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On the robustness of backward stochastic differential equations ☆

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Abstract

In this paper, we study the robustness of backward stochastic differential equations (BSDEs for short) w.r.t. the Brownian motion; more precisely, we will show that if W^n is a martingale approximation of a Brownian motion W then the solution to the BSDE driven by the martingale W^n converges to the solution of the classical BSDE, namely the BSDE driven by W. The particular case of the scaled random walks has been studied in Briand et al. (Electron. Comm. Probab. 6 (2001) 1). Here, we deal with a more general situation and we will not assume that the W^n has the predictable representation property: this yields an orthogonal martingale in the BSDE driven by W^n. As a byproduct of our result, we obtain the convergence of the "Euler scheme" for BSDEs corresponding to the case where W^n is a time discretization of W. © 2002 Published by Elsevier Science B.V.

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1. Introduction

We consider in this paper the following backward stochastic differential equation (BSDE for short in the remaining of the paper) driven by a Brownian motion $W = \{W_t\}_{0 \leqslant t \leqslant T}$

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}W_r, \quad 0 \leqslant t \leqslant T. \tag{1}$$

The solution of such an equation is a process $\{(Y_t, Z_t)\}_{0 \leqslant t \leqslant T}$ which has to be progressively measurable w.r.t. $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$ (the filtration generated by W) even though the condition $Y_T = \xi$ is imposed at the terminal time T. The terminal condition ξ is an $\mathscr{F}_T$-measurable random variable and the random function f is such that, for all

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(y,z), the process $\{f(t,y,z)\}_{0\leqslant t\leqslant T}$ is progressively measurable. It is by now well known that the BSDE (1) has a unique square integrable solution providing that ξ, $\{f(t,0,0)\}_{0\leqslant t\leqslant T}$ are also square integrable and that f is Lipschitz w.r.t. both y and z (see Section 3 for precise assumptions); we refer to the original work of Pardoux and Peng (1990) or to the survey paper by El Karoui et al. (1997).

One of the interesting features of BSDEs are the "a priori" estimates which give the stability of solutions to BSDEs w.r.t. the data (ξ,f); for instance, in El Karoui et al. (1997), the authors used such estimates to get the classical existence and uniqueness result. In this work, we are also interested in some kind of robustness properties of solutions to BSDEs but w.r.t. W instead of (ξ,f). To be more precise, we will consider the solution $\{(Y_t^n,Z_t^n)\}_{0\leqslant t\leqslant T}$ to the BSDE (1) but with W replaced by W^n where the martingale $\{W_t^n\}_{0\leqslant t\leqslant T}$ converges to W uniformly on $[0,T]$ in probability (ucp for short) as well as in L^2 and we will prove that the corresponding solution (Y^n,Z^n) converges to (Y,Z). We should point out that we will not assume that the martingale W^n has the predictable representation property (for this notion, we refer to Jacod, 1979, Chapter XI; Protter, 1990, Chapter 4; or to Revuz and Yor, 1991, Chapter V in the continuous case) so that we will have to find a triple $\{(Y_t^n,Z_t^n,N_t^n)\}_{0\leqslant t\leqslant T}$ such that

$$Y_t^n=\xi^n+\int_t^T f^n(r,Y_{r-}^n,Z_r^n)\,\mathrm{d}\langle W^n\rangle_r-\int_t^T Z_r^n\,\mathrm{d}W_r^n-(N_T^n-N_t^n),\quad 0\leqslant t\leqslant T; \tag{2}$$

in this equation, $\{N_t^n\}_{0\leqslant t\leqslant T}$ is a martingale which is orthogonal to W^n, ξ^n is an $\mathscr{F}_T^n$-measurable random variable where $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ is the filtration generated by W^n, $\{f^n(t,y,z)\}_{0\leqslant t\leqslant T}$ is progressively measurable w.r.t. $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ for each (y,z) and $\{\langle W^n\rangle_t\}_{0\leqslant t\leqslant T}$ denotes the predictable quadratic variation of W^n. BSDEs driven by martingales are also considered in El Karoui and Huang (1997) but in the case where the quadratic variation is continuous; other generalizations of BSDEs can be found in Barles et al. (1997), Pardoux et al. (1997) and Pardoux and Zhang (1998).

The main issue of this paper is to show in Section 4 that if (ξ^n,f^n,W^n) converges to (ξ,f,W), we have the convergence of the solution to the BSDE (2) (Y^n,Z^n) toward the solution to the BSDE (1) (Y,Z). This generalizes the result in Briand et al. (2001) where the case of W^n being the scaled simple random walks were considered.

To motivate this study, let us consider a time discretization of the BSDE (1) in the spirit of the Euler scheme for SDEs. The objective is to solve backwards in time, noting h for T/n, the equation

$$Y_{kh}^n=Y_{(k+1)h}^n+hf((k+1)h,Y_{kh}^n,Z_{kh}^n)-Z_{kh}^n(W_{(k+1)h}-W_{kh}),$$

where $Y_T^n=\xi^n$ is given and under the constraint that the unknown (Y_{kh}^n,Z_{kh}^n) depends only on $W_h,\dots,W_{kh}-W_{(k-1)h}$. This constraint is due to the fact that the solution to the BSDE (1) has to be adapted to the filtration generated by the Brownian motion W. If we consider the case where $f\equiv 0$, we see that the previous equation cannot be solved since the discretization of the Brownian motion has not the predictable representation property. To overcome this difficulty we add an orthogonal martingale term to the equation which becomes

$$Y_{kh}^n=Y_{(k+1)h}^n+hf((k+1)h,Y_{kh}^n,Z_{kh}^n)-Z_{kh}^n(W_{(k+1)h}-W_{kh})+(N_{(k+1)h}^n-N_{kh}^n).$$

This backward equation corresponds to the case where W^n is the time discretization of the Brownian motion W and will be studied in Section 5.1, Proposition 13.

Let us point out that the standard computation of $|Y_t^n - Y_t|^2$ with Itô's formula, which is usually the starting point of proofs of stability results, is not so powerful here: this is mainly due to the fact that W^n and W are not necessarily martingales with respect to a common filtration. As in Briand et al. (2001) the notion of "weak convergence of filtrations" will be very useful to handle this problem and it appears that the case where f and f^n are identically 0 is a key point.

The rest of the paper is organized as follows: in Section 2, after some notations we deal with the case where f and f^n are 0 which is the simplest case and the most important. Section 3 contains all the assumptions and is mainly devoted to the study of BSDEs driven by W^n. In Section 4 we prove our main result and finally in the last section we give some examples and illustrations.

Finally, let us precise that we will use during the paper the classical notations of stochastic calculus that appear for instance in Dellacherie and Meyer (1975) and Revuz and Yor (1991).

2. The martingale context

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space carrying a standard real valued Brownian motion $W = \{W_t\}_{0 \leqslant t \leqslant T}$; we will denote $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$ the right continuous and complete filtration generated by W. On this space, we consider a sequence of càdlàg (right continuous with left limits) square integrable $\{\mathscr{F}_t^n\}_{0 \leqslant t \leqslant T}$-martingales $W^n = \{W_t^n\}_{0 \leqslant t \leqslant T}$, where, for each n, $\{\mathscr{F}_t^n\}_{0 \leqslant t \leqslant T}$ is right continuous and complete.

We will say that a càdlàg process $X = \{X_t\}_{0 \leqslant t \leqslant T}$, with values in $\mathbb{R}^d$, belongs to the space $\mathscr{S}^p(\mathbb{R}^d)$ or simply $\mathscr{S}^p$ where $1 \leqslant p < \infty$, if

$$\|X\|_{\mathscr{S}^p}^p = \mathbb{E}\left[\sup_{t \in [0,T]} |X_t|^p\right] < \infty.$$

In this section, we will treat the simplest linear case namely $f = 0$ and $f^n = 0$. This result together with the Picard procedure and Proposition 11 will allow us to deal with the general case. We start with a result concerning the convergence of the predictable quadratic covariation of martingales. Firstly, we give a theorem concerning the "continuity" of the predictable compensator also called dual predictable projection. The proof of this result is given in the appendix at the end of the paper.

Theorem 1. *Let $\{X_t^n\}_{0 \leqslant t \leqslant T}$ be a sequence of càdlàg $\{\mathscr{F}_t^n\}_{0 \leqslant t \leqslant T}$-integrable processes with finite variation and $X_0^n = 0$ which converges in $\mathscr{S}^1(\mathbb{R})$ to the continuous $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$- adapted process $\{X_t\}_{0 \leqslant t \leqslant T}$. In addition, we assume that $\{\bar{X}_t^n\}_{0 \leqslant t \leqslant T}$, $\bar{X}_t^n = \mathrm{Var}(X^n)_t = \int_0^t |\mathrm{d}X_s^n|$, is* **C***-tight, and the variables $\bar{X}_T^n$ are uniformly integrable.*

Then, the predictable compensator $\{P_t^n\}_{0 \leqslant t \leqslant T}$ of $\{X_t^n\}_{0 \leqslant t \leqslant T}$ converges to $\{X_t\}_{0 \leqslant t \leqslant T}$ in $\mathscr{S}^1(\mathbb{R})$.

As a byproduct, we can obtain a result about the convergence of the predictable covariation of martingales.

Proposition 2. *Let* $\{M_t^n\}_{0\leqslant t\leqslant T}$ *and* $\{N_t^n\}_{0\leqslant t\leqslant T}$ *be two sequences of càdlàg* $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$*-square integrable martingales which are bounded in* $\mathscr{S}^2(\mathbb{R})$. *We assume that* $\{M_t^n\}_{0\leqslant t\leqslant T}$ *converges in* $\mathscr{S}^2(\mathbb{R})$ *to the continuous* $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$*-martingale* $\{M_t\}_{0\leqslant t\leqslant T}$ *and that* $\{N_t^n\}_{0\leqslant t\leqslant T}$ *converges to the martingale* $\{N_t\}_{0\leqslant t\leqslant T}$ *in probability for the Skorokhod topology.*

Then $\{\langle M^n,N^n\rangle_t\}_{0\leqslant t\leqslant T}$ *converges to* $\{\langle M,N\rangle_t\}_{0\leqslant t\leqslant T}$ *in* $\mathscr{S}^1(\mathbb{R})$.

Proof. Under the assumptions, the sequence (M^n,N^n) satisfies the uniform tightness (UT) condition; cf Proposition 1.5 (b) of Mémin and Słomiński (1991) (for an English reference, see Section 7 of Kurtz and Protter, 1996). This implies, by Corollary 1.9 of Mémin and Słomiński (1991), that the sequence of processes with finite variation paths $([M^n,N^n])_{\mathbb{N}}$ converge in ucp to the cross variation of M and N, $[M,N]$ which is a continuous process with finite variation paths. We want to apply Theorem 1 to $[M^n,N^n]$ since $\langle M^n,N^n\rangle$ is the predictable compensator of $[M^n,N^n]$.

