UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC4 about the *uncertainty principle.* The 26/11/2021 (30mn)

Documents are not allowed

Surname : **First name :**

Let $\psi \in \mathcal{S}(\mathbb{R};\mathbb{C})$ be a function in the Schwartz space (the set of functions whose derivatives are rapidly decreasing). We assume that ψ is of norm 1 in $L^2 \equiv L^2(\mathbb{R};\mathbb{C})$. Let X be the position operator (defined by the multiplication by x) and P be the momentum operator (defined by $-i\partial_x$). Given $a \in \mathbb{R}$ and $b \in \mathbb{R}$, introduce

$$\langle \Delta_\psi^a X \rangle := \langle (X-a)\psi, (X-a)\psi \rangle^{1/2} = \Big(\int_{\mathbb{R}} (x-a)^2 \, |\psi(x)|^2 \, dx \Big)^{1/2},$$

$$\langle \Delta_\psi^b P \rangle := \frac{1}{2\pi} \langle (P - 2\pi b)\psi, (P-2\pi b)\psi \rangle^{1/2}.$$

Denote by $\hat{\psi}$ the Fourier transform of ψ. By convention and Plancherel theorem, we have

$$\hat{\psi}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} \, \psi(x) \, dx, \qquad \int_{\mathbb{R}} |\psi(x)|^2 \, dx = \int_{\mathbb{R}} |\psi(\xi)|^2 \, d\xi.$$

1. Determine the value $a_m(\psi)$ of a for which $\langle \Delta_\psi^a X \rangle$ is minimal. Justify the answer.

$a_m(\psi) =$

2.1. Show that $\mathcal{F}(P\psi/2\pi) = \xi \, \hat{\psi}(\xi)$.

2.2. Compute $\langle \Delta_\psi^b P \rangle$ in terms of $\hat{\psi}$. $\langle \Delta_\psi^b P \rangle =$

2.3. Determine the value $b_m(\psi)$ of b for which $\langle \Delta_\psi^b P \rangle$ is minimal.

$b_m(\psi) =$

T.S.V.P. $\Longrightarrow$

3. Determine α and β adjusted in such a way that the unitary operator $U : L^2 \to L^2$ given by $U(\psi) = e^{-2\pi i\alpha x}\psi(x+\beta)$ satisfies $a_m(U\psi) = 0$ and $b_m(U\psi) = 0$.

$\alpha =$ $\qquad\qquad\qquad\qquad$ $\beta =$

4. Prove that the product $\langle \Delta^a_\psi X\rangle\langle \Delta^b_\psi P\rangle$ can be bounded below by a positive constant to be determined.

5. Let $a \in S^m$ be a symbol of order m. We recall that the pseudo-differential operator associated with a is given by

$$\begin{aligned} Op(a) \,:\, \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathcal{S}(\mathbb{R}^n) \\ u &\longmapsto Op(a)(u)(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi \,. \end{aligned}$$

5.1. We denote by $[Op(a), \partial_j]$ the commutator of $Op(a)$ with the derivative with respect to the j^{th} variable. Compute its symbol.

$$[Op(a), \partial_j] = Op\Big(\qquad\qquad \Big).$$

5.2. Same question with the multiplication by x_j.

$$[Op(a), x_j] = Op\Big(\qquad\qquad \Big).$$