A spectral gap property for random walks under unitary representations

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Abstract

Let G be a locally compact group and μ a probability measure on G, which is not assumed to be absolutely continuous with respect to Haar measure. Given a unitary representation $(\pi, \mathcal{H})$ of G, we study spectral properties of the operator $\pi(\mu)$ acting on $\mathcal{H}$. Assume that μ is adapted and that the trivial representation 1_G is not weakly contained in the tensor product $\pi \otimes \overline{\pi}$. We show that $\pi(\mu)$ has a spectral gap, that is, for the spectral radius $r_{\text{spec}}(\pi(\mu))$ of $\pi(\mu)$, we have $r_{\text{spec}}(\pi(\mu)) < 1$. This provides a common generalization of several previously known results. Another consequence is that, if G has Kazhdan's Property (T), then $r_{\text{spec}}(\pi(\mu)) < 1$ for every unitary representation π of G without finite dimensional subrepresentations. Moreover, we give new examples of so-called identity excluding groups.

1 Introduction

Let G be a locally compact group and μ a probability measure defined on the Borel subsets of G. We will always assume that μ is *adapted*; by this, we mean that the subgroup generated by the support $\text{supp}(\mu)$ of μ is dense in G. We will also consider the stronger condition that $\text{supp}(\mu)$ is not contained in the coset of a proper closed subgroup of G. In this case, we say that μ is *strongly adapted*.

Let $(\pi, \mathcal{H})$ be a unitary representation of G, that is, a strongly continuous homomorphism π from G to the unitary group of a Hilbert space $\mathcal{H}$. A bounded operator $\pi(\mu)$ on $\mathcal{H}$ is defined by

$$\langle \pi(\mu)\xi, \eta\rangle = \int_G \langle \pi(x)\xi, \eta\rangle d\mu(x), \qquad \forall \xi, \eta \in \mathcal{H}.$$

Let $r_{\text{spec}}(\pi(\mu))$ be the spectral radius of $\pi(\mu)$. We have $r_{\text{spec}}(\pi(\mu)) \leq 1$, since $\pi(\mu)$ is a contraction.

The main result of this paper is the following theorem. Recall that the (inner) tensor product $\pi \otimes \overline{\pi}$ is the representation acting on the Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$ by $\pi \otimes \overline{\pi}(x) = \pi(x) \otimes \pi(x)$ for all $x \in G$. It can be realized on the space $L^2(\mathcal{H})$ of all Hilbert-Schmidt operators on $\mathcal{H}$ by

$$\pi \otimes \overline{\pi}(x)T = \pi(x)T\pi(x^{-1}), \qquad x \in G,\, T \in L^2(\mathcal{H}).$$

As is well-known and easy to see, $\pi \otimes \overline{\pi}$ has a non-zero invariant vector if and only if π contains a non-zero finite dimensional subreprentation.

Recall also that a unitary representation $(\pi, \mathcal{H})$ of G almost has invariant vectors if, for every compact subset K of G and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\max_{x \in K} \|\pi(x)\xi - \xi\| < \varepsilon.$$

Observe that π almost has invariant vectors if and only if the trivial one-dimensional representation 1_G is weakly contained in π, in the sense of [Fell62].

Theorem 1 *Let G be a locally compact group and μ an adapted probability measure on G. Let $(\pi, \mathcal{H})$ be a unitary representation of G. Assume that the following condition $(*)$ is satified:*

$(*)$ $\pi \otimes \overline{\pi}$ *does not almost have invariant vectors.*

Then $r_{\text{spec}}(\pi(\mu)) < 1$. If μ is strongly adapted, then $\|\pi(\mu)\| < 1$.

Remark 2 (i) Observe that we do *not* assume that μ is absolutely continuous with respect to a Haar measure on G. Indeed, in this case, the result is well-known and easy to prove. In fact, a stronger result is true: if μ is absolutely continuous, and strongly adapted and if, instead of Condition $(*)$, we assume that π does not almost have invariant vectors, then $\|\pi(\mu)\| < 1$ (see [Guiv02, Proposition 4.1] or [BeHV05, Appendix G.4]).

(ii) In general, we cannot replace Condition $(*)$ by the weaker condition that π does not almost have invariant vectors. For a simple counterexample, take $G = \mathbb{S}^1$ the circle group, $\pi = \chi$ a non-trivial character of G, and $\mu = \delta_z$ the Dirac measure at a point z generating a dense subgroup of $\mathbb{S}^1$. In this case, $r_{\text{spec}}(\pi(\mu)) = 1$.