Let us pick $0\leqslant s\leqslant t\leqslant T$; we have:

$$\begin{aligned}|[M^n,N^n]_t-[M^n,N^n]_s| &\leqslant \mathrm{Var}([M^n,N^n])_{[s,t]}\\ &\leqslant ([M^n]_t-[M^n]_s)^{1/2}([N^n]_t-[N^n]_s)^{1/2}.\end{aligned}$$

Since M^n converges to M in $\mathscr{S}^2(\mathbb{R})$, $[M^n]$ converges to $[M]$ in ucp (the UT condition is satisfied) and Scheffé's lemma implies that $[M^n]_T$ converges in L^1 to $[M]_T$; thus the convergence of $[M^n]$ to $[M]$ holds in $\mathscr{S}^1(\mathbb{R})$. The previous estimate, for $t=T$ and $s=0$, yields

$$\sup_{t\in[0,T]}|[M^n,N^n]_t|\leqslant \mathrm{Var}([M^n,N^n])_T\leqslant ([M^n]_T)^{1/2}([N^n]_T)^{1/2}$$

and thus gives, since $[N^n]_T$ is bounded in L^1, the uniform integrability of $\mathrm{Var}([M^n,N^n])_T$ and also the convergence of $[M^n,N^n]$ to $[M,N]$ in $\mathscr{S}^1(\mathbb{R})$. It remains to check the $\mathbf{C}$-tightness of $\mathrm{Var}([M^n,N^n])_t$. But, we have,

$$\mathbb{E}\left[\sup_{0\leqslant t-s\leqslant\theta}\mathrm{Var}([M^n,N^n])_{[s,t]}\right]\leqslant \mathbb{E}[[N^n]_T]^{1/2}\,\mathbb{E}\left[\sup_{t\in[0,T]}[M^n]_{t+\theta}-[M^n]_t\right]^{1/2},$$

and thus the left hand side tends to 0 with $\theta\to 0$ since $[M^n]$ converges to the continuous process $[M]$ in $\mathscr{S}^1(\mathbb{R})$. This implies the $\mathbf{C}$-tightness.

By Theorem 1, we obtain the convergence of $\langle M^n,N^n\rangle$ to $[M,N]=\langle M,N\rangle$ in $\mathscr{S}^1(\mathbb{R})$. □

Before going further, let us precise the assumptions we will work with in the following:

(H1) (i) W^n is a square integrable martingale which converges to W in $\mathscr{S}^2(\mathbb{R})$;

(ii) there exist $\rho:\mathbb{R}_+\to\mathbb{R}_+$ with $\rho(0^+)=0$ and a deterministic sequence $(a_n)_{\mathbb{N}}$ with $\lim_{n\to\infty}a_n=0$ such that, $\mathbb{P}$-a.s.,

$$\forall 0\leqslant s\leqslant t\leqslant T,\quad \langle W^n\rangle_t-\langle W^n\rangle_s\leqslant \rho(t-s)+a_n;$$

(H2) ξ is $\mathscr{F}_T$ measurable and, for all n, ξ^n is $\mathscr{F}^n_T$ measurable such that ξ^n converges to ξ in L^2.

Before stating an important result of this paper, let us recall the notion of "weak convergence of filtrations" which has been studied in Coquet et al. (2001): we say that $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ converges weakly to $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$, if, for each set $A\in\mathscr{F}_T$, the càdlàg martingale $M^n_t := \mathbb{E}(\mathbf{1}_A \mid \mathscr{F}^n_t)$ converges in ucp to the martingale $M_t := \mathbb{E}(\mathbf{1}_A \mid \mathscr{F}_t)$.

If (Y^n, Z^n, N^n) denotes the solution to the BSDE (2) and (Y, Z) the solution to the BSDE (1), we want to prove the convergence of (Y^n, Z^n, N^n) to $(Y, Z, 0)$ and for the first component i.e. the Y's we would like to get at least the convergence in ucp. But when f^n and f are identically 0, we have $Y^n_t = \mathbb{E}(\xi^n \mid \mathscr{F}^n_t)$ and $Y_t = \mathbb{E}(\xi \mid \mathscr{F}_t)$. Since ξ^n is assumed to converge in L^2 to ξ, we get the convergence of Y^n to Y in ucp if and only if $\mathbb{E}(\xi \mid \mathscr{F}^n_t)$ converges to $\mathbb{E}(\xi \mid \mathscr{F}_t)$ in ucp. Thus, the weak convergence of filtrations $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ to $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ is a necessary condition in order to get the convergence of Y^n to Y in ucp.

This comment leads to the following question: under the assumption (H1), do we have the weak convergence of filtrations $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ to $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$? The following proposition, which will be proved in the appendix, gives a positive answer to this question.

Proposition 3. *Let us consider, on the same probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, *a standard Brownian motion* $\{W_t\}_{0\leqslant t\leqslant T}$ *with its natural filtration* $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$, *a sequence of filtrations* $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ *and a sequence* $\{W^n_t\}_{0\leqslant t\leqslant T}$ *of square integrable* $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$-*martingales. We suppose that* W^n *converges to* W *in* $\mathscr{S}^2(\mathbb{R})$.

Then $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ *weakly converges to* $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$.

Remark 4. It is worth noting that the previous result is not true if W^n is not a martingale. To see this, let us choose $T=1$ and W^n_t equal to $W_t + \frac{1}{n}W_1$: we have the convergence of W^n to W in $S^p(\mathbb{R})$ (for all real $p\geqslant 1$), but $\{\mathscr{F}^n_t\}_{0\leqslant t\leqslant T}$ is not weakly convergent to $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$. Indeed, for each $t\in[0,1]$, $\mathscr{F}^n_t = \sigma(W_s,\, s\leqslant t,\, W_1)$ and thus $\mathbb{E}(W_1 \mid \mathscr{F}^n_t) = W_1$ which does not converge to $\mathbb{E}(W_1 \mid \mathscr{F}_t) = W_t$ in ucp.

With this proposition in hands, we can state one of the main results of this paper. It concerns the robustness of the orthogonal decomposition of ξ^n w.r.t. W^n. A similar study is considered in Jacod et al. (2000).

Theorem 5. *Let the assumptions* (H1) *and* (H2) *hold. We consider the orthogonal decomposition w.r.t.* W^n *of* ξ^n, *i.e.* Z^n *is a predictable process and* N^n *a càdlàg martingale* (*with* $N^n_0 = 0$) *which is orthogonal to* W^n *and*

$$M^n_t = \mathbb{E}(\xi^n \mid \mathscr{F}^n_t) = M^n_0 + \int_0^t Z^n_r \,\mathrm{d}W^n_r + N^n_t, \quad 0\leqslant t\leqslant T$$

and the representation of ξ *as a stochastic integral*

$$M_t = \mathbb{E}(\xi \mid \mathscr{F}_t) = M_0 + \int_0^t Z_r \,\mathrm{d}W_r, \quad 0\leqslant t\leqslant T.$$

Then, we have

$$\left(M^n, \int_0^\cdot Z_r^n \,\mathrm{d}W_r^n, N^n\right) \to \left(M, \int_0^\cdot Z_r \,\mathrm{d}W_r, 0\right), \quad \textit{as } n \to \infty$$

in $\mathscr{S}^2(\mathbb{R}^3)$ *and*

$$\left(\int_0^\cdot Z_r^n \,\mathrm{d}\langle W^n\rangle_r, \int_0^\cdot |Z_r^n|^2 \,\mathrm{d}\langle W^n\rangle_r\right) \to \left(\int_0^\cdot Z_r \,\mathrm{d}r, \int_0^\cdot |Z_r|^2 \,\mathrm{d}r\right)$$

in $\mathscr{S}^2(\mathbb{R}) \times \mathscr{S}^1(\mathbb{R})$, *as* n *tends to* ∞.

Proof. Since ξ^n converges to ξ in L^2, we can assume without loss of generality that each ξ^n is bounded by the same constant say k. As a byproduct of this remark, we need only to prove the convergence in probability under the additional assumption that M^n is uniformly bounded by k which implies that

$$\sup_n \mathbb{E}\left[\left(\int_0^T |Z_r^n|^2 \,\mathrm{d}\langle W^n\rangle_r\right)^p + \langle N^n\rangle_T^{\frac{p}{2}}\right]$$

is finite for all real $p \geqslant 1$. Indeed, we have $\sup_n \mathbb{E}[\sup_t |M_t^n|^p] \leqslant k^p$ and then by BDG inequality $\sup_n \mathbb{E}\{[M^n]_T^{p/2}\} < \infty$. Moreover, since $\langle M^n\rangle$ is the predictable compensator of $[M^n]$, we have, see e.g. (Dellacherie and Meyer, 1980, Eq. (100.2) p. 183), $\mathbb{E}\{\langle M^n\rangle_T^{p/2}\} \leqslant (p/2)^{p/2}\mathbb{E}\{[M^n]_T^{p/2}\}$, as soon as $p \geqslant 2$. Thus, since N^n and W^n are orthogonal, we get, for each $t \in [0, T]$,

$$\langle M^n\rangle_t = \int_0^t |Z_r^n|^2 \,\mathrm{d}\langle W^n\rangle_r + \langle N^n\rangle_t$$

from which the result follows.

Since we have the weak convergence of the filtration $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ to $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$, we can apply the second point of Remark 1 in Coquet et al. (2001) to get the convergence of M^n to M in the sense of Skorokhod-topology on $\mathbf{D}$. But Brownian martingales are continuous so that the previous convergence holds also in ucp and in all $\mathscr{S}^p$ spaces. Hence we can apply Proposition 2 to (W^n, M^n) to obtain the ucp convergence of $\langle M^n, W^n\rangle$ and $\langle M^n\rangle$ towards $\langle M, W\rangle$ and $\langle M\rangle$, respectively. Namely, taking into account the fact that W^n and N^n are mutually orthogonal,

$$\left(M^n, \int_0^\cdot Z_r^n \,\mathrm{d}\langle W^n\rangle_r, \int_0^\cdot (Z_r^n)^2 \,\mathrm{d}\langle W^n\rangle_r + \langle N^n\rangle\right) \to \left(M, \int_0^\cdot Z_r \,\mathrm{d}r, \int_0^\cdot (Z_r)^2 \,\mathrm{d}r\right).$$

It remains only to prove that $\langle N^n\rangle$ tends actually to 0. For this, let us first remark that, since $\langle M^n\rangle - \langle N^n\rangle$ is an increasing process, the increments of $\langle N^n\rangle$ are bounded by those of $\langle M^n\rangle$. Thus $\langle N^n\rangle$ is $\mathbf{C}$-tight. By Theorem 4.13 of Chapter 6 in Jacod and Shiryaev (1987), the tightness in $\mathbf{C}$ of $\langle N^n\rangle$ implies the tightness in $\mathbf{D}$ of the sequence of martingales N^n. Let N be a limit point of N^n. Using Skorokhod representation theorem, we assume that we are on the same space and that the convergence of N^n to N holds in probability for the Skorokhod topology. Moreover, the sequence (W^n, N^n, M^n) satisfies the uniform tightness condition since it is bounded in L^2. Thus $[W^n, N^n]$ converges to $[W, N]$ in ucp the limit being obviously continuous and with finite variation paths.

On the other hand, W^n and N^n are orthogonal so that $[W^n, N^n]$ is a martingale. Using the fact that $[N^n]_T^{1/2}$ is bounded in L^2 and the fact that $[W^n]_T$ is uniformly integrable by Scheffé's lemma, we deduce that $[W^n,N^n]_t$ is uniformly integrable and thus that $[W,N]$ is a continuous martingale with finite variation paths. Thus $[W,N]=0$.

Moreover, M is a Brownian martingale which implies that $[M,N]=0$. Taking into account that M^n converges to M in ucp and is bounded and that N^n is bounded in L^2, Proposition 2 shows that $\langle M^n,N^n\rangle$ converges to $\langle M,N\rangle$ in $\mathscr{S}^1(\mathbb{R})$ (we work with the filtration of the past of (W,N)). But $\langle M,N\rangle=[M,N]=0$. Since $\langle M^n,N^n\rangle=\langle N^n,N^n\rangle$, we deduce that N^n converges to 0 in $\mathscr{S}^2$. This concludes the proof. □

3. BSDEs driven by W^n

In this section, we will give some results concerning BSDEs driven by the martingale W^n. One of the goals of this technical part is to get results that are in some sense uniform in n. Firstly, we will give an additional assumption.

