UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the Terminal Examination (2h)

Documents are not allowed

Problem. Let p be a prime number. We denote by $\mathbb{F}_p$ the prime field of order p which may be constructed as the integers modulo p, that is $\mathbb{F}_p \equiv \mathbb{Z}/(p\mathbb{Z})$. Let $\mathbf{H}$ be a complex Hilbert space of finite dimension $d \in \mathbb{N}_*$. We consider two unitary operators $A : \mathbf{H} \to \mathbf{H}$ and $B : \mathbf{H} \to \mathbf{H}$ satisfying $A^p = Id$ and $B^p = Id$, as well as

$$\forall (l,m) \in \mathbb{N}^2, \qquad A^l\, B^m = e^{-2\pi i l m/p}\, B^m\, A^l. \tag{1}$$

We suppose that the only subspaces of $\mathbf{H}$ invariant under both A and B are $\{0\}$ and $\mathbf{H}$.

1. Explain why A has at least one eigenvalue $\lambda \in \mathbb{C}$. Show that λ is of modulus 1.

The characteristic polynomial $P(X) := det(A - XId)$ is of degree $d \geq 1$. It has therefore at least one root $\lambda \in \mathbb{C}$ which is an eigenvalue of A. Let v a non-zero eigenvector of A which is associated to λ. We have

$$\langle Av, Av\rangle = \langle \lambda v, \lambda v\rangle = |\lambda|^2 \parallel v \parallel^2 = \langle v, A^* A v\rangle = \parallel v \parallel^2,$$

which is possible only if $|\lambda| = 1$.

2. Let $0 \neq v \equiv B^0 v \in \mathbf{H}$ be an eigenvector of A of norm 1 that is associated with λ. Show that $B^k v$ is for all $k \in \mathbb{N}$ an eigenvector for A. What is the corresponding eigenvalue ?

For $k = 0$, this is due to the definition of $v \equiv B^0 v$. For $k \in \mathbb{N}_*$, we can apply (1) with $(l,m) = (1,k)$ to obtain

$$AB^k v = e^{-2\pi i k/p}\, B^k A v = \lambda_k\, B^k v, \qquad \lambda_k := \lambda e^{-2\pi i k/p},$$

which says that $B^k v$ is an eigenvector for A associated with the eigenvalue λ_k.

3. What can be said about the (vector) subspace $E \subset \mathbf{H}$ which is generated by the family of vectors $\{B^k v\}_{k\in\mathbb{N}}$?

The subspace E is obviously stable under the action of B. In view of the preceding question, it is also stable under the action of A. It is of dimension greater than 1 (because the non-zero vector v is in E). Thus, by assumption, it must coincide with $\mathbf{H}$.

4. Explain why the vectors $B^k v$ with $0 \leq k \leq p-1$ form an orthonormal basis of $\mathbf{H}$ built with eigenvectors of A. What can be said about the dimension of $\mathbf{H}$? Prove that the eigenspace $E_\lambda := \{f \in \mathbf{H}; Af = \lambda f\}$ is of dimension 1. Explain why $\lambda = 1$ is sure to be an eigenvalue of A.

Since B is unitary and v is of norm 1, $B^k v$ is of norm 1. Since $B^p = Id$, we have

$$\mathbf{H} \equiv E \equiv \left\{ \sum_{k=0}^{p-1} c_k \, B^k v \, ; \, c_k \in \mathbb{C} \right\}.$$

This clearly indicates that the p vectors $B^k v$ with $0 \leq k \leq p-1$ generate $\mathbf{H}$. On the other hand, for $k \neq \tilde{k}$, we have

$$\langle B^k v, AB^{\tilde{k}} v\rangle = \bar{\lambda}_{\tilde{k}} \langle B^k v, B^{\tilde{k}} v\rangle = \langle A^* B^k v, B^{\tilde{k}} v\rangle = \langle A^{-1} B^k v, B^{\tilde{k}} v\rangle = \lambda_k^{-1} \langle B^k v, B^{\tilde{k}} v\rangle.$$

Since $\lambda_k^{-1} = \bar{\lambda}_k$ and $\lambda_k \neq \lambda_{\tilde{k}}$, we must have $\langle B^k v, B^{\tilde{k}} v\rangle = 0$. In other words, the vectors $B^k v$ with $0 \leq k \leq p-1$ form an orthonormal basis of $\mathbf{H}$, and therefore $d = p$.