(iii) In view of (ii), one can ask, whether we have $r_{\text{spec}}(\pi(\mu)) < 1$, if π does not almost have invariant vectors and if we assume moreover that μ is strongly adapted. The answer is still negative as the following example shows. Let $G = \mathbb{S}^1$ and let π be the restriction of the regular representation of $\mathbb{S}^1$ to the subspace $L^2_0(\mathbb{S}^1)$ of all functions in $L^2(\mathbb{S}^1)$ which are orthogonal to the constants. Let μ be any probability measure on $\mathbb{S}^1$ with finite support. Then there exists a sequence $(n_k)_k$ in $\mathbb{Z} \setminus \{0\}$ such that $\lim_k z^{n_k} = 1$ for all $z \in \text{supp}(\mu)$. This implies that 1 is a spectral value of $\pi(\mu)$.

(iv) Condition $(*)$ in the theorem above was studied by the first author in [Bekk90]. More specifically, a unitary representation π is amenable in the sense of this paper if $\pi \otimes \overline{\pi}$ almost has invariant vectors. In other words, the theorem can be stated as follows: given an adapted probability measure μ on G, a unitary representation π of G is amenable if $r_{\text{spec}}(\pi(\mu)) = 1$.

(v) Condition $(*)$ *never* holds if G is amenable. More precisely, it was shown in [Bekk89, Theorem 1] that, if π is any unitary representation of an amenable locally compact group, then $\pi \otimes \overline{\pi}$ almost has invariant vectors.

(vi) Actually, as the proof below shows, we also obtain (under the assumptions of the theorem) the result that $r_{\text{spec}}(\pi \otimes \overline{\pi}(\mu)) < 1$ (and $\|\pi \otimes \overline{\pi}(\mu)\| < 1$ in case μ is strongly adapted).

The theorem above leads to spectral gap properties (see Corollaries 3 and 10) which are somewhat unexpected for general probability measures. This may be seen as an indication of the prevalence of exponential mixing for actions of non amenable groups (compare [Dolgo02]).

We will derive several corollaries of our result. The first one deals wih Kazhdan groups. Recall that a locally compact group G is said to have Kazhdan's Property (T), if whenever a unitary representation π of G almost has invariant vectors, then π has a non-zero invariant vector (see [Kazh67], [HaVa89], or [BeHV05]). Examples of locally compact groups with Property (T) are all simple real Lie of real rank greater or equal to 2. This is for instance the case for the groups $SL_n(\mathbb{R})$ for $n \geq 3$ and $Sp_{2n}(\mathbb{R})$ for $n \geq 2$. An important fact is that Property (T) is inherited by lattices. (Recall that a discrete group Γ is a lattice in a locally compact group G if the homogeneous space G/Γ has a G-invariant probability measure.) Thus, $SL_n(\mathbb{Z})$ for $n \geq 3$ and $Sp_{2n}(\mathbb{Z})$ for $n \geq 2$ have Property (T). Examples of Lie groups which have

Property (T) and are not semisimple are the semi-direct products $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ for $n \geq 3$.

The following corollary is an immediate consequence of the theorem above.

Corollary 3 *Let G be a locally compact group with Property (T), and let μ be an adapted probability measure on G. Let π be unitary representation of G without finite dimensional subrepresentations. Then $r_{\text{spec}}(\pi(\mu)) < 1$. If μ is strongly adapted, then $\|\pi(\mu)\| < 1$.*

Remark 4 (i) Unfortunately, the previous corollary does not apply to the important case of a compact group; indeed, every unitary representation of such a group is a direct sum of finite dimensional ones. It should be mentioned that, for the orthogonal groups $SO(n, \mathbb{R})$, $n \geq 3$, the problem of finding a probability measure μ with a spectral gap for the corresponding operator acting on the subspace orthogonal to the constants in $L^2(\mathbb{S}^{n-1})$ is related to the so-called Banach-Ruziewicz problem (see [Lubo94], [Sarn90]).

(ii) The previous corollary may be reformulated as follows, at least when G is second countable. Let D be a dense subgroup of the Kazhdan group G. For any unitary representation π of G without finite dimensional subrepresentation, the restriction of π to D does not weakly contain the trivial representation 1_D, where D is equipped with the discrete topology (for more details, see Section 4).