Let us recall that we are working on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ carrying a standard real valued Brownian motion $W=\{W_t\}_{0\leqslant t\leqslant T}$. On this space, we consider a sequence of càdlàg square integrable $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-martingales $W^n=\{W_t^n\}_{0\leqslant t\leqslant T}$; $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ is right continuous and complete and $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ stands for the right continuous and complete filtration generated by W.

Let f and f^n be random functions, defined on $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}$ with values in $\mathbb{R}$, such that, for each (y,z), the process $\{f^n(t,y,z)\}_{0\leqslant t\leqslant T}$ (resp. $\{f(t,y,z)\}_{0\leqslant t\leqslant T}$) is progressively measurable w.r.t. $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ (resp. $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$).

We will assume that:

(H3) (i) there exists $K\geqslant 0$ such that, $\mathbb{P}$-a.s., for each n and each $t\in[0,T]$,

$$\forall (y,z),(y',z')\in\mathbb{R}^2,\quad |f^n(t,y,z)-f^n(t,y',z')|\leqslant K(|y-y'|+|z-z'|)$$

and f is also K-Lipschitz;

(ii) for all (y,z), $\{f^n(t,y,z)\}_{0\leqslant t\leqslant T}$ has càdlàg paths and converges to $\{f(t,y,z)\}_{0\leqslant t\leqslant T}$ in $\mathscr{S}^2(\mathbb{R})$.

This section is devoted to the study of the BSDE (2) under the assumptions described above which cover the case of the scaled simple random walks studied in Briand et al. (2001). These assumptions are rather strong in order to get existence and uniqueness of a solution. However, in preparation to further results, we need some uniform (in n) estimates.

Firstly, let us recall that a solution of the BSDE (2) is a triple $\{(Y_t^n,Z_t^n,N_t^n)\}_{0\leqslant t\leqslant T}$ progressively measurable w.r.t. $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ such that Y^n has càdlàg paths, Z^n is predictable and N^n is a càdlàg martingale, orthogonal to W^n and such that $N_0^n=0$. We will use a standard fixed point argument to study such BSDEs as in El Karoui et al. (1997).

$\mathscr{S}_a^2$ and $\mathscr{M}^2$ (a subscript n is omitted) denotes the set of progressively measurable respectively predictable processes w.r.t. $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ such that

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|Y_t|^2\right]<+\infty,\quad\text{respectively,}\quad \mathbb{E}\left[\int_0^T|Z_r|^2\,\mathrm{d}\langle W^n\rangle_r\right]<+\infty.$$

$\mathscr{H}^2$ (n is still omitted) is the Hilbert space of square integrable càdlàg martingales w.r.t. $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ endowed with the scalar product $(M,N)=\mathbb{E}[M_T N_T]$ and $\mathscr{H}_0^2$ the closed subspace of martingales M s.t. $M_0=0$.

We start with an elementary lemma:

Lemma 6. *Let the assumptions* (H1), (H2) *and* (H3) *hold. Let* $\{U_t,V_t\}_{0\leqslant t\leqslant T}$ *in* $\mathscr{S}_a^2\times\mathscr{M}^2$. *Then the BSDE*

$$Y_t^n=\xi^n+\int_t^T f^n(r,U_{r-},V_r)\,\mathrm{d}\langle W^n\rangle_r-\int_t^T Z_r^n\,\mathrm{d}W_r^n-\int_t^T \mathrm{d}N_r^n \tag{3}$$

has a solution in the space $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$ *for each* n.

Proof. Firstly, we set,

$$Y_t^n=\mathbb{E}\left(\xi^n+\int_t^T f^n(r,U_{r-},V_r)\,\mathrm{d}\langle W^n\rangle_r|\mathscr{F}_t^n\right),$$

$\{Y_t^n\}_{0\leqslant t\leqslant T}$ is a càdlàg process and moreover

$$|Y_t^n|\leqslant\mathbb{E}\left(|\xi^n|+\int_0^T |f^n(r,U_{r-},V_r)|\,\mathrm{d}\langle W^n\rangle_r|\mathscr{F}_t^n\right),$$

and thus Doob's inequality yields

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|Y_t^n|^2\right]\leqslant 4\,\mathbb{E}\left[\left(|\xi^n|+\int_0^T |f^n(r,U_{r-},V_r)|\,\mathrm{d}\langle W^n\rangle_r\right)^2\right].$$

The right-hand side of the previous inequality is finite in view of the assumptions noting in particular that $\sup_n\langle W^n\rangle_T$ is finite from (H1)(ii).

Now, $\{Z_t^n,N_t^n\}_{0\leqslant t\leqslant T}$ is given by the orthogonal decomposition w.r.t. W^n—see e.g. (Jacod, 1979, Theorem 4.27 p. 126)—of the martingale

$$\mathbb{E}\left(\xi^n+\int_0^T f^n(r,U_{r-},V_r)\,\mathrm{d}\langle W^n\rangle_r|\mathscr{F}_t^n\right)=Y_0^n+\int_0^t Z_r^n\,\mathrm{d}W_r^n+N_t^n,\quad 0\leqslant t\leqslant T.$$

In addition, we have, since N^n and W^n are orthogonal

$$\mathbb{E}\left[\int_0^T |Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r+\langle N^n\rangle_T\right]\leqslant 10\,\mathbb{E}\left[\left(|\xi^n|+\int_0^T |f^n(r,U_{r-},V_r)|\,\mathrm{d}\langle W^n\rangle_r\right)^2\right],$$

which is finite as already mentioned. It is plain to check that the triple (Y^n,Z^n,N^n) solves the BSDE (3) to conclude the proof. □

We keep on the study by giving an easy a priori estimate.

Proposition 7. *Let* (Y^n,Z^n,N^n) (*resp.* (Y'^n,Z'^n,N'^n)) *be the solution to the BSDE* (3) *associated to* $(\xi^n,U,V)\in\mathrm{L}^2(\mathscr{F}_T^n)\times\mathscr{S}_a^2\times\mathscr{M}^2$ (*resp.* (ξ'^n,U',V')).

Under the assumptions of the previous lemma, we have, for each $0\leqslant\sigma\leqslant\tau\leqslant T$,

$$\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta Y_t^n|^2+\int_\sigma^\tau|\delta Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r+\langle\delta N^n\rangle_\tau-\langle\delta N^n\rangle_\sigma\right]$$

$$\leqslant 42\,\mathbb{E}[|\delta Y_\tau^n|^2]+C(\tau-\sigma,a_n)\,\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta U_t|^2+\int_\sigma^\tau|\delta V_r|^2\,\mathrm{d}\langle W^n\rangle_r\right],$$

where $C(r,a_n)=42K^2\max\{(\rho(r)+a_n)^2,\rho(r)+a_n\}$ *and* δY^n *stands for* $Y^n - Y'^n$ *and so on.*

Proof. The starting point is the equation

$$\delta Y_t^n = \delta Y_\tau^n + \int_t^\tau (f^n(r,U_{r-},V_r) - f^n(r,U'_{r-},V'_r))\,\mathrm{d}\langle W^n\rangle_r - \int_t^\tau \delta Z_r^n\,\mathrm{d}W_r^n - \int_t^\tau \mathrm{d}\delta N_r^n, \quad 0\leqslant t\leqslant \tau.$$

Since f^n is K-Lipschitz, we have

$$|\delta Y_t^n| \leqslant \mathbb{E}\left(|\delta Y_\tau^n| + K\int_t^\tau (|\delta U_{r-}|+|\delta V_r|)\,\mathrm{d}\langle W^n\rangle_r|\mathscr{F}_t^n\right)$$

and then, Doob's inequality gives

$$\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta Y_t|^2\right] \leqslant 4\,\mathbb{E}\left[\left(|\delta Y_\tau^n| + K\int_\sigma^\tau (|\delta U_{r-}|+|\delta V_r|)\,\mathrm{d}\langle W^n\rangle_r\right)^2\right]. \tag{4}$$

Moreover, we have, since W^n and δN^n are orthogonal,

$$\mathbb{E}\left[\int_\sigma^\tau |\delta Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r + \langle\delta N^n\rangle_\tau - \langle\delta N^n\rangle_\sigma\right] = \mathbb{E}\left[\left|\int_\sigma^\tau \delta Z_r^n\,\mathrm{d}W_r^n + \delta N_\tau^n - \delta N_\sigma^n\right|^2\right]$$

and

$$\int_\sigma^\tau \delta Z_r^n\,\mathrm{d}W_r^n + \delta N_\tau^n - \delta N_\sigma^n = \int_\sigma^\tau \{f^n(r,U_{r-},V_r) - f^n(r,U'_{r-},V'_r)\}\,\mathrm{d}\langle W^n\rangle_r + \delta Y_\tau^n - \delta Y_\sigma^n.$$

Using the fact that f^n is K-Lipschitz, we obtain

$$\left|\int_\sigma^\tau \delta Z_r^n\,\mathrm{d}W_r^n + \delta N_\tau^n - \delta N_\sigma^n\right| \leqslant |\delta Y_\tau^n| + K\int_\sigma^\tau (|\delta U_{r-}|+|\delta V_r|)\,\mathrm{d}\langle W^n\rangle_r + \sup_{\sigma\leqslant t\leqslant\tau}|\delta Y_t^n|.$$

From the estimate (4), we get

$$\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta Y_t^n|^2 + \int_\sigma^\tau |\delta Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r + \langle\delta N^n\rangle_\tau - \langle\delta N^n\rangle_\sigma\right] \leqslant 14\,\mathbb{E}\left[\left(|\delta Y_\tau^n| + K\int_\sigma^\tau (|\delta U_{r-}|+|\delta V_r|)\,\mathrm{d}\langle W^n\rangle_r\right)^2\right].$$

Hölder's inequality together with the assumption (H1)(ii) lead to the estimate

$$\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta Y_t^n|^2 + \int_\sigma^\tau |\delta Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r + \langle\delta N^n\rangle_\tau - \langle\delta N^n\rangle_\sigma\right] \leqslant 42\,\mathbb{E}[|\delta Y_\tau^n|^2] + C(\tau-\sigma,a_n)\,\mathbb{E}\left[\sup_{\sigma\leqslant t\leqslant\tau}|\delta U_t|^2 + \int_\sigma^\tau |\delta V_r|^2 d\langle W^n\rangle_r\right],$$

with $C(\tau-\sigma,a_n)=42K^2\max\{(\rho(\tau-\sigma)+a_n)^2,\rho(\tau-\sigma)+a_n\}$. $\square$

Let us remark that, since $\lim_{r\to 0^+}\rho(r)=0$ by the assumption (H1)(ii), there exists $r_0\in(0,T)$ such that $42K^2\max(\rho(r),\rho(r)^2)\leqslant\frac{1}{6}$ as soon as $r\leqslant r_0$. Let us fix $m=[T/r_0]+1$ and consider the regular partition of $[0,T]$ into m intervals. We set, for $0\leqslant k\leqslant m-1$, $I_k=[kT/m,(k+1)T/m]$ and we introduce the following norm on $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$:

$$\|(Y^n,Z^n,N^n)\|_s^2=\sum_{k=0}^{m-1}(5\times 42)^k\,\mathbb{E}\left[\sup_{t\in I_k}|Y_t^n|^2+\int_{I_k}|Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r+\int_{I_k}\mathrm{d}\langle N^n\rangle_r\right].$$

This norm is equivalent to the classical one since we have

$$\|(Y^n,Z^n,N^n)\|^2\leqslant\|(Y^n,Z^n,N^n)\|_s^2\leqslant m(5\times 42)^{m-1}\|(Y^n,Z^n,N^n)\|^2.$$

The estimate of Proposition 7 and a straightforward computation show that, if (Y^n,Z^n,N^n) and (Y'^n,Z'^n,N'^n) are the solutions to the BSDEs (3) with (ξ^n,U,V) and (ξ^n,U',V'), we have, setting as usual $\delta Y^n=Y^n-Y'^n$ and so on,

$$\|(\delta Y^n,\delta Z^n,\delta N^n)\|_s^2\leqslant(1/5)\,\|(\delta Y^n,\delta Z^n,\delta N^n)\|_s^2+C(T/m,a_n)\,\|(\delta U,\delta V,0)\|_s^2.$$

It is really worth noting that, since $a_n\to 0$ as $n\to\infty$, there exists n_0 such that, for all $n\geqslant n_0$, for all $r\leqslant r_0$, $C(r,a_n)\leqslant\frac{1}{5}$; in particular, $C(T/m,a_n)\leqslant\frac{1}{5}$ as soon as $n\geqslant n_0$. Thus, for each $n\geqslant n_0$, we have

$$\|(\delta Y^n,\delta Z^n,\delta N^n)\|_s^2\leqslant(1/4)\,\|(\delta U,\delta V,0)\|_s^2. \tag{5}$$

The previous estimate leads immediately to the following

Remark 8. There exists a unique solution, in the space $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$, to the BSDE (3) where $(U,V)\in\mathscr{S}_a^2\times\mathscr{M}^2$, as soon as $n\geqslant n_0$.

