UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC1 on the continuity of pseudo-differential operators, the 29/10/2021 (30mn)

Documents are not allowed

Surname : **First name :**

Let $n \in \mathbb{N}^*$ and $m \in \mathbb{R}$. We consider a symbol $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$ which is in the class $S^m \equiv S^m_{1,0}$ of symbols of order m.

1. Recall the definition of the symbol class S^m.

$S^m =$

2. We assume that we can find $\tilde{m} < m$ and $R \in \mathbb{R}^*_+$ which are such that

$$\forall\, (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \quad \exists\, C_{\alpha,\beta} \in \mathbb{R}_+; \quad R \leq \parallel \xi \parallel \Longrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}\, (1+ \parallel \xi \parallel)^{\tilde{m}-|\beta|}.$$

Prove that a is in $S^{\tilde{m}}$.

3. We assume in this question that $m < -n$.

3.1 Show that we can find some $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \qquad op(a)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi = \int_{\mathbb{R}^n} K(x,y)\, u(y)\, dy.$$

$\Longrightarrow$ T.S.V.P.

3.2. Let $\alpha \in \mathbb{N}^n$. Prove that $(x-y)^\alpha\, K(x,y) \in L^\infty(\mathbb{R}^n)$.

3.3. Show that, for all $p \in \mathbb{N}^*$, the map $op(a) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is a bounded operator.

4. Let A, B and $C \not\equiv 0$ three self-adjoint operators on a Hilbert space $\mathcal{H}$. We assume that $[A, B] = Id$ and $[A, C] = 0$. Can we assert that $[B, C] \not\equiv 0$? Justify the answer.