UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC2

We work on $\mathbb{R}^n$. We consider the Laplace operator $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial^2_{x_i}}$.

1.1. What is the symbol $a(x,\xi)$ that is associated to the action of the operator $1-\Delta$ through the relation $1-\Delta = op(a) = a(x,D)$.

By definition, the symbol $a(x,\xi)$ associated to the action of $a(x,D)$ is such that

$$a(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi}\, a(x,\xi)\, \hat{u}(\xi)\ d\xi\,,$$

where $\hat{u} \equiv \mathcal{F}u$ is the Fourier transform of u. From

$$(1-\Delta)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi}\, (1+|\xi|^2)\, \hat{u}(\xi)\ d\xi\,,$$

we can see that $a(x,\xi) = 1+|\xi|^2$.

1.2. Determine the symbol $b(x,\xi)$ of the pseudo-differential operator allowing to satisfy the relation $b(x,D)\,a(x,D) = Id$. The operator $b(x,D)$ is denoted by $(1-\Delta)^{-1}$. Indicate on the right its symbol class.

It suffices to remark that

$$\begin{aligned} u(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi}\, (1+|\xi|^2)^{-1}\, (1+|\xi|^2)\, \hat{u}(\xi)\ d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi}\, (1+|\xi|^2)^{-1}\, \mathcal{F}((1-\Delta)u)(\xi)\ d\xi\,, \end{aligned}$$

to deduce that $b(x,\xi) = (1+|\xi|^2)^{-1}$ which is in the symbol class $S^{-2}_{1,0}$.

1.3. Determine the symbol $c(x,\xi)$ of the pseudo-differential operator $1-(1-\Delta)^{-1}$. Indicate on the right its symbol class.

$c(x,\xi) = 1-(1+|\xi|^2)^{-1}\,, \qquad c(x,D) \in S^0_{1,0}\,.$

In fact, a, b and c do not depend on x. They are Fourier multipliers.

1.4. Let Ω be a relatively compact open subset of $\mathbb{R}^n$. Fix $s\in\mathbb{R}$ and $f \in H^s(\Omega)$. Given some $s_0 < s$, we consider a distribution $u \in H^{s_0}(\Omega)$ which is such that $\Delta u = f$.

1.4.1. We recall Peetre's inequality :

$$\langle\xi'\rangle^{\tilde{s}}\ \langle\xi\rangle^{-\tilde{s}} \le 2^{|\tilde{s}|}\ \langle\xi-\xi'\rangle^{|s|}\,, \qquad \forall\,(\tilde{s},\xi,\xi') \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n.$$

We select a cutoff function $\chi \in C_c^\infty(\Omega)$. Show that $\chi\,\Delta u \in H^s(\mathbb{R}^n)$.

The function χ can be extended as a function in the Schwartz space so that

$$\langle \xi \rangle^{\tilde{s}}\ |\chi(\xi)| \in L^2(\mathbb{R}^n)\,, \qquad \forall\, \tilde{s} \in \mathbb{R}.$$

On the other hand, we know that $\langle \xi' \rangle^{-s}\, |\hat{f}(\xi')|$ is in L^2. Now, we have

$$\mathcal{F}(\chi\,\Delta u)(\xi) = \int_{\mathbb{R}^n} \chi(\xi - \xi')\ \hat{f}(\xi')\ d\xi'\,.$$

It follows that

$$\frac{|\mathcal{F}(\chi\,\Delta u)(\xi)|}{\langle \xi \rangle^s} \leq \int_{\mathbb{R}^n} |\chi(\xi - \xi')|\ \frac{\langle \xi' \rangle^s}{\langle \xi \rangle^s}\ \frac{|\hat{f}(\xi')|}{\langle \xi' \rangle^s}\ d\xi' \leq 2^{|s|} \int_{\mathbb{R}^n} \langle \xi - \xi' \rangle^{|s|}\ |\chi(\xi - \xi')|\ \frac{|\hat{f}(\xi')|}{\langle \xi' \rangle^s}\ d\xi'\,.$$

The right hand side is the convolution of two L^2-functions. We can conclude with Young's convolution inequality.

1.4.2. Show that $g := (1-\Delta)^{-1}\,\chi\,\Delta u \in H^{s+2}(\mathbb{R}^n)$.

The operator $(1-\Delta)^{-1}$ is continuous from $H^s(\mathbb{R}^n)$ onto $H^{s+2}(\mathbb{R}^n)$, since it is a Fourier multiplier of order -2. This implies that g is in $H^{s+2}(\mathbb{R}^n)$.