With the help of this result, we can obtain an existence and uniqueness result for the BSDE (2).

Theorem 9. *Under the assumptions* (H1), (H2) *and* (H3), *the BSDE* (2) *has a unique solution* (Y^n,Z^n,N^n) *in the space* $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$, *for* n *large enough.*

Proof. We will use a fixed point argument. Let us consider the application Ψ^n from $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$ into itself which is defined by setting $\Psi^n(U,V,L)=(Y^n,Z^n,N^n)$ where (Y^n,Z^n,N^n) is the solution to the BSDE (3). Notice that L does not appear and that Ψ^n is well defined by Lemma 6 and the remark above. For n large enough (larger than the n_0 constructed before), the estimate (5) says that Ψ_n is a contraction with constant 1/2 if we use the equivalent norm $\|\cdot\|_s$ instead of $\|\cdot\|$ on the Banach space $\mathscr{S}_a^2\times\mathscr{M}^2\times\mathscr{H}_0^2$. □

We finish this section by an easy consequence of the estimate (5) which will be very useful in the sequel. For each p, we consider the approximation of (Y^n,Z^n,N^n) by the Picard procedure on the interval $[0,T]$ i.e. $(Y^{n,0},Z^{n,0})=(0,0)$ and, for $0\leqslant t\leqslant T$,

$$Y_t^{n,p+1}=\xi^n+\int_t^T f^n(r,Y_{r-}^{n,p},Z_r^{n,p})\,\mathrm{d}\langle W^n\rangle_r-\int_t^T Z_r^{n,p+1}\,\mathrm{d}W_r^n-\int_t^T\mathrm{d}N_r^{n,p+1}.$$

We have the following result:

Corollary 10. *Let the assumptions of Theorem 9 hold. There exists a constant C such that*

$$\sup_{n\geqslant n_0} \mathbb{E}\left[\sup\nolimits_{t\in[0,T]} |Y_t^n - Y_t^{n,p}|^2 + \int_0^T |Z_r^n - Z_r^{n,p}|^2 \,\mathrm{d}\langle W^n\rangle_r + |N_T^n - N_T^{n,p}|^2\right]$$
$$\leqslant C\, 4^{-p}.$$

Proof. It is a direct consequence of the fact that the application Ψ^n is a contraction with constant $1/2$ for $n \geqslant n_0$ and for the norm $\|\cdot\|_s$. Indeed, we have, for $n \geqslant n_0$,

$$\|(Y^n,Z^n,N^n) - (Y^{n,p},Z^{n,p},N^{n,p})\|^2 \leqslant \|(Y^n,Z^n,N^n) - (Y^{n,p},Z^{n,p},N^{n,p})\|_s^2$$
$$\leqslant 4^{1-p} m(5\times 42)^{m-1}\|(Y^{n,1},Z^{n,1},N^{n,1})\|^2.$$

To conclude, let us notice that a standard computation leads to the inequality

$$\|(Y^{n,1},Z^{n,1},N^{n,1})\|^2 \leqslant 28\,\mathbb{E}\left[|\xi^n|^2 + (\rho(T)+a_n)^2 \sup_{0\leqslant t\leqslant T}|f^n(t,0,0)|^2\right],$$

from which we deduce, in view of the assumptions, that $\sup_{n\geqslant n_0}\|(Y^{n,1},Z^{n,1},N^{n,1})\|^2$ is finite. The proof is complete. □

4. Stability of BSDEs

4.1. A technical result

We begin this section by a technical result which will be useful for proving the convergence of the second term of the right-hand side of (2).

Proposition 11. *Let $(\mu_n)_{\mathbb{N}}$ be a sequence of non-negative measures on $I=[0,T]$ and $(U_n)_{\mathbb{N}}$ a sequence of measurable functions on I; we assume that for some non-negative finite measure μ and some measurable function U:*

$$\sup_{t\in[0,T]}\left|\int_0^t U_n(s)^k \mu_n(\mathrm{d}s) - \int_0^t U(s)^k\mu(\mathrm{d}s)\right| \to 0, \quad k=0,1,2.$$

Then for any sequence of functions $F_n(s,u)$, làdlàg w.r.t. s, such that for some C, and any t,u,v,

$$|F_n(t,u)-F_n(t,v)| \leqslant C|u-v|, \tag{6}$$

converging uniformly in s to some làdlàg function F i.e.

$$\sup_{s\in[0,T]}|F_n(s,u)-F(s,u)| \to 0, \tag{7}$$

one has

$$\sup_{t\leqslant T}\left|\int_0^t F_n(s,U_n(s))\mu_n(\mathrm{d}s) - \int_0^t F(s,U(s))\mu(\mathrm{d}s)\right| \to 0.$$

Proof. For any step function $\varphi(t)$ and $k=0,1,2$, one has

$$\sup_{t\leqslant T}\left|\int_0^t \varphi(t)U_n(t)^k\mu_n(\mathrm{d}t)-\int_0^t \varphi(t)U(t)^k\mu(\mathrm{d}t)\right|\to 0. \tag{8}$$

This extends to bounded làdlàg functions (uniform limit of a sequence of step functions). Consider now a continuous function $\tilde{U}(s)$ such that

$$\int_0^T (\tilde{U}(s)-U(s))^2\mu(\mathrm{d}s)<\varepsilon^2.$$

We can write

$$\begin{aligned}
&\int_0^t F_n(s,U_n(s))\mu_n(\mathrm{d}s)-\int_0^t F(s,U(s))\mu(\mathrm{d}s)\\
&\quad=\int_0^t (F_n(s,U_n(s))-F_n(s,\tilde{U}(s)))\mu_n(\mathrm{d}s)+\int_0^t (F_n(s,\tilde{U}(s))-F(s,\tilde{U}(s))\mu_n(\mathrm{d}s)\\
&\qquad+\int_0^t F(s,\tilde{U}(s))(\mu_n(\mathrm{d}s)-\mu(\mathrm{d}s))+\int_0^t (F(s,\tilde{U}(s))-F(s,U(s))\mu(\mathrm{d}s).
\end{aligned}$$

The last term is smaller than $C\varepsilon\sqrt{\mu(I)}$. The third one tends to zero because of Eq. (8) with $k=0$. For the second one, notice first that condition (6) implies that convergence (7) is uniform w.r.t. u in a compact set, and since $\{\tilde{U}(s),\ s\in[0,T]\}$ is compact, this term tends to zero. For the first one, notice that:

$$\begin{aligned}
&\left(\int_0^t (F_n(U_n(s))-F_n(\tilde{U}(s)))\mu_n(\mathrm{d}s)\right)^2\\
&\quad\leqslant \mu_n(I)\int_0^t (F_n(U_n(s))-F_n(\tilde{U}(s)))^2\mu_n(\mathrm{d}s)\\
&\quad\leqslant C^2\mu_n(I)\int_0^T (U_n(s)-\tilde{U}(s))^2\mu_n(\mathrm{d}s)\\
&\quad=C^2\mu_n(I)\int_0^T (U_n(s)^2+\tilde{U}(s)^2-2U_n(s)\tilde{U}(s))\,\mu_n(\mathrm{d}s).
\end{aligned}$$

Hence, thanks to (8), we get

$$\begin{aligned}
&\limsup_n\sup_t\left(\int_0^t (F_n(s,U_n(s))-F_n(s,\tilde{U}(s)))\mu_n(\mathrm{d}s)\right)^2\\
&\quad\leqslant C^2\mu(I)\int_0^T (U(s)-\tilde{U}(s))^2\mu(\mathrm{d}s)\\
&\quad\leqslant C^2\mu(I)\varepsilon^2,
\end{aligned}$$

which gives the result. □

4.2. *Main result*

We now state and prove our main result.

Theorem 12. *Let the assumptions* (H1), (H2) *and* (H3) *hold. Let* (Y^n, Z^n, N^n) *be the solution of the BSDE* (2) *and* (Y, Z) *the solution of the BSDE* (1); *then we have*

$$\left(Y^n, \int_0^\cdot Z_r^n \,\mathrm{d}W_r^n, N^n\right) \to \left(Y, \int_0^\cdot Z_r \,\mathrm{d}W_r, 0\right),$$

as n *tends to infinity in* $\mathscr{S}^2(\mathbb{R}^3)$ *and*

$$\left(\int_0^\cdot Z_r^n \,\mathrm{d}\langle W^n\rangle_r, \int_0^\cdot |Z_r^n|^2 \mathrm{d}\langle W^n\rangle_r\right) \to \left(\int_0^\cdot Z_r \,\mathrm{d}r, \int_0^\cdot |Z_r|^2 \,\mathrm{d}r\right),$$

in $\mathscr{S}^2(\mathbb{R}) \times \mathscr{S}^1(\mathbb{R})$.

Proof. The method of the proof is the same as in Briand et al. (2001) and it is based on the approximation of the solutions by the Picard method. Let us recall the notations. $(Y^{n,0}, Z^{n,0}, N^{n,0}) = (0,0,0)$, $(Y^{\infty,0}, Z^{\infty,0}) = (0,0)$ and we define recursively for all $p \in \mathbb{N}$,

$$Y_t^{n,p+1} = \xi^n + \int_t^T f^n(r, Y_{r-}^{n,p}, Z_r^{n,p}) \,\mathrm{d}\langle W^n\rangle_r$$
$$- \int_t^T Z_r^{n,p+1} \,\mathrm{d}W_r^n - \int_t^T \mathrm{d}N_r^{n,p+1}, \quad 0 \leqslant t \leqslant T$$

and, similarly,

$$Y_t^{\infty,p+1} = \xi + \int_t^T f(r, Y_r^{\infty,p}, Z_r^{\infty,p}) \,\mathrm{d}r - \int_t^T Z_r^{\infty,p+1} \mathrm{d}W_r, \quad 0 \leqslant t \leqslant T.$$

Since, by Corollary 10, $(Y^{n,p}, \int_0^\cdot Z_r^{n,p} \,\mathrm{d}W_r^n, N^{n,p})$ converges to $(Y^n, \int_0^\cdot Z_r^n \,\mathrm{d}W_r^n, N^n)$ as $p \to \infty$ in $\mathscr{S}^2(\mathbb{R}^3)$ uniformly in n, it is enough to check that for each integer p, $(Y^{n,p}, Z^{n,p}, N^{n,p})$ converges to $(Y^{\infty,p}, Z^{\infty,p}, 0)$ in the sense described in the statement of the result.