Corollary 3 can be generalized to the case where G has a closed subgroup H such that the pair (G, H) has Property (T). Recall that (G, H) has Property (T) (or, as some authors say, G has the relative Property (T) with respect to H) if, whenever a unitary representation π of G almost has invariant vectors, then π has a non-zero H-invariant vector (see [HaVa89] or [BeHV05]). Observe that G has Property (T) if and only if the pair (G, G) has Property (T). Another example of a pair with Property (T) is the pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$, for the semi-direct product $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ with respect to the natural action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$.

Corollary 5 *Let G be a locally compact group and H a closed subgroup such that the pair (G, H) has Property (T), and let μ be an adapted probability measure on G. Let π be unitary representation of G. Assume that the restriction of π to H has no finite dimensional subrepresentations. Then $r_{\text{spec}}(\pi(\mu)) < 1$. If μ is strongly adapted, then $\|\pi(\mu)\| < 1$.*

As we now see, Theorem 1 above unifies and generalizes several previously known results. When π is the left regular representation λ_G of G acting on $L^2(G)$, we recover the result [DeGu73, Théorème] by Derrienic and the second author (for a similar result, see also [BeCh73]). The case where G is discrete and μ is symmetric is due to Kesten [Kest59]. Observe that $\lambda_G(\mu)$ is the operator $\xi \mapsto \mu * \xi$ on $L^2(G)$ given by left convolution with μ.

Corollary 6 ***([DeGu73])*** *Let G be a non-amenable locally compact group, and let μ be an adapted probability measure on G. Then $r_{\text{spec}}(\lambda_G(\mu)) < 1$. If μ is strongly adapted, then $\|\lambda_G(\mu)\| < 1$.*

Our proof of Theorem 1 may be seen as a non-commutative version of the proof of [DeGu73, Théorème]. It relies on the characterization of amenable representations given in [Bekk90] in terms of existence of invariant means on appropriate operator algebras (see Theorem 13 below).

The previous corollary generalizes to homogeneous spaces which are not amenable in the sense of Eymard [Eyma72]. Even more generally, let (X, ν) be a measure space with a σ-finite measure ν. Assume that the separable locally compact group G acts measurably on X and that ν is quasi-invariant under the action of G. We say that the action of G on X is *co-amenable* if there exists a G-invariant mean on $L^\infty(X, \nu)$, that is, a positive linear functional M on $L^\infty(X, \nu)$ such that that $M(1_X) = 1$ and $M({}_g\varphi) = M(\varphi)$ for all g in G and φ in $L^\infty(X, \nu)$, where ${}_g\varphi(x) = \varphi(g^{-1}x)$. (This notion should not be confused with the well-established notion of an amenable action of G on X due to Zimmer; see [Zimm84].) A unitary representation λ_X of G is defined on $L^2(X, \nu)$ by

$$\lambda_X(g)\xi(x) = \sqrt{\frac{d\nu(g^{-1}x)}{d\nu(x)}}\xi(g^{-1}x), \qquad g \in G,\ x \in X,\ \xi \in L^2(X).$$

The following result was obtained in [Guiv80, Proposition 1] (see also [Guiv02, Theorem 4.15]).

Corollary 7 ***([Guiv80])*** *Assume that the action of G on X is not co-amenable and let μ be an adapted probability measure on G. Then $r_{\text{spec}}(\lambda_X(\mu)) < 1$. If μ is strongly adapted, then $\|\lambda_X(\mu)\| < 1$.*

The previous corollary is a direct consequence of Theorem 1 and of the following proposition which shows that Condition $(*)$ is equivalent to the non co-amenability of the action.

Proposition 8 *Let G be separable locally compact group acting measurably on a measure space (X, ν), where ν is a σ-finite measure on X which is quasi-invariant under the action of G. The following properties are equivalent:*

(i) the action of G on X is co-amenable;

(ii) the representation λ_X almost has invariant vectors;

(iii) the representation $\lambda_X \otimes \overline{\lambda}_X$ almost has invariant vectors.

The equivalence of (i) and (ii) in the previous proposition was also shown in [Guiv02, Theorem 4.15].