Now, let $\tilde{v} \in E_\lambda$ be a vector which is not colinear with v and which is given by

$$\tilde{v} = \sum_{k=0}^{p-1} c_k \, B^k v.$$

By applying A, we deduce that (since $\lambda^{-1} = \bar{\lambda}$)

$$\tilde{v} = \sum_{k=0}^{p-1} c_k \, \bar{\lambda} \, \lambda_k B^k v.$$

And therefore

$$0 = \sum_{k=1}^{p-1} c_k \, (1 - \varepsilon^{-2\pi i k/p}) B^k v,$$

which implies that $c_k = 0$ for $k \neq 1$ and yields the expected contradiction. We must have

$$E_\lambda = \{cv \, ; \, c \in \mathbb{C}\}, \qquad \dim E_\lambda = 1.$$

On the other hand, since $A^p = Id$, we must have $\lambda_k^p = \lambda^p = 1$. Thus λ is a p^{th} root of unity which implies that $\lambda_k = 1$ for some k.

5. The Hilbert space $L^2(\mathbb{F}_p)$ is provided with the counting measure on $\mathbb{F}_p$ which means that, given $f \in L^2(\mathbb{F}_p)$ and $g \in L^2(\mathbb{F}_p)$, we work with the inner product

$$\langle f, g\rangle := \sum_{n=0}^{p-1} f(n) \, \bar{g}(n).$$

5.1. Prove that the *modulation operator* $U : L^2(\mathbb{F}_p) \longrightarrow L^2(\mathbb{F}_p)$ and the *translation operator* $V : L^2(\mathbb{F}_p) \longrightarrow L^2(\mathbb{F}_p)$ which are given by

$$\begin{array}{rcl} U(f) : \mathbb{F}_p & \longrightarrow & \mathbb{C} \\ n & \longmapsto & e^{-2\pi i n/p} \, f(n), \end{array} \qquad \begin{array}{rcl} V(f) : \mathbb{F}_p & \longrightarrow & \mathbb{C} \\ n & \longmapsto & f(n-1), \end{array}$$

are unitary operators on $L^2(\mathbb{F}_p)$.

It suffices to note that U and V preserve the L^2-norm. Indeed

$$\| U(f) \|^2_{L^2(\mathbb{F}_p)} = \sum_{n=0}^{p-1} |e^{-2\pi i n/p} f(n)|^2 = \sum_{n=0}^{p-1} |f(n)|^2 = \| f \|^2_{L^2(\mathbb{F}_p)} .$$

On the other hand

$$\| V(f) \|^2_{L^2(\mathbb{F}_p)} = \sum_{n=0}^{p-1} |f(n-1)|^2 = \sum_{n=0}^{p-1} |f(n)|^2 = \| f \|^2_{L^2(\mathbb{F}_p)},$$

where we have used the property according to which the map $n \mapsto n-1$ is a bijection on the field $\mathbb{F}_p$ (since $-1 \equiv p-1$).

5.2. Verify that we have $U^p = Id$ and $V^p = Id$, as well as

$$\forall (l,m) \in \mathbb{N}^2_*, \qquad U^l\, V^m = e^{-2\pi i l m/p}\, V^m\, U^l.$$

The two first properties come from the relations

$$\big(e^{-2\pi i n/p}\big)^p = 1, \qquad n - p = n \ (in\ \mathbb{F}_p).$$

On the other hand, we have

$$\begin{aligned} U^l V^m(f)(n) &= U^l\big(n \mapsto f(n-m)\big) = e^{-2\pi i n l/p} f(n-m), \\ V^m U^l(f)(n) &= V^m\big(n \mapsto e^{-2\pi i l n/p} f(n)\big) = e^{-2\pi i l(n-m)/p} f(n-m), \end{aligned}$$

which leads to the last relation.