The proof will be done by induction. For sake of clarity, we drop the superscript p, so that the previous equations become

$$Y_t'^n = \xi^n + \int_t^T f^n(r, Y_{r-}^n, Z_r^n) \,\mathrm{d}\langle W^n\rangle_r - \int_t^T Z_r'^n \,\mathrm{d}W_r^n - \int_t^T \mathrm{d}N_r'^n, \quad 0 \leqslant t \leqslant T,$$
$$Y_t' = \xi + \int_t^T f(r, Y_r, Z_r) \,\mathrm{d}r - \int_t^T Z_r' \,\mathrm{d}W_r, \quad 0 \leqslant t \leqslant T.$$

The assumption is that $\{Y_t^n, Z_t^n\}_{0 \leqslant t \leqslant T}$ converges to $\{Y_t, Z_t\}_{0 \leqslant t \leqslant T}$ in the sense of Theorem 12 (without the N term) and we have to prove that $\{Y_t'^n, Z_t'^n, N_t'^n\}_{0 \leqslant t \leqslant T}$ converges to $\{Y_t', Z_t', 0\}_{0 \leqslant t \leqslant T}$ in the same sense.

The process, defined by

$$M_t^n = Y_t'^n + \int_0^t f^n(r, Y_{r-}^n, Z_r^n) \,\mathrm{d}\langle W^n\rangle_r, \quad 0 \leqslant t \leqslant T, \tag{9}$$

satisfies

$$M_t^n = M_0^n + \int_0^t Z_r'^n \,\mathrm{d}W_r^n + N_t'^n. \tag{10}$$

Hence M^n is an $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-martingale and, since $Y_T^n=\xi^n$,

$$M_t^n=\mathbb{E}(M_T^n\,|\,\mathscr{F}_t^n),\quad M_T^n=Y_T^n+\int_0^T f^n(r,Y_{r-}^n,Z_r^n)\,\mathrm{d}\langle W^n\rangle_r. \tag{11}$$

If we want to apply Theorem 5, we have to prove the L^2 convergence of M_T^n. We know that $Y_T^n=\xi^n$ converges to $Y_T=\xi$ so that it remains to prove that $\int_0^T f^n(r,Y_{r-}^n,Z_r^n)\,\mathrm{d}\langle W^n\rangle_r$ converges to $\int_0^T f(r,Y_r,Z_r)\,\mathrm{d}r$ in L^2. To do this we will apply Proposition 11. Indeed, from Proposition 2, we know that $\langle W^n\rangle_t$ converges to t in ucp and in all $\mathscr{S}^p(\mathbb{R})$. Moreover, the induction assumption gives the convergence in $\mathscr{S}^2(\mathbb{R})\times\mathscr{S}^1(\mathbb{R})$

$$\left(\int_0^\cdot Z_r^n\,\mathrm{d}\langle W^n\rangle_r,\int_0^\cdot |Z_r^n|^2\,\mathrm{d}\langle W^n\rangle_r\right)\to\left(\int_0^\cdot Z_r\,\mathrm{d}r,\int_0^\cdot |Z_r|^2\,\mathrm{d}r\right).$$

Since $\{f^n\}_n$ is an equi-Lipschitz family, it remains only to prove that, for each u,

$$\sup_{t\in[0,T]}|f^n(t,Y_{t-}^n,u)-f(t,Y_t,u)|$$

goes to 0 in probability. But since f^n is K-Lipschitz, we have

$$\sup_{t\in[0,T]}|f^n(t,Y_{t-}^n,u)-f(t,Y_t,u)|$$
$$\leqslant K\sup_{t\in[0,T]}|Y_t^n-Y_t|+\sup_{t\in[0,T]}|f^n(t,Y_t,u)-f(t,Y_t,u)|.$$

The first term tends to zero by the assumption of induction and the second by the assumption (H3). Indeed let $\varepsilon>0$ and $a>0$. We consider a finite number of points of $[-a,a]$, $\mathscr{Y}=(y_i)_{1\leqslant i\leqslant N(\varepsilon,a)}$, such that, if $|y|\leqslant a$, $\mathrm{dist}(y,\mathscr{Y})\leqslant\varepsilon/(4K)$. On the set $\{\sup_t|Y_t|\leqslant a\}$, we have, writing h^n for f^n-f,

$$|h^n(t,Y_t,u)|\leqslant 2K\,\mathrm{dist}(Y_t,\mathscr{Y})+\max_{y\in\mathscr{Y}}|h^n(t,y,u)|\leqslant\varepsilon/2+\max_{y\in\mathscr{Y}}|h^n(t,y,u)|.$$

Thus, we obtain,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}|h^n(t,Y_t,u)|>\varepsilon\right)\leqslant\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\max_{y\in\mathscr{Y}}|h^n(t,y,u)|>\varepsilon/2\right)$$
$$+\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}|Y_t|>a\right),$$

from which we deduce, in view of (H3), that

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}|f^n(t,Y_t,u)-f(t,Y_t,u)|>\varepsilon\right)\leqslant\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}|Y_t|>a\right).$$

Sending a to infinity, we get the result since $\sup_{0\leqslant t\leqslant T}|Y_t|$ is square integrable.

Since $f^n(t,Y_t^n,u)$ has càdlàg paths, Proposition 4.1 implies that

$$\int_0^\cdot f^n(r,Y_{r-}^n,Z_r^n)\,\mathrm{d}\langle W^n\rangle_r\to\int_0^\cdot f(r,Y_r,Z_r)\,\mathrm{d}r$$

as $n \to \infty$ in ucp and in $\mathscr{S}^2(\mathbb{R})$ since

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t f^n(r, Y^n_{r-}, Z^n_r)\, \mathrm{d}\langle W^n \rangle_r \right|$$

$$\leqslant C \left(\sup_{0 \leqslant t \leqslant T} (|Y^n_t| + |f^n(t,0,0)|) + \int_0^T |Z^n_r|\, \mathrm{d}\langle W^n \rangle_r \right).$$

We set, for $0 \leqslant t \leqslant T$,

$$M_t = \mathbb{E}\left(\xi + \int_0^T f(r, Y_r, Z_r)\, \mathrm{d}r \mid \mathscr{F}_t \right) = Y_t + \int_0^t f(r, Y_r, Z_r)\, \mathrm{d}r = M_0 + \int_0^t Z'_r\, \mathrm{d}W_r$$

and we apply Theorem 5, taking into account Eqs. (10) and (11), to get

$$\left(M^n, \int_0^{\cdot} Z^n_r\, \mathrm{d}W^n_r, N'^n \right) \to \left(M, \int_0^{\cdot} Z_r\, \mathrm{d}W_r, 0 \right)$$

in $\mathscr{S}^2(\mathbb{R}^3)$ and

$$\left(\int_0^{\cdot} Z'^n_r\, \mathrm{d}\langle W^n \rangle_r, \int_0^{\cdot} |Z'^n_r|^2\, \mathrm{d}\langle W^n \rangle_r \right) \to \left(\int_0^{\cdot} Z'_r\, \mathrm{d}r, \int_0^{\cdot} |Z'_r|^2\, \mathrm{d}r \right)$$

as $n \to \infty$ in $\mathscr{S}^2(\mathbb{R}) \times \mathscr{S}^1(\mathbb{R})$.

We deduce from the previous convergence that

$$\sup_{0 \leqslant t \leqslant T} |Y'^n_t - Y'_t| \to 0 \quad \text{in } \mathscr{S}^2(\mathbb{R}).$$

Since we have already proved that

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t f^n(r, Y^n_{r-}, Z^n_r)\, \mathrm{d}\langle W^n \rangle_r - \int_0^t f(r, Y_r, Z_r)\, \mathrm{d}r \right| \to 0$$

in L^2. This concludes the proof. □

5. Examples

5.1. Discretization of Brownian motion

In this section, we give a special emphasis to the case of the discretization of Brownian motion which is in fact a attempt to write down an Euler scheme for BSDEs as mentioned in the introduction. This example can be handled by Theorem 12 as we will see in the sequel.

Let us start with the setup: W is a Brownian motion and $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$ denotes its augmented filtration. We will assume in this section that

(i) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists K s.t. for all t,

$$\forall (x,y,z), (x',y',z'),$$

$$|f(t,x,y,z) - f(t,x',y',z')| \leqslant K(|x-x'| + |y-y'| + |z-z'|);$$

(ii) ξ be square integrable $\mathscr{F}_T$-random variable.

We denote by (E) this assumption.

We consider $\{(Y_t, Z_t)\}_{0 \leqslant t \leqslant T}$ the solution to the BSDE

$$Y_t = \xi + \int_t^T f(r, W_r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}W_r, \quad 0 \leqslant t \leqslant T. \tag{12}$$

We want to construct an approximation of this solution, say $\{(Y_t^n, Z_t^n)\}_{0 \leqslant t \leqslant T}$, in the spirit of the approximation of the solution to a SDE by the Euler scheme. To do this, let us consider $(\pi_n)_n$ a refining sequence of partitions of $[0, T]$ such that $\mathrm{mesh}(\pi_n) \to 0$. By $W^n = \{W_t^n\}_{0 \leqslant t \leqslant T}$ we denote the discretization of W associated to π_n namely, if $\pi_n = (t_k^n)_{k=0,p_n}$ $t_0^n = 0$, $t_{p_n}^n = T$,

$$W_t^n = W_{t_k^n}, \quad \text{if } t_k^n \leqslant t < t_{k+1}^n, \quad W_T^n = W_T,$$

so that W^n is a càdlàg martingale. $\{\mathscr{F}_t^n\}_{0 \leqslant t \leqslant T}$ stands for the filtration generated by W^n.

