UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC3 on the *quantum harmonic oscillator*

Documents are not allowed

Surname : **First name :**

A coherent state $\psi_\alpha \in L^2 \equiv L^2(\mathbb{R};\mathbb{C})$ of type $\alpha \in \mathbb{C}$ is a quantum state (that is a ray of L^2 defined modulo the multiplication by a non-zero complex number) which is a non-zero eigenstate of the annihilation operator a associated with the eigenvalue α. In other words

$$a\psi_\alpha = \alpha\psi_\alpha, \quad \psi_\alpha \neq 0, \quad a = \frac{1}{\sqrt{2}}(X + iP) = \frac{1}{\sqrt{2}}(x + \partial_x), \quad X = x\times, \quad P = -i\partial_x.$$

1. Complete the line below and then reply to the question (circle the appropriate answers).

$$a^* := \frac{1}{\sqrt{2}}(x - \partial_x), \qquad [a, a^*] = aa^* - a^*a = 1.$$

Is the spectrum of a (necessarily) contained in the real line ? *NO*

Indeed, the operator a is not self-adjoint, and there is no reason why its spectrum should be contained in the real line. In fact, we will see below that it is not the case.

2. We can identify ψ_α with any of its representatives (modulo the multiplication by a non-zero complex number).

2.1. By solving some ODE on $\ln \psi_\alpha$, make explicit a coherent state of type α.

Assuming that ψ_α is smooth and positive, it must satisfy the differential equation

$$\frac{1}{\sqrt{2}}(x + \partial_x)\psi_\alpha = \alpha\psi_\alpha \quad \Longleftrightarrow \quad \partial_x \ln \psi_\alpha = \sqrt{2}\alpha - x.$$

After integration, we find the special solution

$$\psi_\alpha(x) = C\, e^{\sqrt{2}\alpha x - (x^2/2)}, \qquad C \in \mathbb{C}^*.$$

2.2. Explain why the set of coherent sates of type α is made of only one (non-trivial) ray.

Let ψ_α some L^2-solution of $\partial_x\psi_\alpha = \sqrt{2}\,\alpha\,\psi_\alpha - x\,\psi_\alpha$. The right hand side is in L^2_{loc} and therefore ψ_α must be in H^1_{loc}. Then, by an induction argument, it must be in H^m_{loc} for all $m \in \mathbb{N}$. It is therefore a smooth C^∞-function.

By Cauchy-Lipschitz theorem (the C^1-regularity is in fact sufficient to this end), the Cauchy problem associated with the initial data $\psi_\alpha(0) = C$ has only one maximal solution which must coincide with the one given before. This means that the set of coherent sates of type α is made of a complex line. In other words, there is only one (non-trivial) ray.

3. We recall that the number (or occupation) operator is defined by $N := a^* a$. Express the hamiltonian operator $H := (P^2 + X^2)/2$ in terms of N.

$$\begin{aligned} H &= (-\partial_x^2 + x^2)/2 = (x^2 + x\partial_x - x\partial_x - 1 - \partial_x^2 + 1)/2 \\ &= \frac{1}{\sqrt{2}}(x - \partial_x)\frac{1}{\sqrt{2}}(x + \partial_x) + \frac{1}{2} = N + (1/2). \end{aligned}$$

4. Use the above formula to compute (in terms of α) the mean energy of the coherent state ψ_α, that is

$$\frac{\langle \psi_\alpha, H\psi_\alpha \rangle}{\langle \psi_\alpha, \psi_\alpha \rangle} = \frac{\langle \psi_\alpha, a^* a \psi_\alpha \rangle + \langle \psi_\alpha, \psi_\alpha \rangle /2}{\langle \psi_\alpha, \psi_\alpha \rangle} = \frac{\langle a\psi_\alpha, a\psi_\alpha \rangle}{\langle \psi_\alpha, \psi_\alpha \rangle} + \frac{1}{2} = |\alpha|^2 + \frac{1}{2}.$$

5. Given $\theta \in \mathbb{R}$, consider the operator $U(\theta) := e^{-i\theta N}$. Show that :

$$\frac{d}{d\theta}[U(\theta)^* a U(\theta)] = -iU(\theta)^* a U(\theta).$$

The operator N is self-adjoint so that (by functional calculus) $U(\theta)^* = e^{i\theta N^*} = e^{i\theta N}$. Then, we can compute

$$\begin{aligned} \frac{d}{d\theta}[U(\theta)^* a U(\theta)] &= \Big(\frac{d}{d\theta}e^{i\theta N}\Big) a U(\theta) + U(\theta)^* a \frac{d}{d\theta} e^{-i\theta N}\Big) \\ &= ie^{i\theta N} N a U(\theta) - iU(\theta)^* a N e^{-i\theta N} = iU(\theta)^*[N, a]U(\theta). \end{aligned}$$

On the other hand, we have

$$[N, a] = a^* a a - a a^* a = -[a, a^*]a = -a.$$

Thus, there remains the expected result.

6. Deduce from the above that : $U(\theta)^* a U(\theta) = a e^{-i\theta}$.

The two operators $U(\theta)^* a U(\theta)$ and $a e^{-i\theta}$ satisfy the same ODE (with values in the Banach space $\mathcal{L}(L^2)$) with the same initial condition, namely

$$\frac{d}{d\theta} Z = -iZ, \qquad Z(0) = a.$$

Thus, they must coincide.

7. Prove that $U(\theta)\psi_\alpha = C\,\psi_{\alpha e^{-i\theta}}$ for some $C \in \mathbb{C}^*$.

From 6. applied to ψ_α, we get

$$aU(\theta)\psi_\alpha = \left(U(\theta)^*\right)^{-1}(ae^{-i\theta}\psi_\alpha) = e^{-i\theta}U(\theta)a\psi_\alpha = \alpha e^{-i\theta}U(\theta)\psi_\alpha.$$

This means that $U(\theta)\psi_\alpha$ is an eigenstate of the annihilation operator a which is associated with the eigenvalue $\alpha e^{-i\theta}$. From 2., we can assert that

$$\exists C \in \mathbb{C}; \qquad U(\theta)\psi_\alpha = C\psi_{\alpha e^{-i\theta}}.$$

But since the unitary operator $U(\theta)$ does preserve the norms, we must have

$$0 \neq \parallel \psi_\alpha \parallel = \parallel U(\theta)\psi_\alpha \parallel = |C| \, \parallel \psi_{\alpha e^{-i\theta}} \parallel .$$

This implies that $C \neq 0$, and therefore $U(\theta)\psi_\alpha$ is indeed (modulo the multiplication by a non-zero complex number) the same as $\psi_{\alpha e^{-i\theta}}$.