Remark 9 Assume that the action of G on the measure space (X, ν) is amenable in Zimmer's sense. The following remarkable result was shown in [Anan03, Corollary 3.2.2]: $r_{\text{spec}}(\lambda_X(\mu)) = r_{\text{spec}}(\lambda_G(\mu))$ for any adapted probability measure μ on G (the case of a discrete group was previously treated in [Kuhn94]). In particular, if G is non amenable, then $r_{\text{spec}}(\lambda_X(\mu)) < 1$, by Corollary 6.

We now turn to semisimple Lie groups. In this case, using [Bekk98, Lemma 4], we immediately obtain the following strengthening of Theorem 1. This result was shown in [Shal00, Theorem C]. The case of a semisimple Lie group with Property (T) was treated in [Nevo98, Theorem 1].

Corollary 10 ***([Shal00])*** *Let G be a semisimple real Lie group with finite centre and without compact factors. Let μ be an adapted probability measure on G. Let π be unitary representation of G which does not almost have invariant vectors. Then $r_{\text{spec}}(\pi(\mu)) < 1$. If μ is strongly adapted, then $\|\pi(\mu)\| < 1$.*

Actually, the previous result is proved in [Shal00] under the weaker assumption that the projection of μ to every simple factor of G is not supported on a closed amenable subgroup. It is worth mentioning that the articles [FuSh99] and [Shal98] contain interesting applications of the condition $\|\pi(\mu)\| < 1$ (or $\|\pi \otimes \overline{\pi}(\mu)\| < 1$) to random ergodic theorems.

Let G be a separable locally compact group acting measurably on a measure space (X, ν), where ν is now an invariant probability measure. As above, let λ_X be the natural representation of G on $L^2(X)$. Observe that $L^2(X)$ contains the constants functions $\mathbb{C}1_X$. Let λ_X^0 denote the restriction of λ_X to the closed invariant subspace

$$L_0^2(X) = \{\xi \in L^2(X) \ : \ \int_G f(x)d\nu(x) = 0\} = (\mathbb{C}1_X)^\perp.$$

As is well-known, the action of G is ergodic if and only if λ_X^0 has no non-zero invariant vectors, and the action is mixing if and only if $\lambda_X^0 \otimes \overline{\lambda}_X^0$ has no non-zero invariant vectors. Important examples of probability spaces X as above are the spaces $X = G/\Gamma$, where Γ is a lattice in a locally compact group G with action of G by left translations. When G is a semisimple real Lie group, it was shown in Lemma 3 of [Bekk98] that, for any lattice Γ, the representation λ_X^0 does not almost have invariant vectors. As a consequence of this fact, we obtain from Corollary 10 the following result, which was also observed in [Shal00] and [FuSh99].

Corollary 11 *Let G be a semisimple real Lie group with finite centre and without compact factors. Let μ be an adapted probability measure on G. Let Γ be a lattice in G and denote by $\lambda_{G/\Gamma}^0$ the representation of G on $L_0^2(G/\Gamma)$ Then $r_{\rm spec}(\lambda_{G/\Gamma}^0(\mu)) < 1$.*

As a further appplication of our results, we now show that some classes of groups have the so-called identity excluding property. This property appeared in several articles (see, for instance, [JoRT94], [LiWi95], [JaRW96]) in connection with the study of the behaviour of convolution powers of a probability measure. It also plays an important role in the study of equidistribution properties of random walks (see [Guiv73]). It is defined as follows. A locally compact group G is *identity excluding* if, for every irreducible unitary representation π of G with $\pi \neq 1_G$ and every dense subgroup D of G, the restriction of π to D does not almost have invariant vectors (D being equipped with the discrete topology).

Part (i) and Part (ii) of the following result are direct consequences of Corollary 3 and Corollary 10, and Part (iii) requires a further argument (see Section 4).

Corollary 12 *A secound countable locally compact group G is identity excluding in the following cases:*

(i) *G has Property (T);*

(ii) *G is a semisimple real Lie group with finite centre and without compact factors;*

(iii) $G = \mathbb{G}(\mathbb{K})$ *is the group of* $\mathbb{K}$*-rational points of a* $\mathbb{K}$*-isotropic simple algebraic group* $\mathbb{G}$ *over a local field* $\mathbb{K}$.

This paper is organized as follows In Section 2, we give the proof of Theorem 1, in Section 3 the proof of Proposition 8, and in Section 4 the proof of Corollary 12.