A naive idea consists in trying to solve the following equation, where we write W_k^n in place of $W_{t_k^n}^n$ and so on:

$$Y_k^n = Y_{k+1}^n + f(t_{k+1}^n, W_k^n, Y_k^n, Z_k^n)(t_{k+1}^n - t_k^n) - Z_k^n(W_{k+1}^n - W_k^n),$$

where Y_{k+1}^n is $\mathscr{F}_{t_{k+1}^n}^n$-measurable. The unknowns (Y_k^n, Z_k^n) are required to be $\mathscr{F}_{t_k^n}^n$-measurable. Thus if we want to solve the previous equation, we first set

$$Z_k^n = \frac{1}{t_{k+1}^n - t_k^n}\mathbb{E}(Y_{k+1}^n(W_{k+1}^n - W_k^n) \,|\, \mathscr{F}_{t_k^n}^n), \tag{13}$$

the last formula being obtained by multiplying the equation by $(W_{k+1}^n - W_k^n)$ and then take the conditional expectation w.r.t $\mathscr{F}_{t_k^n}^n$. To find Y_k^n, we use a fixed point argument which requires $\mathrm{mesh}(\pi_n)K < 1$ in order to solve the equation

$$Y_k^n = \mathbb{E}(Y_{k+1}^n \,|\, \mathscr{F}_{t_k^n}^n) + (t_{k+1}^n - t_k^n) f(t_{k+1}^n, W_k^n, Y_k^n, Z_k^n). \tag{14}$$

However, there is a drawback: the equation has no reason to be satisfied since the predictable representation property does not hold for W^n. Indeed, in the discrete case, a real martingale with independent increments has the predictable representation property if and only if the increments are supported by two points. It follows that, for the BSDE (2), we have to add an orthogonal martingale term so that the equation we really solve is

$$Y_k^n = Y_{k+1}^n + f(t_{k+1}^n, W_k^n, Y_k^n, Z_k^n)(t_{k+1}^n - t_k^n) - Z_k^n(W_{k+1}^n - W_k^n) - (N_{k+1}^n - N_k^n),$$
$$k = 0, p_n - 1, \tag{15}$$

with the requirement that N^n is a càdlàg martingale orthogonal to W^n with $N_0^n = 0$. Of course, we have to specify the value of Y^n at time T. In order to cover all cases, we choose ξ^n to be $\mathbb{E}(\xi \,|\, \mathscr{F}_T^n)$. We can remark that if ξ is given as $g(W)$ with a smooth g then we can take ξ^n as $g(W^n)$. Nevertheless, our first choice of ξ^n is not so bad. Indeed, since Proposition 3 ensures that we have weak convergence of the filtration $\{\mathscr{F}_t^n\}_{0 \leqslant t \leqslant T}$ to $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$ which implies that ξ^n goes to ξ in L^2. For solving Eq. (15), we use (13) and (14) and then set $N_{k+1}^n - N_k^n = Y_{k+1}^n - Y_k^n - Z_k^n(W_{k+1}^n - W_k^n)$. If we set moreover,

$$Y_t^n = Y_k^n, \quad N_t^n = N_k^n, \quad \text{if } t_k^n \leqslant t < t_{k+1}^n, \qquad Z_{t_{k+1}^n}^n = Z_k^n,$$

then we obtain exactly the solution of the BSDE (2) (the support of Z^n is the point of π_n).

We have $\langle W^n \rangle_t = t_k^n$ on $[t_k^n, t_{k+1}^n[$ so that the assumption (H1) (ii) is satisfied with $\rho(x) = x$ and $a_n = \text{mesh}(\pi_n)$. Hence the assumptions (H1), (H2) and (H3) are satisfied. Thus we have

$$\left(|Y_t^n - Y_t|^2, \left| \int_0^t Z_r^n \, \mathrm{d}\langle W^n \rangle_r - \int_0^t Z_r \, \mathrm{d}r \right|^2, \right.$$

$$\left. \left| \int_0^t |Z_r^n|^2 \, \mathrm{d}\langle W^n \rangle_r - \int_0^t |Z_r|^2 \, \mathrm{d}r \right|, |N_t^n|^2 \right)$$

tends to 0 as $n \to \infty$ in $\mathscr{S}^1(\mathbb{R}^4)$.

If we define Z_t^n for all $t \in [0,T]$, by setting

$$Z_0^n = 0, \qquad Z_t^n = Z_k^n, \quad \text{if } t_k^n < t \leqslant t_{k+1}^n,$$

we deduce, from these uniform (in t) convergences, that

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t Z_r^n \, \mathrm{d}r - \int_0^t Z_r \, \mathrm{d}r \right| \xrightarrow{\mathrm{P}} 0,$$

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t |Z_r^n|^2 \, \mathrm{d}r - \int_0^t |Z_r|^2 \, \mathrm{d}r \right| \xrightarrow{\mathrm{P}} 0.$$

Extracting a subsequence (still indexed by n), we have for almost every ω,

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t Z_r^n(\omega) \, \mathrm{d}r - \int_0^t Z_r(\omega) \, \mathrm{d}r \right| \to 0,$$

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t |Z_r^n|^2(\omega) \, \mathrm{d}r - \int_0^t |Z_r|^2(\omega) \, \mathrm{d}r \right| \to 0,$$

which implies the convergence of $Z_\cdot^n(\omega)$ to $Z_\cdot(\omega)$ weakly in $\mathrm{L}^2([0,T],\lambda)$. Since we have the uniform integrability of the sequence $\int_0^T |Z_r^n|^2 \, \mathrm{d}r = \int_0^T |Z_r^n|^2 \, \mathrm{d}\langle W^n \rangle_r$, we finally get for this model

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Y_t^n - Y_t|^2 + \int_0^T |Z_r^n - Z_r|^2 \, \mathrm{d}r + \sup_{t \in [0,T]} |N_t^n|^2 \right] \to 0$$

as n tends to infinity i.e. the convergence of the approximation to the solution in the classical norm used for BSDEs. Let us resume what we have proved

Proposition 13. *Let the assumption* (E) *hold. We consider W^n the discretization of W associated to the grid π^n. Let (Y^n, Z^n, N^n) be the solution to the BSDE*

$$Y_t^n = \xi^n + \int_t^T f(r, W_r^n, Y_{r-}^n, Z_r^n) \, \mathrm{d}\langle W^n \rangle_r - \int_t^T Z_r^n \, \mathrm{d}W_r^n - (N_T^n - N_t^n), \quad 0 \leqslant t \leqslant T,$$

where $\xi^n = \mathbb{E}(\xi \mid \mathscr{F}_T^n)$. Then we have when $\text{mesh}(\pi_n) \to 0$,

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Y_t^n - Y_t|^2 + \int_0^T |Z_r^n - Z_r|^2 \, \mathrm{d}r + \sup_{t \in [0,T]} |N_t^n|^2 \right] \to 0,$$

where (Y,Z) is the solution to the BSDE (12).

5.2. The invariance principle

In Theorem 12, the BSDEs (1) and (2) were solved on the same probability space. But, we can also consider these equations on different probability spaces and obtain the convergence of solutions in law. As an example, we will treat the case of the invariance principle which is a generalization of Corollary 3.3 in Briand et al. (2001).

Let us consider a standard real Brownian motion W defined on a probability space and a sequence of independent and identically distributed real random variables $\{\zeta_k\}_{k\geqslant 1}$ defined on a possibly different probability space. We assume that ζ_1 is in $\mathrm{L}^{2+\delta}$ for some $\delta > 0$ with $\mathbb{E}[\zeta_1]=0$ and $\mathbb{E}[|\zeta_1|^2]=1$. We define, for each n, the scaled random walks

$$S_t^n = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\zeta_k, \quad 0\leqslant t\leqslant T.$$

We denote by $\mathbf{D}$ (resp. $\mathbf{B}$) the space of càdlàg functions from $[0,T]$ in $\mathbb{R}$ (resp. làdcàg) endowed with the topology of uniform convergence and we assume that:
(H4) There exists $K\geqslant 0$ such that:
(i) $g:\mathbf{D}\to\mathbb{R}$ is K-Lipschitz;
(ii) $f:[0,T]\times\mathbf{B}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and, for each t, for each $X\in\mathbf{B}$, $f(t,X,y,z)$ depends on X only up to time t, and moreover

$$|f(t,X,y,z)-f(t,X',y',z')|\leqslant K\left(\sup_{0\leqslant s\leqslant t}|X_s-X'_s|+|y-y'|+|z-z'|\right),$$

for all $(X,X')\in\mathbf{B}$, for all (y,z), (y',z').

Let $\{(Y_t,Z_t)\}_{0\leqslant t\leqslant T}$ be the solution to the BSDE

$$Y_t = g(W)+\int_t^T f(r,W,Y_r,Z_r)\,\mathrm{d}r-\int_t^T Z_r\,\mathrm{d}W_r, \quad 0\leqslant t\leqslant T,$$

and let $\{(Y_t^n,Z_t^n,N_t^n)\}_{0\leqslant t\leqslant T}$ be the solution to the discrete BSDE

$$Y_t^n = g(S^n)+\int_t^T f(r,S_-^n,Y_{r-}^n,Z_r^n)\,\mathrm{d}\langle S^n\rangle_r-\int_t^T Z_r^n\,\mathrm{d}S_r^n-\int_t^T \mathrm{d}N_r^n, \quad 0\leqslant t\leqslant T.$$

We work with the natural filtrations of S^n and W. Let us mention that, in this context, $N^n\equiv 0$ if and only if the real random variable ζ_1 takes only two values. We have the following result

Corollary 14. *Let the assumptions* (H4) *hold. The sequence* $\{(Y^n,\int_0^\cdot Z_r^n\,\mathrm{d}S_r^n,N^n)\}_n$ *converges in law to* $(Y,\int_0^\cdot Z_r\,\mathrm{d}W_r,0)$ *for the topology of uniform convergence on* $\mathbf{D}(\mathbb{R}^3)$.

Proof. Let us notice that the laws of the solutions (Y,Z) and (Y^n,Z^n,N^n) to the previous BSDEs depend only on $(\mathbb{P}_W,g^{-1}(\mathbb{P}_W),f)$ and $(\mathbb{P}_{S^n},g^{-1}(\mathbb{P}_{S^n}),f)$ where $g^{-1}(\mathbb{P}_W)$ (resp. $g^{-1}(\mathbb{P}_{W^n})$) is the law of $g(W)$ (resp. $g(S^n)$). So, as far as the convergence in law is concerned, we can consider these equations on any probability space.

But, from Donsker's theorem and Skorokhod representation theorem, there exists a probability space, with a Brownian motion W and a sequence of i.i.d. sequences $(\zeta^n)_n$

such that the processes

$$W_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \zeta_k^n, \quad 0 \leqslant t \leqslant T,$$

satisfy

$$\sup_{0 \leqslant t \leqslant T} |W_t^n - W_t| \to 0, \quad \text{as } n \to \infty,$$

in probability as well as in L^2 since ζ is in $L^{2+\delta}$.

It remains to solve the BSDEs on this space with respect to the filtrations generated by W^n and W and to apply Theorem 12 to obtain the convergence of $(Y^n, \int_0^\cdot Z_r^n \, dW_r^n, N^n)$ to $(Y, \int_0^\cdot Z_r \, dW_r, 0)$ in $\mathscr{S}^2(\mathbb{R}^3)$. Indeed, (H1)(ii) is satisfied with $\rho(x) = x$ and $a_n = 1/n$ since $\langle W^n \rangle_t = [nt]/n$.

This convergence implies the convergence of $\{(Y^n, \int_0^\cdot Z_r^n \, dS_r^n, N^n)\}_n$ to $(Y, \int_0^\cdot Z_r \, dW_r, 0)$ in law for the topology of uniform convergence on $\mathbf{D}(\mathbb{R}^3)$. □

5.3. Other examples

5.3.1. Approximation by diffusions

We give an example where the approximation of the Brownian motion does not come from a discrete model. Here, we are on a fixed complete probability space. Let us consider $\sigma^n : \mathbb{R} \to \mathbb{R}$ which is bounded by K and K-Lipschitz. We assume that $\sigma^n \to 1$ uniformly on compact sets of $\mathbb{R}$. W^n is the solution to the SDE

$$W_t^n = \int_0^t \sigma^n(W_r^n) \, dW_r, \quad t \geqslant 0.$$

It follows from stability results on SDEs see e.g. (Protter, 1990, p. 207) that $\{W_t^n\}_{0 \leqslant t \leqslant T}$ converges to $\{W_t\}_{0 \leqslant t \leqslant T}$ in $\mathscr{S}^p(\mathbb{R})$ for each real $p \geqslant 1$. The assumption (H1)(ii) is satisfied with $\rho(x) = Kx$ and $a_n = 0$.

Thus, we can apply Theorem 12 to this situation if for instance (H4) is satisfied.

5.3.2. Approximation by Poisson processes

For this last example, we consider a Poisson process $\{P_t\}_{t \geqslant 0}$ with intensity 1 and we define a process $\{W_t^n\}_{0 \leqslant t \leqslant T}$ as follows

$$W_t^n = \frac{1}{\sqrt{n}}(P_{nt} - nt), \quad 0 \leqslant t \leqslant T.$$

The martingale $\{W_t^n\}_{0 \leqslant t \leqslant T}$ has the predictable representation property and converges weakly to a Brownian motion $\{W_t\}_{0 \leqslant t \leqslant T}$.

