UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC2 on the *inversion of elliptic operators*

Documents are not allowed

Surname : **First name :**

Let $n \in \mathbb{N}$ as well as m and m' in $\mathbb{R} \cup \{-\infty\}$. Given two symbols $a \in S^m \equiv S^m_{1,0}(\mathbb{R}^n)$ and $b \in S^{m'}$, we admit that the composition of $Op(a)$ with $Op(b)$ is a pseudo-differential operator whose symbol is in $S^{m+m'}$. More precisely

$$Op(a)Op(b) = Op(a\#b), \qquad a\#b = ab + r, \qquad ab \in S^{m+m'}, \qquad r \in S^{m+m'-1}.$$

1. Take $n = 1$, $a = i\xi$ and $b = x$. What is r ?

$$r(x, \xi) = 1.$$

We have $Op(a) = \partial_x$ and $Op(b) = x\times$ so that $Op(a)Op(b) = \partial_x(x \times \cdot) = x\partial_x + Id$ whose corresponding symbol is $ix\xi + 1$. Note that $a \in S^1$, $b \in S^0$ while $ab = ix\xi \in S^1$. As expected, we find that $r \in S^{1+0-1} \equiv S^0$. This is in accordance with the formula provided by the symbolic calculus since

$$\begin{aligned} a\#b(x,\xi) = \sum_{\alpha \in \mathbb{N}} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a(x,\xi)\, \partial_x^\alpha b(x,\xi) &= (ab)(x,\xi) + \frac{1}{i}\, \partial_\xi a(x,\xi)\, \partial_x b(x,\xi) \\ &= ix\xi + \frac{1}{i}\, i \times 1 = ix\xi + 1. \end{aligned}$$

2. Take $n = 1$, $a = i\xi$ and $b = x$. What is the symbol of the adjoint of $Op(a)Op(b)$?

$$(a\#b)^*(x, \xi) = -ix\xi.$$

From $Op(a\#b) = \partial_x(x \times \cdot)$, an integration by parts furnishes $Op(a\#b)^* = -x\partial_x$ whose corresponding symbol is as indicated above. Note that we have

$$\begin{aligned} (a\#b)^*(x,\xi) = \sum_{\alpha \in \mathbb{N}} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a\#b}(x,\xi) &= \overline{a\#b}(x,\xi) + \frac{1}{i}\, \partial^2_{x\xi} \overline{a\#b}(x,\xi) \\ &= (-ix\xi + 1) + \frac{1}{i}\,(-i) = -ix\xi. \end{aligned}$$

3. Let $a \in S^m$. We assume here that we can find some $b \in S^{-m}$ which is such that $a\#b - 1$ is in the class $S^{-\infty}$.

3.1. Prove that : $\exists\, R \in \mathbb{R}_+^*; \quad |\xi| \geq R \implies |(ab)(x,\xi)| \geq 1/2$.
By construction, we have

$$ab - 1 = (a\#b - 1) - r \in S^{-\infty} + S^{-1} \subset S^{-1}$$

and therefore

$$|(ab)(x,\xi) - 1| \leq C\,(1 + \parallel \xi \parallel)^{-1}\,.$$

In particular, for $R \leq \parallel \xi \parallel$ with R large enough, we have

$$|(ab)(x,\xi) - 1| \leq \frac{1}{2} \quad \Longrightarrow \quad \frac{1}{2} \leq |a(x,\xi)|\,|b(x,\xi)|.$$

3.2. Prove that :

$$\exists (R,c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*; \qquad |\xi| \geq R \quad \Longrightarrow \quad c\,(1 + \parallel \xi \parallel)^m \leq |a(x,\xi)|. \tag{1}$$

For $R \leq \parallel \xi \parallel$, the value of $b(x,\xi)$ is not zero and, since $b \in S^{-m}$, we can find some $C \in \mathbb{R}_+^*$ such that

$$|b(x,\xi)| \leq C\,(1 + \parallel \xi \parallel)^{-m} \quad \Longrightarrow \quad \frac{1}{C}\,(1 + \parallel \xi \parallel)^m \leq \frac{1}{|b(x,\xi)|},$$

and therefore we have (1) with $c = 1/(2C)$.

4. Let $a \in S^m$. We assume here that we have the property (1) for some $R \in \mathbb{R}_+^*$.
4.1. Find $b_0 \in S^{-m}$ which is such that $Op(a)\,Op(b_0) = Id + \mathcal{R}_0$ with $\mathcal{R}_0 \in Op(S^{-1})$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ in the ball $B(0,1]$. Consider

$$b_0(x,\xi) = (1 - \chi(\xi/R))/a(x,\xi).$$

With (1), it is easy to infer that $b_0 \in S^{-m}$. On the other hand

$$Op(a)\,Op(b_0) = Op(a\#b_0) = Op(ab_0 + r_0) = Id - Op\,\chi(\xi/R) + Op(r_0), \qquad r_0 \in S^{-1}.$$

Just remark that

$$\mathcal{R}_0 := -Op\,\chi(\xi/R) + Op(r_0) \in Op(S^{-\infty}) + Op(S^{-1}) \subset Op(S^{-1}).$$

4.2. Find $b_1 \in S^{-m-1}$ which is such that $Op(a)\,Op(b_0 + b_1) = Id + \mathcal{R}_1$ with $\mathcal{R}_1 \in Op(S^{-2})$.

With b_0 as above, we have to seek b_1 in such a way that

$$Op(a)\,Op(b_0 + b_1) = Op(a)\,Op(b_0) + Op(a)\,Op(b_1) = Id + \mathcal{R}_1, \qquad \mathcal{R}_1 \in Op(S^{-2}).$$

Interpreted in terms of symbols, this means that

$$ab_0 + r_0 + ab_1 + \tilde{r}_1 = 1 + r_1, \qquad r_0 \in S^{-1}, \quad \tilde{r}_1 \in S^{m-m-1-1} = S^{-2}, \quad r_1 \in S^{-2}.$$

It suffices to adjust b_1 in such a way that $ab_0 + r_0 + ab_1 = 1$, that is

$$b_1 := (1 - \chi)(\xi/R)\,\big[(1/a) - b_0 - (r_0/a)\big] = (1 - \chi)(\xi/R)\,\big[(\chi(\xi/R) - r_0)/a\big] \in S^{-m-1}.$$

Then, the computation of $a\#b_1$ yields some $\tilde{r}_1 \equiv r_1 \in S^{-2}$.