2 Proof of Theorem 1

Our proof of Theorem 1 is modelled after the proof of [DeGu73, Théorème]. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, that is, a norm closed self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$. Assume that $\mathcal{A}$ is unital, that is, it contains the identity operator I. A state M on $\mathcal{A}$ is a linear functional on $\mathcal{A}$ which is positive (that is, $M(T^*T) \geq 0$ for all $T \in \mathcal{A}$) and normalized (that is, $M(I) = 1$). In view of the analogy with the classical case, one should think of M as a mean on $\mathcal{A}$.

Let now π be a unitary representation of the locally compact group G on $\mathcal{H}$. We denote by Ad the representation of G on the vector space $\mathcal{L}(\mathcal{H})$ given by

$$\mathrm{Ad}(x)(T) = \pi(x)T\pi(x^{-1}) \qquad x \in G,\ T \in \mathcal{L}(\mathcal{H}).$$

Observe that this representation is not necessarily strongly continuous, when $\mathcal{L}(\mathcal{H})$ is endowed with the norm topology; this means that, for fixed $T \in \mathcal{L}(\mathcal{H})$, the mapping

$$G \to \mathcal{L}(\mathcal{H}), \quad x \mapsto \pi(x)T\pi(x^{-1})$$

is in general not continuous (as an easy example, take $G = \mathbb{R}$, $\pi = \lambda_{\mathbb{R}}$ the regular representation of $L^2(\mathbb{R})$, and T the mutiplication operator defined by a bounded function which is not uniformly continuous).

Let $\mathcal{X}(\mathcal{H})$ be the subspace of $\mathcal{L}(\mathcal{H})$ consisting of all G-continuous operators, that is, all $T \in \mathcal{L}(\mathcal{H})$ such that the mapping $x \mapsto \pi(x)T\pi(x^{-1})$ is continuous. It is easy to show that $\mathcal{X}(\mathcal{H})$ is an $\mathrm{Ad}(G)$-invariant unital C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. Using Cohen's factorization theorem (see, e.g., [HeRo70, Theorem (32.22)] we have the following description of $\mathcal{X}(\mathcal{H})$:

$$\mathcal{X}(\mathcal{H}) = \{\mathrm{Ad}(f)T \ : \ f \in L^1(G),\ T \in \mathcal{L}(\mathcal{H})\},$$

where $\mathrm{Ad}(f)T$ is defined by means of the weakly convergent integral

$$\mathrm{Ad}(f)T = \int_G f(x)\pi(x)T\pi(x^{-1})dx.$$

Let $\mathcal{A}$ be an $\mathrm{Ad}(G)$-invariant unital C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. A state M on $\mathcal{A}$ is $\mathrm{Ad}(G)$-invariant if

$$M(\pi(x)T\pi(x^{-1}) = M(T) \qquad \forall\, T \in \mathcal{A},\ x \in G.$$

The essential tool for our proof of Theorem 1 is the following result from [Bekk90] which is a generalization of the Hulanicki-Reiter characterization of amenable groups (see [Gree69, 3.2.5]).

Theorem 13 *(**[Bekk90]**) Let π be a unitary representation of the locally compact group G on $\mathcal{H}$. The following properties are equivalent:*

(i) $\pi \otimes \overline{\pi}$ *almost has invariant vectors;*

(ii) there exists an $\mathrm{Ad}(G)$*-invariant state on* $\mathcal{L}(\mathcal{H})$;

(iii) there exists an $\mathrm{Ad}(G)$*-invariant state on the algebra* $\mathcal{X}(\mathcal{H})$ *of all G-continuous operators.*

For the proof of Theorem 1, we need two elementary lemmas. The first one is well-known and easy to prove (see the beginning of the proof of Théorème 1 in [DeGu73]).

Lemma 14 *Let T be a linear contraction on a Hilbert space $\mathcal{H}$. Let $c \in \mathbb{C}$ with $|c| = 1$ be a spectral value of T. Then c is an approximate eigenvalue of T, that is, there exists a sequence $(\xi_n)_n$ of unit vectors in $\mathcal{H}$ such that* $\lim_n \|T\xi_n - c\xi_n\| = 0$.

For a Hilbert space $\mathcal{H}$, recall that the inner product on the space $L^2(\mathcal{H})$ of all Hilbert-Schmidt operators on $\mathcal{H}$ is defined by

$$\langle S, T\rangle = \mathrm{Trace}(T^*S), \qquad S, T \in L^2(\mathcal{H}),$$

where Trace is the usual trace of operators. We denote by $T \mapsto \|T\|_2$ the corresponding norm. Recall that the absolute value of an operator $T \in \mathcal{L}(\mathcal{H})$ is the positive operator $|T| = (T^*T)^{1/2}$.