Let us consider $\{Y_t^n, Z_t^n\}_{0 \leqslant t \leqslant T}$ (there is no N^n here because W^n has the predictable representation property) the solution to the BSDE

$$Y_t^n = g(W^n) + \int_t^T f(r, W_-^n, Y_{r-}^n, Z_r^n) \, dr - \int_t^T Z_r^n \, dW_r^n, \qquad 0 \leqslant t \leqslant T.$$

Under the assumption (H4), we can prove that the sequence $\{(Y_t^n, \int_0^t Z_r^n \, dW_r^n)\}_{0 \leqslant t \leqslant T}$ converges in law for the topology of uniform convergence to the solution $\{(Y_t, \int_0^t Z_r \, dW_r)\}_{0 \leqslant t \leqslant T}$ to the BSDE (12) with $\xi = g(W)$.

The method of proof is the same as in Section 5.2 since $\langle W^n \rangle_t = t$.

6. Concluding remark

A possible extension of this work consists in replacing the standard Brownian motion by a more general martingale. If this martingale is assumed to be continuous and to have the predictable representation property, the results of the paper still hold under the property of convergence of filtrations. However, in this context, we do not know if the result of Proposition 3 concerning the weak convergence of filtrations still holds. Nevertheless, this property is satisfied in many examples: see e.g. (Coquet et al., 2001, Propositions 1–6).

The possibility of further extension lies on the corresponding extension of Theorem 5. But this seems to be a difficult problem as pointed out in Jacod et al. (2000).

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Appendix A. Theorem 1, Proposition 3: proofs

In this section, we give the proof of Theorem 1 and the proof of Proposition 3. Let us first recall the statement.

Theorem 1. *Let $\{X_t^n\}_{0\leqslant t\leqslant T}$ be a sequence of càdlàg $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-integrable processes with finite variation and $X_0^n=0$ which converges in $\mathscr{S}^1(\mathbb{R})$ to the continuous $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$-adapted process $\{X_t\}_{0\leqslant t\leqslant T}$. In addition, we assume that $\{\bar{X}_t^n\}_{0\leqslant t\leqslant T}$, $\bar{X}_t^n=\mathrm{Var}(X^n)_t=\int_0^t|\,\mathrm{d}X_s^n|$, is* **C***-tight, and the variables $\bar{X}_T^n$ are uniformly integrable.*

Then the predictable compensator $\{P_t^n\}_{0\leqslant t\leqslant T}$ of $\{X_t^n\}_{0\leqslant t\leqslant T}$ converges to $\{X_t\}_{0\leqslant t\leqslant T}$ in $\mathscr{S}^1(\mathbb{R})$.

Proof. Firstly, let us show that the jumps of P^n go to 0 in ucp. Indeed, ΔP^n is indistinguishable from the predictable projection of ΔX^n (see e.g. Jacod and Shiryaev, 1987, p. 33); this means that, for each predictable stopping time τ, $\Delta P_\tau^n=\mathbb{E}(\Delta X_\tau^n\,|\,\mathscr{F}_{\tau-}^n)$. Let us fix $\varepsilon>0$ and consider the following stopping time $\tau=\inf\{t>0,\ |\Delta P_t^n|\geqslant\varepsilon\}\wedge T$. Since $\{t>0,\ |\Delta P_t^n|\geqslant\varepsilon\}$ is a finite set (P^n is a process with finite variation), $[\![\tau]\!]=\{(\omega,\tau(\omega)),\ \omega\in\Omega\}$ is a subset of $H=\{(\omega,t),\ |\Delta P_t^n|\geqslant\varepsilon\}\cup\{(\omega,T),\ \omega\in\Omega\}$. Since τ is the début of the predictable set H and $[\![\tau]\!]\subset H$, τ is a predictable stopping time: see e.g. (Dellacherie and Meyer, Remark (d) after the corrections) or (Jacod, 1979, Theorem 1.14). Thus, we have

$$\mathbb{P}\left(\sup_{t\in[0,T]}|\Delta P_t^n|\geqslant\varepsilon\right)=\mathbb{P}(|\Delta P_\tau^n|\geqslant\varepsilon)=\mathbb{P}\{|\mathbb{E}(\Delta X_\tau^n\,|\,\mathscr{F}_{\tau-}^n)|\geqslant\varepsilon\},$$

this inequality yields

$$\mathbb{P}\left(\sup_{t\in[0,T]}|\Delta P_t^n|\geqslant\varepsilon\right)\leqslant\mathbb{E}\left[\sup_{t\in[0,T]}|\Delta X_t^n|\right]\Big/\varepsilon,$$

which tends to 0 since X^n converges in $\mathscr{S}^1(\mathbb{R})$ to the continuous process X.

To get the **C**-tightness of the sequence (P^n), it is sufficient to prove that this sequence is **D**-tight. For this, we will use Aldous' criterion; see e.g. (Jacod and Shiryaev, 1987, p. 320). Note first that, by construction, if we denote by P_+^n and P_-^n the predictable compensators of the increasing processes $(\bar{X}^n+X^n)/2$ and $(\bar{X}^n-X^n)/2$, then P_+^n and P_-^n are increasing and $P^n=P_+^n-P_-^n$. This implies that the process $V^n-\mathrm{Var}(P^n)$ is increasing where $V_s^n=(P_+^n+P_-^n)_s$. Let us fix $\theta>0$. If σ and τ are two stopping times such that $0\leqslant\sigma\leqslant\tau\leqslant\sigma+\theta\leqslant T$, we have, since $]\!]\sigma,\tau]\!]$ is predictable and P^n is càd,

$$\mathbb{E}[|P_\tau^n-P_\sigma^n|]\leqslant\mathbb{E}\left[\int\mathbf{1}_{\sigma<t\leqslant\tau}\,\mathrm{d}\mathrm{Var}(P^n)_t\right]\leqslant\mathbb{E}\left[\int\mathbf{1}_{\sigma<t\leqslant\tau}\,\mathrm{d}V_t^n\right].$$

On the other hand, $V^n=P_+^n+P_-^n$ is the predictable compensator of $\bar{X}^n$ and thus

$$\mathbb{E}[|P_\tau^n-P_\sigma^n|]\leqslant\mathbb{E}[\bar{X}_\tau^n-\bar{X}_\sigma^n]=\mathbb{E}\left[\int\mathbf{1}_{\sigma<t\leqslant\tau}\,\mathrm{d}V_t^n\right],$$

the right-hand side tends uniformly to 0 as θ tends to 0 since $\bar{X}^n$ is **C**-tight and uniformly integrable. Hence P^n is **D**-tight and thus **C**-tight.

Define the $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-martingale $\{M_t^n\}_{0\leqslant t\leqslant T}$ by setting, for $0\leqslant t\leqslant T$, $M_t^n=X_t^n-P_t^n$. The sequence (X^n,P^n,M^n) is **C**-tight. Consider a limit point of $\{X^n,P^n,M^n\}$, say (X,P,M) which is a continuous process in view of the **C**-tightness. In order to verify that M is a martingale it is enough to check, according to Proposition 1.12 in Jacod and Shiryaev (1987, p. 484), that $\sup_t|M_t^n|$ is uniformly integrable. To see this, let us remark first that $\sup_t|X_t^n|$ is uniformly integrable since X^n converges to X in $\mathscr{S}^1(\mathbb{R})$. Moreover $\sup_t|P_t^n|$ is also uniformly integrable; indeed from Dellacherie and Meyer (1980, remarques 100, p. 182), we have $\mathbb{E}[(V_T^n-\lambda)^+]\leqslant\mathbb{E}[\bar{X}_T^n\mathbf{1}_{V_T^n\geqslant\lambda}]$. But we know already that $\bar{X}_T^n$ is uniformly integrable and thus we have, since $\mathbb{E}[V_T^n]=\mathbb{E}[\bar{X}_T^n]$,

$$\lim_{\lambda\to\infty}\sup_n\mathbb{E}[(V_T^n-\lambda)^+]=0,$$

which gives the uniform integrability of V_T^n (Lemma 1.11 in Jacod and Shiryaev (1987, p. 482); since, as we seen before, $V_T^n\geqslant\mathrm{Var}(P^n)_T\geqslant\sup_t|P_t^n|$ we get the uniform integrability of $\sup_t|P^n|$.

Using Skorokhod representation theorem, we may construct a probability space on which $(\tilde{X}^n,\tilde{P}^n,\tilde{M}^n)$ converges to $(\tilde{X},\tilde{P},\tilde{M})$ in ucp (the converging subsequence is still indexed by n) where $(\tilde{X}^n,\tilde{P}^n,\tilde{M}^n)$ (resp. $(\tilde{X},\tilde{P},\tilde{M})$) and (X^n,P^n,M^n) (resp. (X,P,M)) have the same law. We have $\tilde{X}=\tilde{M}+\tilde{P}$ and all these processes are predictable (since continuous and adapted) w.r.t. $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$ the filtration of the past of $(\tilde{X},\tilde{P})$. Since $\tilde{X}$ and $\tilde{P}$ are processes with finite variation paths, the uniqueness of the $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-Doob-Meyer decomposition leads to $\tilde{X}=\tilde{P}$, $\tilde{M}=0$.

It follows that M^n converges in distribution to 0 for the topology of uniform convergence and then that M^n converges to 0 in ucp. Hence $P^n=X^n-M^n$ converges in ucp to X. We get also the convergence in $\mathscr{S}^1(\mathbb{R})$ since we have already proved that $\sup_t|P_t^n|$ is uniformly integrable. □

Remark A.1. It is worth noticing that the continuity of the limit X is very important in the previous result. One can construct a sequence of processes (X^n) such that X^n converges to X in $\boldsymbol{D}$, X^n is bounded by 1 and for which the predictable compensator P^n does not converge. We refer to Jacod et al. (1983, Counter-example 2.9).

We proceed now to the proof of Proposition 3 that we also recall.

Proposition 3. *Let us consider, on the same probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, *a standard Brownian motion* $\{W_t\}_{0\leqslant t\leqslant T}$ *with its natural filtration* $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$, *a sequence of filtrations* $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ *and a sequence* $\{W_t^n\}_{0\leqslant t\leqslant T}$ *of square integrable* $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$*-martingales. We suppose that* W^n converges to W in $\mathscr{S}^2(\mathbb{R})$.

Then $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$ *weakly converges to* $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$.

