UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC3 about the uncertainty principle, the 25/11/2022 (30mn)

Documents are not allowed

Surname : **First name :**

We consider on the Hilbert space $\mathcal{H} := L^2([-1,1])$ the *position operator* A and the *momentum operator* B defined by

$$A\psi(x) = x\,\psi(x)\,, \qquad B\psi(x) = -i\hbar\,\psi'(x) = -i\hbar\,\frac{d\psi}{dx}(x)\,.$$

1.1. Prove that A is a bounded operator and compute its operator norm $\|A\|$.

1.2. We look at B as an unbounded operator with domain

$$\mathrm{Dom}(B) := \{\,\psi \in C^1([-1,1])\,;\,\psi(-1) = \psi(1)\,\}\,.$$

Check that B is symmetric.

1.3. For $n \in \mathbb{Z}$, define $\psi_n(x) := e^{\pi i n x}/\sqrt{2}$. Show that ψ_n is in $\mathrm{Dom}(B)$, and that $(\psi_n)_{n\in\mathbb{Z}}$ constitutes an orthonormal basis of eigenvectors for B with real eigenvalues.

$\Longrightarrow$ T.S.V.P.

1.4. Let $\psi \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$. Complete the following formula

$$AB\psi - BA\psi =$$

1.5. Given a self-adjoint operator A on $\mathcal{H}$ and a unit vector $\psi \in \mathcal{H}$, recall that the *uncertainty* of A in the state ψ is defined by

$$\Delta_\psi A := \sqrt{\langle A^2 \rangle_\psi - \langle A \rangle_\psi^2}\,, \qquad \langle A \rangle_\psi := \langle \psi, A\psi \rangle\,.$$

Explain why $\Delta_{\psi_n} A$ and $\Delta_{\psi_n} B$ are both unambiguously defined, and compute $\Delta_{\psi_n} B$.

1.6. Recall the content of the *uncertainty principle* in the case of (A, B, ψ).

1.7. Explain why the uncertainty principle is not verified when dealing with (A, B, ψ_n).