5.3. What can be said about the family of Dirac functions $\{\delta_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$ given by

$$\mathbb{F}_p \ni n \longmapsto \delta_\ell(n) := \begin{cases} 1 & \text{if} \quad n = \ell, \\ 0 & \text{if} \quad n \neq \ell, \end{cases} \qquad \ell \in \mathbb{F}_p$$

first from the viewpoint of $L^2(\mathbb{F}_p)$ and secondly from the perspective of U ?

The family $\{\delta_l\}_l$ forms an orthonormal basis of $L^2(\mathbb{F}_p)$. Moreover

$$U(\delta_l)(n) = e^{-2\pi i n/p}\, (\delta_l)(n) = e^{-2\pi i l/p}\, (\delta_l)(n),$$

which indicates that δ_l is an eigenvector of U associated with the eigenvalue $e^{-2\pi i l/p}$.

5.4. Find a self-adjoint operator R on $L^2(\mathbb{F}_p)$ which is such that $e^{-2\pi i R/p} = U$. Compute the mean value of R along δ_ℓ, that is the quantity $\langle \delta_\ell, R\delta_\ell \rangle$. What could be a possible interpretation of R ?

It suffices to adjust R in such a way that $R(\delta_\ell) = \ell \delta_\ell$ to ensure that $e^{-2\pi i R/p}\delta_\ell = e^{-2\pi i \ell/p}\delta_\ell$ which, in view of the above, guarantees that $e^{-2\pi i R/p} = U$. Thus

$$R(f)(n) = R\Big(\sum_{k=0}^{p-1} f(k)\, \delta_k\Big)(n) = \sum_{k=0}^{p-1} f(k)\, R(\delta_k)(n) = \sum_{k=0}^{p-1} f(k)\, k\, \delta_k(n) = n f(n),$$

which make R appears as a position operator. This is confirmed by the relation

$$\langle \delta_\ell, R\delta_\ell \rangle = \langle \delta_\ell, \ell\delta_\ell \rangle = \ell,$$

which says that a state concentrated at the position ℓ returns the value of ℓ.

5.5. What can be said about the family of functions $\{g_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$ given by

$$\mathbb{F}_p \ni n \longmapsto g_\ell(n) := \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i \ell k/p} \, \delta_k(n), \qquad \ell \in \mathbb{F}_p$$

first from the perspective of V and secondly from the viewpoint of $L^2(\mathbb{F}_p)$?

First, compute

$$\begin{aligned} V(g_\ell)(n) &= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i \ell k/p} \, \delta_k(n-1) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i \ell k/p} \, \delta_{k+1}(n) \\ &= \frac{1}{\sqrt{p}} \sum_{k=1}^{p} e^{-2\pi i \ell (k-1)/p} \, \delta_k(n) = e^{2\pi i \ell/p} \, g_\ell(n). \end{aligned}$$

This means that g_ℓ is an eigenvector of V associated with the eigenvalue $e^{2\pi i \ell/p}$. Since these eigenvalues are distinct and have a total number of p, the corresponding eigenvectors g_ℓ form an orthogonal basis of $L^2(\mathbb{F}_p)$. Moreover, we can remark that $g_\ell(n) = e^{-2\pi i \ell n/p}/\sqrt{p}$ so that g_ℓ is a function of norm 1 in $L^2(\mathbb{F}_p)$. The basis is orthonormal.

5.6. Using the family of functions $\{g_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$, determine a self-adjoint operator S on $L^2(\mathbb{F}_p)$ which is such that $e^{2\pi i S/p} = V$.

As in question 5.4, we can argue on the eigenspaces. It suffices to define S through the conditions $Sg_\ell = \ell g_\ell$ for all $\ell \in \mathbb{F}_p$.

5.7. Do the operators R and S commute ? Justify the answer.

NO. By contradiction. Assume that R and S commute. Then, U and V must commute which is not the case because from (1) with $(l, m) = (1, 1)$, we have

$$[U, V] = (1 - e^{-2\pi i/p}) \, VU \not\equiv 0.$$

6. Show that we can construct a unitary (surjective) map W from $\mathbf{H}$ onto $L^2(\mathbb{F}_p)$ which is such that

$$WAW^{-1} = U, \qquad WBW^{-1} = V.$$

Mention the name of the theorem which is associated with the above relation.