Proof. We start the proof by showing that we can reduce to the case where $\sup_{\mathbb{N}} \langle W^n\rangle_T \leqslant M$ for some real M. This is done in two steps. Let us show firstly that we can assume that $\sup_n \sup_t |\Delta W_t^n| \leqslant 2$. Indeed, let us denote $A_t^n = \sum_{s\leqslant t} \Delta W_s^n \mathbf{1}_{|\Delta W_s^n|>1}$ and let us consider $\widetilde{A^n}$ its predictable compensator. We write W^n as follows:

$$W_t^n = A_t^n - \widetilde{A_t^n} + M_t^n, \quad 0 \leqslant t \leqslant T.$$

$A^n - \widetilde{A^n}$ and M^n are $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-martingales and it is not hard to check that $\sup_t |\Delta\widetilde{A_t^n}| \leqslant 1$, $\sup_t |\Delta M_t^n| \leqslant 2$ (see e.g. (Jacod, 1979, p. 30–31)). Since $\mathbb{E}[|\widetilde{A^n}_T|^2] \leqslant 4\,\mathbb{E}[|A_T^n|^2]$ (see e.g. Dellacherie and Meyer, 1980, Eq. (100.1)), we have moreover

$$\mathbb{E}[[A^n - \widetilde{A^n}]_T] \leqslant 10\mathbb{E}\left[\sum_{s\leqslant T} (\Delta W_s^n)^2 \mathbf{1}_{|\Delta W_s^n|>1}\right] \leqslant 10\mathbb{E}[[W^n]_T].$$

Hence $A^n - \widetilde{A^n}$ and M^n are square integrable martingales. Moreover, the convergence of W^n to W in $\mathscr{S}^2(\mathbb{R})$ together with the continuity of W imply that $\sum_{s\leqslant T} (\Delta W_s^n)^2\, \mathbf{1}_{|\Delta W_s^n|>1}$ converges to 0 in probability as well as in L^1; for the convergence in L^1, the sum is bounded by $[W^n]_T$ which converges to $[W]$ in L^1 since W^n converges to W in $\mathscr{S}^2(\mathbb{R})$. Thus, $A^n - \widetilde{A^n}$ converges to 0 and M^n to W both of them in $\mathscr{S}^2(\mathbb{R})$. So, we will assume now that the jumps of W^n are bounded by 2.

Let us consider, for each n the $\{\mathscr{F}_t^n\}_{t\in[0,T]}$-stopping time $T_n = \inf\{t \in [0,T], \langle W^n\rangle_t \geqslant 2T\}$ $(\inf \emptyset = T)$. Since the jumps of W^n are bounded by 2, those of $[W^n]$ are bounded by 4 and thus the jumps of $\langle W^n\rangle$ are also bounded by 4. It follows that

$$\langle W^{n,T_n}\rangle_T = \langle W^n\rangle_{T_n} \leqslant 2T + 4.$$

But since $\langle W^n\rangle_t \to t$, it is clear that $\mathbb{P}[T_n < T] \to 0$ and that W^{n,T_n} converges to W in $\mathscr{S}^2(\mathbb{R})$. Thus we will assume that $\sup_{\mathbb{N}} \langle W^n\rangle_T$ is bounded by some real M.

Using Lemma 3 from Coquet et al. (2001), it is sufficient to prove that, for $k \in \mathbb{N}^*$ and $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$,

$$\mathbb{E}\left(\exp\left\{ \mathrm{i}\sum_{l=1}^k \lambda_l W_{t_l} \right\} \middle| \mathscr{F}_\cdot^n \right) \to \mathbb{E}\left(\exp\left\{ \mathrm{i}\sum_{l=1}^k \lambda_l W_{t_l} \right\} \middle| \mathscr{F}_\cdot \right) \quad \text{in ucp.}$$

Since the limit process is a continuous martingale, it is enough to prove pointwise convergence (see Aldous, 1989) that is to say, for each $t \in [0,T]$,

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{l=1}^{k}\lambda_l W_{t_l}\right\}\middle|\mathscr{F}_t^n\right) \to \mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{l=1}^{k}\lambda_l W_{t_l}\right\}\middle|\mathscr{F}_t\right) \quad \text{in probability.}$$

Moreover, as we have the convergence of W^n to W in ucp, we can replace W by W^n in the first conditional expectation, and finally we have to prove that, for each real λ and for each (s,t) s.t $0 \leqslant s < t \leqslant T$, we have, in probability, as $n \to \infty$,

$$\mathbb{E}(\exp\{\mathrm{i}\lambda(W_t^n - W_s^n)\}\,|\mathscr{F}_s^n) \to \mathbb{E}(\exp\{\mathrm{i}\lambda(W_t - W_s)\}\,|\,\mathscr{F}_s) = \exp\{-\lambda^2(t-s)/2\}.$$

If $s > 0$, we can consider $\bar{W}_t^n = W_{t+s}^n - W_s^n$, $\bar{W}_t = W_{t+s} - W_s$, $\bar{\mathscr{F}}_t^n = \mathscr{F}_{t+s}^n$ and $\bar{\mathscr{F}}_t = \mathscr{F}_{t+s}$, to reduce to the case $s=0$. So we will assume, without loss of generality, that $s=0$.

To prove that, let us fix (λ,t) and let us denote $\mathscr{E}(\mathrm{i}\lambda W^n)$ the Doléans-exponential of $\mathrm{i}\lambda W^n$, namely, for $0 \leqslant t \leqslant T$,

$$\mathscr{E}(\mathrm{i}\lambda W^n)_t = \exp\{\mathrm{i}\lambda W_t^n + \lambda^2\langle W^{n,c}\rangle_t/2\}\prod_{0<u\leqslant t}(1+\mathrm{i}\lambda\Delta W_u^n)\mathrm{e}^{-\mathrm{i}\lambda\Delta W_u^n}.$$

$\mathscr{E}(\mathrm{i}\lambda W^n)$ is a square integrable complex $\{\mathscr{F}_t^n\}_{0\leqslant t\leqslant T}$-martingale and moreover there exists a constant C such that

$$\forall n \in \mathbb{N},\ \ \forall 0 \leqslant t \leqslant T, \quad \mathbb{E}[|\mathscr{E}(\mathrm{i}\lambda W^n)_t|^2] \leqslant C.$$

Indeed, $\{\mathscr{E}(\mathrm{i}\lambda W^n)_t\}_{0\leqslant t\leqslant T}$ is a local martingale and we have $\mathscr{E}(\mathrm{i}\lambda W^n)_0 = 1$. Moreover,

$$|\mathscr{E}(\mathrm{i}\lambda W^n)_t|^2 = \exp\{\lambda^2\langle W^{n,c}\rangle_t\}\prod_{0<u\leqslant t}(1+\lambda^2|\Delta W_u^n|^2) = \mathscr{E}(\lambda^2[W^n])_t.$$

Now $[W^n]$ is an integrable semimartingale whose decomposition is $[W^n] = ([W^n] - \langle W^n\rangle) + \langle W^n\rangle$. It follows from Jacod (1979, Corollaire 6.35) that $\mathscr{E}(\lambda^2[W^n])$ has the following multiplicative decomposition $\mathscr{E}(\lambda^2[W^n]) = \mathscr{E}(N)\,\mathscr{E}(\lambda^2\langle W^n\rangle)$ where N is a local martingale (N is given explicitly by the formula $N_\cdot = \int_0^\cdot \lambda^2(1+\lambda^2\Delta\langle W^n\rangle_s)^{-1}\mathrm{d}([W^n] - \langle W^n\rangle)_s$). Since $\langle W^n\rangle$ is an increasing process, we have $\mathscr{E}(\lambda^2\langle W^n\rangle)_t \leqslant \exp(\lambda^2\langle W^n\rangle_t) \leqslant C$ in view of the boundedness of $\langle W^n\rangle_T$. Thus,

$$|\mathscr{E}(\mathrm{i}\lambda W^n)_t|^2 \leqslant C\,\mathscr{E}(N)_t.$$

Moreover, $\mathscr{E}(N)$ is a local martingale with $\mathscr{E}(N)_0 = 1$. Fatou's lemma implies that

$$\forall t \geqslant 0, \quad \mathbb{E}[|\mathscr{E}(\mathrm{i}\lambda W^n)_t|^2] \leqslant C.$$

It follows that $\{\mathscr{E}(\mathrm{i}\lambda W^n)_t\}_{0\leqslant t\leqslant T}$ is a square integrable martingale; then $\mathbb{E}(\mathscr{E}(\mathrm{i}\lambda W^n)_t\,|\,\mathscr{F}_0^n) = 1$.

Let us set

$$U_n = \exp\{-\lambda^2\langle W^{n,c}\rangle_t/2\}\prod_{0<u\leqslant t}(1+\mathrm{i}\lambda\Delta W_u^n)^{-1}\mathrm{e}^{\mathrm{i}\lambda\Delta W_u^n},$$

and write $\exp\{\mathrm{i}\lambda W_t^n\} = \mathscr{E}(\mathrm{i}\lambda W^n)_t\,U_n$. With this notation we have to prove that

$$\mathbb{E}(\exp\{\mathrm{i}\lambda W_t^n\}\,|\,\mathscr{F}_0^n) = \mathbb{E}(\mathscr{E}(\mathrm{i}\lambda W^n)_t\,U_n\,|\,\mathscr{F}_0^n) \to \exp\{-\lambda^2 t/2\}.$$

We have

$$\mathbb{E}[|\mathbb{E}(\mathscr{E}(\mathrm{i}\lambda W^n)_t\, U_n \mid \mathscr{F}_0^n) - \mathrm{e}^{-\lambda^2 t/2}|] \leqslant \mathbb{E}[|\mathscr{E}(\mathrm{i}\lambda W^n)_t\,(U_n - \mathrm{e}^{-\lambda^2 t/2})|]$$

$$\leqslant \mathbb{E}[|\mathscr{E}(\mathrm{i}\lambda W^n)_t|^2]^{1/2}\mathbb{E}[|U_n - \mathrm{e}^{-\lambda^2 t/2}|^2]^{1/2}.$$

Notice that obviously $|U_n| \leqslant 1$. Let us prove that U_n converges to $\exp\{-\lambda^2 t/2\}$ in probability. For this we write

$$U_n = \exp\{-\lambda^2[W^n]_t/2\} \prod_{0<u\leqslant t} (1 + \mathrm{i}\lambda\Delta W_u^n)^{-1}\mathrm{e}^{\mathrm{i}\lambda\Delta W_u^n}\,\mathrm{e}^{\lambda^2|\Delta W_u^n|^2/2}.$$

Since W^n converges to W in ucp, $[W^n]$ converges to t in ucp and then

$$\exp\{-\lambda^2[W^n]_t/2\} \to \exp\{-\lambda^2 t/2\} \quad \text{in probability.}$$

It remains only to prove that, as $n \to \infty$,

$$\prod_{0<u\leqslant t} (1 + \mathrm{i}\lambda\Delta W_u^n)^{-1}\mathrm{e}^{\mathrm{i}\lambda\Delta W_u^n}\,\mathrm{e}^{\lambda^2|\Delta W_u^n|^2/2} \to 1 \quad \text{in probability.}$$

Let us write

$$\prod_{0<u\leqslant t} (1 + \mathrm{i}\lambda\Delta W_u^n)^{-1}\mathrm{e}^{\mathrm{i}\lambda\Delta W_u^n}\,\mathrm{e}^{\lambda^2|\Delta W_u^n|^2/2}$$

$$= \exp\left\{-\sum_{0<u\leqslant t} (\log(1 + \mathrm{i}\lambda\Delta W_u^n) - \mathrm{i}\lambda\Delta W_u^n - \lambda^2|\Delta W_u^n|^2/2)\right\}.$$

Since we have, for all $x \in \mathbb{R}$, $|\log(1 + \mathrm{i}x) - \mathrm{i}x - x^2/2| \leqslant |x|^3/3$, we get the inequality

$$\sum_{0<u\leqslant t} |\log(1+\mathrm{i}\lambda\Delta W_u^n) - \mathrm{i}\lambda\Delta W_u^n - \lambda^2|\Delta W_u^n|^2/2| \leqslant C \sup_{0\leqslant u\leqslant T} |\Delta W_u^n| \sum_{0<u\leqslant T} |\Delta W_u^n|^2$$

$$\leqslant C \sup_{0\leqslant u\leqslant T} |\Delta W_u^n|\,[W^n]_T.$$

Since W^n converges to W in ucp and $[W^n]_T$ converges to T in probability we deduce that the last sum goes to 0 in probability. The proof of this proposition is complete. □

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