Microlocal Analysis

CC3 about the *quantum harmonic oscillator.* The 19/11/2021 (30mn)

Documents are not allowed

Surname : **First name :**

A coherent state $\psi_\alpha \in L^2 \equiv L^2(\mathbb{R};\mathbb{C})$ of type $\alpha \in \mathbb{C}$ is a quantum state (that is a ray of L^2 defined modulo the multiplication by a non-zero complex number) which is a non-zero eigenstate of the annihilation operator a associated with the eigenvalue α. In other words

$$a\psi_\alpha = \alpha\psi_\alpha, \quad \psi_\alpha \neq 0, \quad a = \frac{1}{\sqrt{2}}(X + iP) = \frac{1}{\sqrt{2}}(x + \partial_x), \quad X = x\times, \quad P = -i\partial_x.$$

1. Complete the line below and then reply to the question (circle the appropriate answer).

$a^* :=$ $\qquad\qquad\qquad$ $[a, a^*] = aa^* - a^*a =$

Is the spectrum of a (necessarily) contained in the real line ? YES NO

2. We can identify ψ_α with any of its representatives (modulo the multiplication by a non-zero complex number).

2.1. By solving some ODE on $\ln \psi_\alpha$, make explicit a coherent state of type α.

$\psi_\alpha(x) =$

2.2. Explain why the set of coherent sates of type α is made of only one (non-trivial) ray.

3. We recall that the number (or occupation) operator is defined by $N := a^*a$. Express the hamiltonian operator $H := (P^2 + X^2)/2$ in terms of N.

$H =$

T.S.V.P. $\Longrightarrow$

4. Use the preceding formula to compute (in terms of α) the mean energy of the coherent state ψ_α, that is

$$\frac{\langle \psi_\alpha, H\psi_\alpha \rangle}{\langle \psi_\alpha, \psi_\alpha \rangle} =$$

5. Given $\theta \in \mathbb{R}$, consider the operator $U(\theta) := e^{-i\theta N}$. Show that :

$$\frac{d}{d\theta}[U(\theta)^* a U(\theta)] = -iU(\theta)^* a U(\theta).$$

6. Deduce from the above that : $U(\theta)^* a U(\theta) = a e^{-i\theta}$.

7. Prove that $U(\theta)\psi_\alpha = C\, \psi_{\alpha e^{-i\theta}}$ for some $C \in \mathbb{C}^*$.