We have seen in question 1.4 that we can always assume that $\lambda = 1$. Then, define W through the relation $W(B^\ell v) = \delta_\ell$ for all $\ell \in \mathbb{F}_p$. Such map W exchanges two orthonormal basis, and therefore it is a unitary operator. By this way, we also find that, for all $\ell \in \mathbb{F}_p$, we have

$$WA(B^\ell v) = W(e^{-2\pi i \ell/p} B^\ell v) = e^{-2\pi i \ell/p} \, \delta_\ell = U\delta_\ell = UW(B^\ell v) \quad \Longrightarrow \quad WA = UW,$$

as well as

$$WB(B^\ell v) = WB^{\ell+1}v = \delta_{\ell+1} = V\delta_\ell = VW(B^\ell v) \quad \Longrightarrow \quad WB = VW.$$

It suffices to compose with W^{-1} on the right to recover the expected result. In this exercice, we have developed a discrete version of the Stone-von Neumann Theorem.

Problem 2. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R})$ be a smooth compactly supported function with $\chi \equiv 1$ in a neighbourhood of the position $\xi = 0$. Consider the symbol

$$K(\xi) := i\ |\xi|\ (1-\chi(\xi))\,, \qquad |\xi| := (\xi_1^2 + \cdots + \xi_n^2)^{1/2}\,, \qquad \xi \in \mathbb{R}^n\,.$$

1. Explain why the function $\xi \longmapsto K(\xi)$ is a symbol in the class $S^1(\mathbb{R}^n)$.

Definet $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. By definition, the function K is in $S^1(\mathbb{R}^n)$ if and only if

$$\forall\,(\alpha,\beta) \in (\mathbb{N}^n)^2\,, \quad \exists\, C_{\alpha,\beta} \in \mathbb{R}_+^*\,; \quad |\partial_\xi^\alpha \partial_x^\beta K(\xi)| \leq C_{\alpha,\beta}\ \langle\xi\rangle^{1-|\alpha|}\,.$$

This is evident when $|\beta| \neq 0$. When $|\beta| = 0$, we can exploit the Leibniz formula that yields

$$\partial_\xi^\alpha K(\xi) \,=\, i \sum_{0\leq\gamma\leq\alpha} C_\alpha^\gamma\ \partial^\gamma(|\xi|)\ \partial^{\alpha-\gamma}[1-\chi(\xi)] \,=\, i\ \partial^\alpha(|\xi|)\ [1-\chi(\xi)] \,+\, f(\xi)$$

For $\gamma < \alpha$, the function $\partial^{\alpha-\gamma}[1-\chi(\xi)]$ is in $\mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R})$ and it is equal to 0 in a neighbourhood of $\xi = 0$. Thus, we have

$$\partial_\xi^\alpha K(\xi) \,=\, i\ \partial^\alpha(|\xi|)\ [1-\chi(\xi)] \,+\, f(\xi)\,, \qquad f \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R})\,.$$

To conclude, it suffices to remark that $\partial^\alpha(|\xi|)$ is homogeneous of degree $1-|\alpha|$.

2. Let $K(D)$ be the pseudo-differential operator associated to the symbol K, that is

$$[K(D)\,u](x) \,=\, (2\,\pi)^{-n/2} \int_{\mathbf{R}^n} e^{i\,x\cdot\xi}\ K(\xi)\ \hat{u}(\xi)\ d\xi\,.$$

2.1. We select some function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Prove that $K(D)u$ is a bounded function. What more needs to be said about $K(D)u$?

We have

$$[K(D)\,u](x) \,=\, (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,x\cdot\xi}\ \langle\xi\rangle^{-1}\,K(\xi)\ \langle\xi\rangle^{-n-1}\,\langle\xi\rangle^{n+2}\,\hat{u}(\xi)\ d\xi\,.$$

The function $\langle\xi\rangle^{-1}\,K(\xi)$ is bounded because $K \in S^1$. The function $\langle\xi\rangle^{n+2}\,\hat{u}(\xi)$ is bounded because $\hat{u} \in \mathcal{S}$. On the other hand, the function $\langle\xi\rangle^{-n-1}$ is integrable on $\mathbb{R}^n$. Thus, the above integral is convergent (with a uniform bound). As viewed in the course (and as can be proved directly), the function $K(D)u$ is in fact in $\mathcal{S}(\mathbb{R}^n)$.

