)^{\star m} \\ \text{On } U_3 & (z_1^3 e_1 + z_2^3 e_2 + e_3)^{\star m-1} \mapsto z_1^3(z_1^2 e_1 + e_2 + z_3^2 e_3)^{\star m} \end{array}$$

patchs on the overlap, defines a map of sheaves $\mathcal{O}_{\mathbb{P}^2}(m-1) \to \mathcal{O}_{\mathbb{P}^2}(m)$ and may be written

$$(Z_1e_1 + Z_2e_2 + Z_3e_3)^{\star m-1} \mapsto Z_1(Z_1e_1 + Z_2e_2 + Z_3e_3)^{\star m}.$$

Is is injective on germs. Its cokernel is empty on U_1 and is $(\mathcal{O}_{U_2}/(z_1^2)) \otimes \mathcal{O}_{\mathbb{P}^2}(m)$ on U_2. It is therefore $\mathcal{O}_L(m)$ the structure sheaf of the line L defined by $Z_1 = 0$ tensorised by $\mathcal{O}_{\mathbb{P}^2}(m)$. We get the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^2}(m-1) \to \mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_L(m) \to 0.$$

3 Let m be a non negative integer. Show that $\check{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$ and $\check{H}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$ vanish.

1. Depending on the setting, algebraic or analytic, we choose we use polynomials or power series.

Solution : From the previous sequence, we infer a long exact sequence

$$
\begin{array}{ccccc}
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-1)) & \hookrightarrow & H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) & \longrightarrow & H^0(L, \mathcal{O}_L(m)) \\
\to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-1)) & \longrightarrow & H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) & \longrightarrow & H^1(L, \mathcal{O}_L(m)) \\
\to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-1)) & \longrightarrow & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) & \longrightarrow & H^2(L, \mathcal{O}_L(m))
\end{array}
$$

that simplifies

$$
\begin{array}{ccccc}
S^{m-1}V^\star & \hookrightarrow & S^m V^\star & \longrightarrow & S^m W^\star \\
\to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-1)) & \longrightarrow & H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) & \longrightarrow & 0 \\
\to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-1)) & \longrightarrow & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) & \longrightarrow & 0
\end{array}
$$

As the maps $S^m V^\star = S^m(\mathbb{C}Z_1 \oplus \mathbb{C}Z_2 \oplus \mathbb{C}Z_3) \to S^m W^\star = (\mathbb{C}Z_2 \oplus \mathbb{C}Z_3)$ are surjective, we infer that all the $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$ are isomorphic to $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$, and the same for H^2.

We will now compute $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$ and $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$. Denote $U_i := \{Z_i \neq 0\}$, and $z^i_j := \frac{Z_j}{Z_i}$ on U_i. The spaces of regular algebraic functions identify with polynomial rings $\mathbb{C}[U_0] = \mathbb{C}[z^0_1, z^0_2]$, $\mathbb{C}[U_1] = \mathbb{C}[z^1_0, z^1_2] = \mathbb{C}[\frac{1}{z^0_1}, \frac{z^0_2}{z^0_1}]$ and $\mathbb{C}[U_{01}] = \mathbb{C}[z^0_1, z^0_2, \frac{1}{z^0_1}]$. Hence, all 1-Cech cocycles are coboundaries : $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. Noting $\mathbb{C}[U_{012}] = \mathbb{C}[z^0_1, z^0_2, \frac{1}{z^0_1}, \frac{1}{z^0_2}]$, we also infer that $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$.

4 Compute $\check{H}^0(\mathbb{P}^2, T\mathbb{P}^2)$, $\check{H}^1(\mathbb{P}^2, T\mathbb{P}^2)$ and $\check{H}^2(\mathbb{P}^2, T\mathbb{P}^2)$.