1.5. Show that $g = (1-\Delta)^{-1}\,[\chi, \Delta]u - \big(1 - (1-\Delta)^{-1}\big)\,\chi u$.

We can start from the right hand side

$$\begin{aligned}(1-\Delta)^{-1}\,[\chi,\Delta]u - \big(1-(1-\Delta)^{-1}\big)\,\chi u &= g - (1-\Delta)^{-1}\,\Delta(\chi u) - \big(1-(1-\Delta)^{-1}\big)\,\chi u\\ &= g - (1-\Delta)^{-1}\,(\Delta - 1)(\chi u) - (1-\Delta)^{-1}\,\chi u - \big(1-(1-\Delta)^{-1}\big)\,\chi u\\ &= g + \chi u - \chi u = g\,.\end{aligned}$$

1.6. Explain why $R := [\chi, \Delta]$ is a first-order differential operator.

This is just because $R = -2\,\nabla\chi \cdot \nabla - \Delta\chi$ which is a differential operator of order 1.

1.7. Let $\psi \in C_c^\infty(\Omega)$ be a non negative function which is equal to 1 on the ball $\{\xi; |\xi| \leq 1\}$ and equal to 0 out of the ball $\{\xi; |\xi| \leq 2\}$. Prove that the pseudo-differential operator $e(x,D) := \psi(D) + c(x,D)$ (where c is as in question 1.2) is an isomorphism of $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

The symbol of $e(x,D)$ is $e(x,\xi) = \psi(\xi) + 1 - (1+|\xi|^2)^{-1}$. We find that

$$e(x,0) = \psi(0) + 1 - 1 = 1\,, \qquad \lim_{|\xi| \longrightarrow +\infty} e(x,\xi) = 0\,,$$

as well as

$$e(x,\xi) \geq 1 - (1+|\xi|^2)^{-1} > 0\,, \qquad \forall \xi \neq 0\,.$$

It follows that

$$\exists\,(c,C)\in(\mathbb{R}_+^*)^2\,;\quad 0<c\le e(x,\xi)\le C\,,\quad \forall\xi\in\mathbb{R}^n\,. \tag{1}$$

The symbol $e(x,\xi)$ is in $S^0_{1,0}$. This implies that the Fourier multiplier $e(x,D)$ acts continuously from $H^s(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$.

From condition (1) - of ellipticity, we can define $e(x,\xi)^{-1}\in S^0_{1,0}$. The pseudo-differential operator $e(x,D)$ is invertible with inverse $e(x,D)^{-1}$ which acts continuously from $H^s(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$.

1.8. Show that $\chi u = e(x,D)^{-1}\left[(1-\Delta)^{-1}\right)Ru-g+\psi(D)\,(\chi u)]$, and deduce from this relation that χu is in $H^{s_0+1}(\mathbb{R}^n)$.

From quastion 1.4, we have

$$\left(1-(1-\Delta)^{-1}\right)\chi u = (1-\Delta)^{-1}\,[\chi,\Delta]u-g\,,$$

so that

$$e(x,D)\,(\chi u)=\big(c(x,D)+\psi(D)\big)\,(\chi u)=(1-\Delta)^{-1}\,Ru-g+\psi(D)\,(\chi u)\,.$$

It suffices to compose on the left by $e(x,D)^{-1}$ to recover the expected formula. Then

$$\left.\begin{array}{l} u\in H^{s_0}(\mathbb{R}^n)\\ R\in Op(S^1_{1,0})\end{array}\right\}\quad\Longrightarrow\quad Ru\in H^{s_0-1}(\mathbb{R}^n)\,.$$

This gives rise to

$$\left.\begin{array}{l} Ru\in H^{s_0-1}(\mathbb{R}^n)\\ (1-\Delta)^{-1}\in Op(S^{-2}_{1,0})\end{array}\right\}\quad\Longrightarrow\quad (1-\Delta)^{-1}\,Ru\in H^{s_0+1}(\mathbb{R}^n)\,.$$

On the other hand

$$\left.\begin{array}{l} \chi u\in H^{s_0}(\mathbb{R}^n)\\ \psi(D)\in Op(S^{-\infty}_{1,0})\end{array}\right\}\quad\Longrightarrow\quad \psi(D)u\in H^{+\infty}(\mathbb{R}^n)\,.$$

From question 1.3, we know already that $g\in H^{s+2}(\mathbb{R}^n)$. After summation, since $s_0<s$, there remains

$$\chi u\in H^{\min(s_0+1,+\infty,s+2)}(\mathbb{R}^n)=H^{s_0+1}(\mathbb{R}^n)\,.$$

Concluding remarks. *Due to the pde $\Delta u=f$ and the regularity assumption on the source term f, the regularity of u goes (locally) one step up, from $H^{s_0}_{loc}(\Omega)$ to $H^{s_0+1}_{loc}(\Omega)$. This information comes also from the ellipticity of the operator Δ. Refined (microlocal) versions of this principle do exist : they distinguish between the various directions ξ. If moreover we know that $f\in C^\infty(\Omega)$, then, by induction we recover that $u\in C^\infty(\Omega)$. This is called hypoellipticity, see this reference.*