2.2. Show that $K(D)$ is (formally) skew-symmetric in the sense that

$$\langle u, K(D)v\rangle = \int_{\mathbf{R}^n} u(x)\ \overline{K(D)v}(x)\,dx = -\langle K(D)u, v\rangle, \qquad \forall (u,v)\in \mathcal{S}(\mathbf{R}^n).$$

We know that $K(D)$ and $K(D)^*$ are in $OP^1(\mathbf{R}^n)$. From Plancherel theorem, we have

$$\langle u, K(D)v\rangle_{\mathcal{S}\times\mathcal{S}} = \langle \hat u, K(\xi)\hat v\rangle_{\mathcal{S}\times\mathcal{S}} = \int \hat u(\xi)\ \bar K(\xi)\ \bar{\hat v}(\xi)\ d\xi\,.$$

Since $K(\xi)\in i\,\mathbf{R}$, this leads to

$$\langle u, K(D)v\rangle_{\mathcal{S}\times\mathcal{S}} = -\,\langle K(\xi)\hat u, \hat v\rangle_{\mathcal{S}\times\mathcal{S}} = \langle -K(D)u, v\rangle_{\mathcal{S}\times\mathcal{S}} = \langle K(D)^* u, v\rangle_{\mathcal{S}\times\mathcal{S}}\,.$$

Just compare the two last terms.

3. We consider the Cauchy problem

$$(\mathcal{PC}) \qquad \{\ \partial_t u \,-\, K(D)\,u = 0\,, \qquad u_{|t=0} = u_0 \in H^s(\mathbb{R}^n)\,, \qquad s\in\mathbb{R}\,.$$

We denote by $\hat u(t,\xi)$ the Fourier transform of $u(t,\cdot)$ with respect to $x\in\mathbb{R}^n$.

3.1. Compute $\hat u(t,\xi)$ and deduce from the formula thus obtained that

$$(\mathcal{I}) \qquad u(t,\cdot)\in H^s(\mathbb{R}^n)\,, \qquad \|\,u(t,\cdot)\,\|_{H^s(\mathbb{R}^n)} = \|\,u_0(\cdot)\,\|_{H^s(\mathbb{R}^n)}\,, \qquad \forall\, t\in\mathbb{R}_+^*\,.$$

We have $\hat u(t,\xi) = e^{tK(\xi)}\,\hat u_0(\xi)$. Since $e^{tK(\xi)}$ is of modulus 1, we have

$$\begin{aligned}\|\,u(t,x)\,\|_{H^s(\mathbf{R}^n)} = \|\,\langle\xi\rangle^s\,\hat u(t,\xi)\,\|_{L^2(\mathbf{R}^n)} &= \Big(\int \langle\xi\rangle^{2\,s}\ |e^{t\,K(\xi)}|\ |\hat u_0(\xi)|^2\ d\xi\Big)^{1/2} \\ &= \Big(\int \langle\xi\rangle^{2\,s}\ |\hat u_0(\xi)|^2\ d\xi\Big)^{1/2} = \|\,u_0\,\|_{H^s(\mathbf{R}^n)}\,.\end{aligned}$$

3.2. Prove that the identity $(\mathcal{I})$ can also be recovered through energy estimates performed at the level of $(\mathcal{PC})$.