Solution : From the Euler sequence, we infer the long exact sequence

$$
\begin{array}{ccccc}
H^0(\mathcal{O}_{\mathbb{P}^2}) & \hookrightarrow & H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes V & \longrightarrow & H^0(T\mathbb{P}^2) \\
\to H^1(\mathcal{O}_{\mathbb{P}^2}) & \longrightarrow & H^1(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes V & \longrightarrow & H^1(T\mathbb{P}^2) \\
\to H^2(\mathcal{O}_{\mathbb{P}^2}) & \longrightarrow & H^2(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes V & \longrightarrow & H^2(T\mathbb{P}^2) \\
\to 0 & & & &
\end{array}
$$

We find $\check{H}^0(\mathbb{P}^2, T\mathbb{P}^2) = (V^\star \otimes V)/\mathbb{C}$, $\check{H}^1(\mathbb{P}^2, T\mathbb{P}^2) = 0$ and $\check{H}^2(\mathbb{P}^2, T\mathbb{P}^2) = 0$.

5 Infer from these computations that $\mathbb{P}^2$ has no non-trivial deformations.

Solution : As $\check{H}^2(\mathbb{P}^2, T\mathbb{P}^2) = 0$, the Kodaira-Nirenberg-Spencer tells that the Kuranishi family of $\mathbb{P}^2$ is an open set in $\check{H}^1(\mathbb{P}^2, T\mathbb{P}^2) = 0$: hence, $\mathbb{P}^2$ has no non-trivial deformations.

Exercise 4

Let m be an integer and $F_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ the Hirzebruch surface. Take for granted that

- $\dim \check{H}^1(F_m, TF_m) = m - 1$ for $m \geq 2$, and 0 for $m = 0$ and $m = 1$.
- $\dim \check{H}^2(F_m, TF_m) = 0$ for all m.

Let k be an integer. Consider the family obtained by patching two copies of $\mathbb{C} \times \mathbb{C} \times \mathbb{P}^1$ by

$$(t_1, z_1, \zeta_1) \equiv (t_2, z_2, \zeta_2) \iff \begin{cases} t_1 &= t_2 \\ z_1 &= \frac{1}{z_2} \\ \zeta_1 &= z_2^m \zeta_2 + t_2 z_2^k \end{cases}$$

Recall that the fiber over $t = 0$ is F_m and that the other fibers are isomorphic to F_{m-2k}.

1 For which m and k could this family be at $t = 0$ the Kuranishi family of F_m (i.e. complete at each point of the base $\mathbb{C}$ and versal at $t = 0$).

Solution : As this family has a base of dimsension one, and as by Kodaira-Nirenberg-Spencer theorem ($\dim \check{H}^2(F_m, TF_m) = 0$= the base is an open set in $\check{H}^1(F_m, TF_m)$, we see that m has to be 2. As the family is trivial for $k = 0$, k has to be 1.

2 Compute the Kodaira-Spencer map of this family at $t = 0$.

Solution : For $m = 2$ and $k = 1$, the Kodaira-Spencer map reads

$$\begin{aligned} \kappa(\frac{\partial}{\partial t}) &= \frac{\partial z_1}{\partial t}(t, z_2, \zeta_2)\frac{\partial}{\partial z_1} + \frac{\partial \zeta_1}{\partial t}(t, z_2, \zeta_2)\frac{\partial}{\partial \zeta_1} \\ &= -(z_1)^2 \frac{\partial}{\partial z_1} + \frac{1}{z_1}\frac{\partial}{\partial \zeta_1}. \end{aligned}$$

3 Conclude.

Solution : As $\frac{\partial}{\partial \zeta_1}$ writes in the other chart $\frac{\partial}{\partial \zeta_1} = \frac{\partial \zeta_2}{\partial \zeta_1}\frac{\partial}{\partial \zeta_2} + \frac{\partial z_2}{\partial \zeta_1}\frac{\partial}{\partial z_2} + \frac{\partial t_2}{\partial \zeta_1}\frac{\partial}{\partial t_2} = \frac{1}{z_1^2}\frac{\partial}{\partial \zeta_2} + \cdots$ the image of the Kodaira-Spencer is not a coboundary. Hence, the Kodaira-Spencer map is surjective ewerywhere and an isomorphism at 0 : the family is hence the Kuranishi family of F_2.