UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC4 about the *uncertainty principle*

Documents are not allowed

Surname : **First name :**

Let $\psi \in \mathcal{S}(\mathbb{R};\mathbb{C})$ be a function in the Schwartz space (the set of functions whose derivatives are rapidly decreasing). We assume that ψ is of norm 1 in $L^2 \equiv L^2(\mathbb{R};\mathbb{C})$. Let X be the position operator (defined by the multiplication by $X = x\times$). Let P be the momentum operator (defined by $P = -i\partial_x$). Given $a \in \mathbb{R}$ and $b \in \mathbb{R}$, introduce

$$\langle \Delta_\psi^a X \rangle := \langle (X-a)\psi, (X-a)\psi \rangle^{1/2} = \Big(\int_{\mathbb{R}} (x-a)^2 \, |\psi(x)|^2 \, dx \Big)^{1/2},$$

$$\langle \Delta_\psi^b P \rangle := \frac{1}{2\pi} \, \langle (P-2\pi b)\psi, (P-2\pi b)\psi \rangle^{1/2}.$$

Denote by $\hat\psi$ the Fourier transform of ψ. By convention and Plancherel theorem, we have

$$\hat\psi(\xi) \equiv \mathcal{F}\psi(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} \, \psi(x) \, dx, \qquad \int_{\mathbb{R}} |\psi(x)|^2 \, dx = \int_{\mathbb{R}} |\hat\psi(\xi)|^2 \, d\xi.$$

1. Determine the value a_m of a for which $\langle \Delta_\psi^a X \rangle$ is minimal. Justify the answer.

$$a_m(\psi) = \langle \psi, X\psi \rangle = \int_{\mathbb{R}} x \, |\psi(x)|^2 \, dx.$$

With a_m as above, this is because

$$\langle \Delta_\psi^a X \rangle^2 = (a-a_m)^2 + \int_{\mathbb{R}} x^2 \, |\psi(x)|^2 \, dx - a_m^2.$$

2. Show that $\mathcal{F}(P\psi/2\pi) = \xi \, \hat\psi(\xi)$. Then, compute $\langle \Delta_\psi^b P \rangle$ in terms of $\hat\psi$ and determine the value $b_m(\psi)$ of b for which $\langle \Delta_\psi^b P \rangle$ is minimal.

$$\langle \Delta_\psi^b P \rangle = \Big(\int_{\mathbb{R}} (\xi-b)^2 \, |\hat\psi(\xi)|^2 \, d\xi \Big)^{1/2}, \qquad b_m(\psi) = \langle \psi, P\psi \rangle = \int_{\mathbb{R}} \xi \, |\hat\psi(\xi)|^2 \, d\xi.$$

Since $\mathcal{F}(P\psi/2\pi) = \xi \, \hat\psi(\xi)$, by Plancherel theorem, we have

$$\langle \Delta_\psi^b P \rangle^2 = \langle \mathcal{F}((P/2\pi - b)\psi), \mathcal{F}((P/2\pi - b)\psi) \rangle = \int_{\mathbb{R}} (\xi-b)^2 \, |\hat\psi(\xi)|^2 \, d\xi.$$

Then, the same arguments as above give rise to the value of $b_m(\psi)$.

3. Find a unitary operator $U : L^2 \to L^2$ which is adjusted in such a way that $a_m(U\psi) = 0$ and $b_m(U\psi) = 0$.

$$U(\psi) = e^{-2\pi i\alpha x}\, U(x+\beta), \qquad \alpha = b_m(\psi), \qquad \beta = a_m(\psi).$$

Indeed, we have

$$a_m(U\psi) = \int_{\mathbb{R}} x\, |U(\psi)(x)|^2\, dx = \int_{\mathbb{R}} (x+\beta)\, |\psi(x+\beta)|^2\, dx - \beta = a_m(\psi) - \beta,$$

as well as

$$\begin{aligned} b_m(U\psi) \ &= \int_{\mathbb{R}} \xi\, |\mathcal{F}(U(\psi))(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}} \xi\, |\hat{\psi}(\xi+\alpha)|^2\, d\xi = \int_{\mathbb{R}} (\xi+\alpha)\, |\hat{\psi}(\xi+\alpha)|^2\, d\xi - \alpha = b_m(\psi) - \alpha. \end{aligned}$$

4. We assume here that $a_m(\psi) = 0$ and $b_m(\psi) = 0$. Compute :

$$-\int_{\mathbb{R}} x\, \psi(x)\, \psi'(x)\, dx = -\frac{1}{2}\int_{\mathbb{R}} x\, \partial_x[\psi(x)^2]\, dx = \frac{1}{2}\int_{\mathbb{R}} \psi(x)^2\, dx = \frac{1}{2}.$$

and prove that the product $\langle \Delta^0_\psi X\rangle \langle \Delta^0_\psi P\rangle$ can be bounded below by a positive constant to be determined.

It suffices to apply the Cauchy-Schwartz inequality to obtain

$$\frac{1}{2} = |\int_{\mathbb{R}} (X\psi)(x)\, (P\psi)(x)\, dx| \leq 2\pi\, \langle \Delta^0_\psi X\rangle\, \langle \Delta^0_\psi P\rangle.$$

In other words

$$\langle \Delta^0_\psi X\rangle\, \langle \Delta^0_\psi P\rangle = \Big(\int_{\mathbb{R}} x^2\, |\psi(x)|^2\, dx\Big)^{1/2}\, \Big(\int_{\mathbb{R}} \xi^2\, |\hat{\psi}(\xi)|^2\, d\xi\Big)^{1/2} \geq \frac{1}{4\pi}.$$

5. Let $a \in S^m$ be a symbol of order m. We recall that the pseudo-differential operator associated with a is given by

$$\begin{aligned} Op(a)\, :\, \mathcal{S}(\mathbb{R}^n) \ &\longrightarrow \ \mathcal{S}(\mathbb{R}^n) \\ u \ &\longmapsto \ Op(a)(u)(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi\,. \end{aligned}$$

5.1. We denote by $[Op(a), \partial_j]$ the commutator of $Op(a)$ with the derivative with respect to the j^{th} variable. Compute its symbol.

$$[Op(a), \partial_j] = Op\,(-\partial_{x_j} a).$$

5.2. Same question with the multiplication by x_j.

$$[Op(a), x_j] = Op\,(-i\partial_{\xi_j} a).$$