We can follow the following steps : :

∘ *We compose $(\mathcal{PC})$ on the left with $op\,\langle\xi\rangle^s \equiv \langle D\rangle^s$. Since $[\langle D\rangle^s ; K(D)] \equiv 0$, the expression $w(t,x) := \langle D\rangle^s u(t,x)$ must be a solution of*

$$\{\ \partial_t w \,-\, K(D)\,w = 0\,, \qquad w_{|t=0} = w_0 := \langle D\rangle^s u_0 \in L^2(\mathbb{R}^n)\,.$$

∘ *We perform L^2-energy estimates on this equation. In other words, we multiply the equation on the left by $2\,{}^t\bar w(t,x)$ and then we integrate in x to obtain*

$$\frac{d}{dt}\,\|\,w(t,\cdot)\,\|^2_{L^2(\mathbf{R}^n)} \ + 2\ \langle w, K(D)\,w\rangle_{L^2\times L^2} \ = \ 0\,.$$

Since $K(D)$ is skew-symmetric, we have

$$\langle w, K(D)\,w\rangle_{L^2\times L^2} = \langle K(D)^* w, w\rangle_{L^2\times L^2} = -\,\langle K(D)\,w, w\rangle_{L^2\times L^2} = -\,\overline{\langle w, K(D)\,w\rangle}_{L^2\times L^2}\,.$$

This means that the number $\langle w, K(D)\,w\rangle_{L^2\times L^2}$ is purely imaginary. There remains

$$\frac{d}{dt}\,\|\,w(t,\cdot)\,\|^2_{L^2(\mathbf{R}^n)} = \ 0$$

which after integration in time, between 0 and t furnishes

$$\| u(t,\cdot) \|^2_{H^s(\mathbb{R}^n)} = \| w(t,\cdot) \|^2_{L^2(\mathbb{R}^n)} = \| w_0(\cdot) \|^2_{L^2(\mathbb{R}^n)} = \| u_0(\cdot) \|^2_{H^s(\mathbb{R}^n)} \, .$$

4. Let δ_0 be the Dirac mass located at the position $x = 0$. Show that $\delta_0 \in H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ satisfying $s < -(n/2)$.

The Fourier transform of δ_0 *coincides with* $1_{\mathbb{R}^n}$. *Thus*

$$\| \delta_0 \|^2_{H^s(\mathbb{R}^n)} = \| \langle \xi \rangle^s \|^2_{L^2(\mathbb{R}^n)} = \Big(\int_{\mathbb{S}^n} \int_{\mathbb{R}_+} (1+r^2)^s \; r^{n-1} \; dr \, d\theta \Big)^{1/2} .$$

This becomes integrable on condition that $2\,s + n - 1 < -1$ *which yields the expected condition* $s < -(n/2)$.

5. We start with $u_0 = \delta_0$. Recall the definition of the wave front set $WF(\delta_0)$ of the distribution δ_0. Then describe the content of $WF(\delta_0)$.

We want to show that

$$WF(\delta_0) \; = \; \{ (0,\xi) \, ; \; \xi \in \mathbb{R}^n \setminus \{0\} \} \, .$$

It is clear that $\delta \equiv 0$ *on all open sets that do not contain* $0 \in \mathbf{R}^n$ *which implies that* $WF(\delta) \subset \{0\} \times \mathbf{R}^n$. *Then, if* $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ *satisfies* $\varphi(0) \neq 0$, *we find that (modulo multiplicative constants)*

$$\widehat{\varphi \, \delta}(\xi) \; = \; (\hat\varphi \star \hat\delta)(\xi) \; = \int_{\mathbf{R}^n} \hat\varphi(\xi - \eta) \; \mathbb{I}_{\mathbf{R}^N}(\eta) \; d\eta \; = \int_{\mathbf{R}^n} \hat\varphi(\eta) \; d\eta \; = \; \varphi(0) \neq 0.$$

The fonction $\widehat{\varphi \, \delta}(\cdot)$ *is a (non-zero) constant. There is no conic neighboorhood of a direction* $\xi \in \mathbf{R}^n \setminus \{0\}$ *where it can be rapidly decreasing. Thus, all positions* $(0,\xi)$ *are in the wave front set of* δ_0.

