UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Terminal Examination - the 10/12/2021 (2h)

Documents are not allowed

Problem 1. Let p be a prime number. We denote by $\mathbb{F}_p$ the prime field of order p which may be constructed as the integers modulo p, that is $\mathbb{F}_p \equiv \mathbb{Z}/(p\mathbb{Z})$. Let $\mathbf{H}$ be a complex Hilbert space of finite dimension $d \in \mathbb{N}_*$. We consider two unitary operators $A : \mathbf{H} \to \mathbf{H}$ and $B : \mathbf{H} \to \mathbf{H}$ satisfying $A^p = Id$ and $B^p = Id$, as well as

$$\forall (l,m) \in \mathbb{N}^2, \qquad A^l \, B^m = e^{-2\pi i l m/p} \, B^m \, A^l .$$

We suppose that the only subspaces of $\mathbf{H}$ invariant under both A and B are $\{0\}$ and $\mathbf{H}$.

1. Explain why A has at least one eigenvalue $\lambda \in \mathbb{C}$. Show that λ is of modulus 1.

2. Let $0 \neq v \equiv B^0 v \in \mathbf{H}$ be an eigenvector of A of norm 1 that is associated with λ. Show that $B^k v$ is for all $k \in \mathbb{N}$ an eigenvector for A. What is the corresponding eigenvalue ?

3. What can be said about the stability (under the action of A and B) of the (vector) subspace $E \subset \mathbf{H}$ which is generated by the family of vectors $\{B^k v\}_{k\in\mathbb{N}}$?

4. What can be deduced from these observations about the inclusion $E \subset \mathbf{H}$?

5. Explain why the vectors $B^k v$ with $0 \leq k \leq p-1$ form an orthonormal basis of $\mathbf{H}$ built with eigenvectors of A. What is the dimension of $\mathbf{H}$ (in terms of p) ?

6. We consider the eigenspace $E_\lambda := \{f \in \mathbf{H}; Af = \lambda f\}$.

6.1. Prove that E_λ is of dimension 1. *Indication. You can work by contradiction. Let $\tilde{v}$ and v be two non trivial eigenvectors of A with $\tilde{v}$ not colinear to v. Use question 5 to decompose $\tilde{v}$ as a sum of $B^k v$ and then conclude.*

6.2. Explain why $\lambda = 1$ is sure to be an eigenvalue of A.

7. The Hilbert space $L^2(\mathbb{F}_p)$ is provided with the counting measure on $\mathbb{F}_p$ which means that, given $f \in L^2(\mathbb{F}_p)$ and $g \in L^2(\mathbb{F}_p)$, we work with the inner product

$$\langle f, g \rangle := \sum_{n=0}^{p-1} f(n) \, \bar{g}(n).$$

7.1. Prove that the *modulation operator* $U : L^2(\mathbb{F}_p) \longrightarrow L^2(\mathbb{F}_p)$ and the *translation operator* $V : L^2(\mathbb{F}_p) \longrightarrow L^2(\mathbb{F}_p)$ which are given by

$$U(f) : \begin{array}{ccc} \mathbb{F}_p & \longrightarrow & \mathbb{C} \\ n & \longmapsto & e^{-2\pi i n/p} \, f(n), \end{array} \qquad V(f) : \begin{array}{ccc} \mathbb{F}_p & \longrightarrow & \mathbb{C} \\ n & \longmapsto & f(n-1), \end{array}$$

are unitary operators on $L^2(\mathbb{F}_p)$.

$\Longrightarrow$ T.S.V.P.

7.2. Verify that we have $U^p = Id$ and $V^p = Id$, as well as

$$\forall (l, m) \in \mathbb{N}_*^2, \qquad U^l \, V^m = e^{-2\pi i l m/p} \, V^m \, U^l.$$

7.3. We consider the (finite) family of Dirac functions $\{\delta_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$ given by

$$\mathbb{F}_p \ni n \longmapsto \delta_\ell(n) := \begin{cases} 1 & \text{if} \quad n = \ell, \\ 0 & \text{if} \quad n \neq \ell, \end{cases} \qquad \ell \in \mathbb{F}_p.$$

7.3.a. What can be said about the family of vectors δ_ℓ with $\ell \in \mathbb{F}_p$?

7.3.b. What can be said about the action of U on the δ_ℓ ?

7.4. Use the preceding question **7.3** (that is look at the δ_ℓ) to exhibit a self-adjoint operator R on $L^2(\mathbb{F}_p)$ which is such that $e^{-2\pi i R/p} = U$.

7.5. Compute the mean value of R along δ_ℓ, that is the quantity $\langle \delta_\ell, R\delta_\ell \rangle$.

7.6. What could be a possible interpretation of R ?

7.7. We consider the (finite) family of functions $\{g_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$ given by

$$\mathbb{F}_p \ni n \longmapsto g_\ell(n) := \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i \ell k/p} \, \delta_k(n), \qquad \ell \in \mathbb{F}_p.$$

7.7.a. What can be said about the action of V on the g_ℓ ?