6. We consider $(\mathcal{PC})$ for the choice $u_0 = \delta_0$. We denote by u the corresponding solution. We fix some $t \in \mathbb{R}_+^*$ as well as some $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Show that we can find a fonction ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ giving rise to

$$\widehat{\varphi u}(t,\xi) \; = \; \int_{\mathbb{R}^n} \hat\varphi(\xi - \eta) \; e^{i\,t\,|\eta|} \; d\eta \; + \; \psi(\xi) \, .$$

Let again $\chi \in C_0^\infty(\mathbb{R}^n)$ *with* $\chi \equiv 1$ *in a neighboorhood of* 0. *Define*

$$\tilde\chi(\eta) := e^{i\,t\,|\eta|\,(1-\chi(\eta))} - e^{i\,t\,|\eta|} .$$

The function $\tilde\chi$ *is smooth, bounded and compactly supported. Retain that*

$$\exists \, R \in [1,+\infty[\, ; \qquad |\eta| \geq R \quad \Longrightarrow \quad \tilde\chi(\eta) = 0 \, .$$

We take

$$\psi(\xi) \; := \int_{\mathbb{R}^n} \hat\varphi(\xi - \eta) \; \chi(\eta) \; d\eta \; = \; (\hat\varphi \star \chi)(\xi) \, , \qquad \hat\varphi \in \mathcal{S}(\mathbb{R}^n) \, , \quad \chi \in L^1(\mathbb{R}^n) \, .$$

The function ψ *is smooth (of class* $\mathcal{C}^\infty$*) since the same applies to* φ. *The integrale in* η *concerns only the ball* $B(0,R]$. *For* $|\eta| \leq R$ *and* $|\xi| \geq 4\,R$, *we have*

$$1+|\xi-\eta|^2 \geq 1+(|\xi|-R)^2 \geq 1+|\xi|^2-2\,R\,|\xi|+R^2 \geq \frac{1}{2}\ (1+|\xi|^2)\,.$$

On the other hand, since $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, *we have the control*

$$\forall\, k \in \mathbf{N}\,, \qquad \exists\, C_k \in \mathbf{R}_+^*\,; \qquad |\hat{\varphi}(\zeta)| \leq \langle \zeta \rangle^{-k}\,.$$

It follows that

$$|\xi| \geq 4\,R \quad \Longrightarrow \quad |\psi(\xi)| \leq C_k\ \langle \xi-\eta \rangle^{-k}\ \parallel \chi \parallel_{L^1(\mathbb{R}^n)} \leq 2^{-k/2}\, C_k\ \parallel \chi \parallel_{L^1(\mathbb{R}^n)}\ \langle \xi \rangle^{-k},$$

which means that ψ *is indeed rapidly decreasing.*

7. In this question, we consider the Cauchy problem

$$(\mathcal{PC}\delta) \qquad \{\, \partial_t \tilde{u} \,-\, i\,|D|\,\tilde{u}=0\,, \qquad \tilde{u}_{|t=0}=\delta\,.$$

Let $t \in \mathbb{R}_+$. Describe the wave front set $WF\big(\tilde{u}(t,\cdot)\big)$ of the distribution $\tilde{u}(t,\cdot)$. Justify the answer.

The theorem describing the propagation of the wave front set says that

$$WF\big(\tilde{u}(t,\cdot)\big) \;=\; \Phi_t(WF(\delta))$$

where Φ_t *is the diffeomorphism induced by the Hamiltonian field* $H(x,\xi) \equiv H(\xi) := |\xi|$. *We have to look at*

$$\begin{cases} \dfrac{d}{dt}X(t,y,\eta) \;=\; -\nabla_\xi H(X,\Xi) \;=\; -\dfrac{\Xi}{|\Xi|}\,, & X(0,y,\eta) \;=\; y\,, \\[2ex] \dfrac{d}{dt}\Xi(t,y,\eta) \;=\; +\nabla_x H(X,\Xi) \;=\; 0\,, & \Xi(0,y,\eta) \;=\; \eta\,. \end{cases}$$

This furnishes $\Xi(t,y,\eta)=\eta$ *and then*

$$X(t,y,\eta) \;=\; y \;-\; t\,\frac{\eta}{|\eta|}\,.$$

A the time t, *the wave front set is contained in the cone with center* 0 *and radius* t *(the so-called light cone). In other words*

$$WF\big(\tilde{u}(t,\cdot)\big) \;=\; \Big\{\big(-t\,\frac{\eta}{|\eta|},\eta\big)\,;\ \eta \in \mathbf{R}^n\Big\}\,.$$