7.7.b. Show that the vectors g_ℓ with $\ell \in \mathbb{F}_p$ form an orthonormal basis of $L^2(\mathbb{F}_p)$.

7.8. Using the family of functions $\{g_\ell\}_\ell \in L^2(\mathbb{F}_p)^{\mathbb{F}_p}$, determine a self-adjoint operator S on $L^2(\mathbb{F}_p)$ which is such that $e^{2\pi i S/p} = V$.

7.9. Do the operators R and S commute ? Justify the answer *(for instance by looking at the consequences on U and V).*

8. Let v be an eigenvector of A of norm 1 that is associated with the eigenvalue $\lambda = 1$ (such v does exist in view of question **6.2**).

8.1. Explain why the p relations $W(B^\ell v) = \delta_\ell$ with $\ell \in \mathbb{F}_p$ allow to define some unitary surjective map W from $\mathbf{H}$ onto $L^2(\mathbb{F}_p)$.

8.2. Prove that $WAW^{-1} = U$ and $WBW^{-1} = V$. What is the name of the theorem which is associated with this relation.

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Problem 2. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R})$ be a smooth compactly supported function with $\chi \equiv 1$ in a neighbourhood of the position $\xi = 0$. Consider the symbol

$$K(\xi) \,:=\, i\ |\xi|\ (1-\chi(\xi))\,, \qquad |\xi| := (\xi_1^2 + \cdots + \xi_n^2)^{1/2}\,, \qquad \xi \in \mathbb{R}^n\,.$$

1. Explain why the function $\xi \longmapsto K(\xi)$ is a symbol in the class $S^1(\mathbb{R}^n)$.

2. Let $K(D)$ be the pseudo-differential operator associated to the symbol K, that is

$$\big[K(D)\,u\big](x) \;=\; (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,x\cdot\xi}\ K(\xi)\ \hat{u}(\xi)\ d\xi\,.$$

2.1. We select some function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Prove that $K(D)u$ is a bounded function. What more needs to be said about $K(D)u$?

2.2. Show that $K(D)$ is (formally) skew-symmetric in the sense that

$$\langle u, K(D)v\rangle = \int_{\mathbb{R}^n} u(x)\ \overline{K(D)v}(x)\,dx = -\langle K(D)u, v\rangle, \qquad \forall (u,v) \in \mathcal{S}(\mathbb{R}^n).$$

3. We consider the Cauchy problem

$$(\mathcal{PC}) \qquad \{\,\partial_t u \,-\, K(D)\,u = 0\,, \qquad u_{|t=0} = u_0 \in H^s(\mathbb{R}^n)\,, \qquad s \in \mathbb{R}\,.$$

We denote by $\hat{u}(t,\xi)$ the Fourier transform of $u(t,\cdot)$ with respect to $x \in \mathbb{R}^n$.

3.1. Compute $\hat{u}(t,\xi)$ and deduce from the formula thus obtained that

$$(\mathcal{I}) \qquad u(t,\cdot) \in H^s(\mathbb{R}^n)\,, \qquad \|\, u(t,\cdot)\,\|_{H^s(\mathbb{R}^n)} = \|\, u_0(\cdot)\,\|_{H^s(\mathbb{R}^n)}\,, \qquad \forall\, t \in \mathbb{R}_+^*\,.$$

3.2. Prove that the identity $(\mathcal{I})$ can also be recovered through energy estimates performed at the level of $(\mathcal{PC})$.

4. Let δ_0 be the Dirac mass located at the position $x = 0$. Show that $\delta_0 \in H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ satisfying $s < -(n/2)$.

5. We start with $u_0 = \delta_0$. Recall the definition of the wave front set $WF(\delta_0)$ of the distribution δ_0. Then describe the content of $WF(\delta_0)$.

6. We consider $(\mathcal{PC})$ for the choice $u_0 = \delta_0$. We denote by u the corresponding solution. We fix some $t \in \mathbb{R}_+^*$ as well as some $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Show that we can find a fonction ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ giving rise to

$$\widehat{\varphi\, u}(t,\xi) \;=\; \int_{\mathbb{R}^n} \hat{\varphi}(\xi-\eta)\ e^{i\,t\,|\eta|}\ d\eta \;+\; \psi(\xi)\,.$$

7. In this question, we consider the Cauchy problem

$$(\mathcal{PC}\delta) \qquad \{\,\partial_t \tilde{u} \,-\, i\ |D|\ \tilde{u} = 0\,, \qquad \tilde{u}_{|t=0} = \delta\,.$$

Let $t \in \mathbb{R}_+$. Exploit a course result to describe the wave front set $WF\big(\tilde{u}(t,\cdot)\big)$ of the distribution $\tilde{u}(t,\cdot)$. Justify the answer.