Lemma 15 *Let $S, T \in L^2(\mathcal{H})$. Then*

$$|\langle S, T\rangle|^2 \leq \langle |S|, |T|\rangle\langle |S^*|, |T^*|\rangle.$$

Proof Let U and V be the partial isometries on $\mathcal{H}$ arising from the polar decompositions

$$S = U|S|, \qquad T = V|T|.$$

Observe that the polar decompositions of S^* and T^* are

$$S^* = U^*(U|S|U^*), \qquad T = V^*(V|T|V^*),$$

so that $|S^*| = U|S|U^*$ and $|T^*| = V|T|V^*$. Using the Cauchy-Schwarz inequality in $L^2(\mathcal{H})$, we have

$$\begin{aligned}
|\langle S, T\rangle| &= |\mathrm{Trace}(V^*|T^*|U|S|)| \\
&= |\mathrm{Trace}(V^*|T^*|^{1/2}|T^*|^{1/2}U|S|^{1/2}|S|^{1/2})| \\
&= |\mathrm{Trace}\left((|S|^{1/2}V^*|T^*|^{1/2})(|T^*|^{1/2}U|S|^{1/2})\right)| \\
&\leq \||S|^{1/2}V^*|T^*|^{1/2}\|_2 \; \||T^*|^{1/2}U|S|^{1/2}\|_2 \\
&= \left(\mathrm{Trace}(|T^*|^{1/2}V|S|V^*|T^*|^{1/2})\right)^{1/2} \left(\mathrm{Trace}(|S|^{1/2}U^*|T^*|U|S|^{1/2})\right)^{1/2} \\
&= \left(\mathrm{Trace}(|V^*|T^*|V|S)\right)^{1/2} \left(\mathrm{Trace}(|U|S|U^*|T^*)\right)^{1/2} \\
&= \left(\mathrm{Trace}(|T||S|)\right)^{1/2} \left(\mathrm{Trace}(|S^*||T^*|)\right)^{1/2} \\
&= \left(\langle |S|, |T|\rangle\right)^{1/2} \left(\langle |S^*|, |T^*|\rangle\right)^{1/2}.
\end{aligned}$$

■

We now proceed with the proof of Theorem 1. This will be done in several steps.

• *First step:* We claim that it suffices to show that $r_{\mathrm{spec}}((\pi \otimes \overline{\pi})(\mu)) < 1$. To show this, we use the following argument which appears in the proof of Theorem 1 in [Nevo98]. For every $n \in \mathbb{N}$ and every vector $\xi \in \mathcal{H}$, denoting by μ^{*n} the n-fold convolution product of μ by itself, we have

$$\begin{aligned}
|\langle(\pi(\mu))^n\xi,\xi\rangle|^2 &= |\langle\pi(\mu^{*n})\xi,\xi\rangle|^2\\
&= \left|\int_G \langle\pi(x)\xi,\xi\rangle d\mu^{*n}(x)\right|^2\\
&\leq \int_G |\langle\pi(x)\xi,\xi\rangle|^2 d\mu^{*n}(x)\\
&= \int_G \langle\pi\otimes\overline{\pi}(x)\xi\otimes\overline{\xi},\xi\otimes\overline{\xi}\rangle d\mu^{*n}(x)\\
&= \langle(\pi\otimes\overline{\pi}(\mu))^n\xi\otimes\overline{\xi},\xi\otimes\overline{\xi}\rangle,
\end{aligned}$$

where we used Jensen's inequality. It follows that

$$\|\pi(\mu)^n\| \leq \|(\pi\otimes\overline{\pi}(\mu))^n\|^{1/2}$$

for all $n \in \mathbb{N}$ and hence

$$r_{\text{spec}}((\pi)(\mu)) = \lim_n \|\pi(\mu)^n\|^{1/n} \leq \left(r_{\text{spec}}((\pi\otimes\overline{\pi})(\mu))\right)^{1/2}.$$

This proves the claim.

From now on, we assume that $r_{\text{spec}}((\pi\otimes\overline{\pi})(\mu)) = 1$. We will show that $\pi\otimes\overline{\pi}$ almost has invariant vectors.

• *Second step:* We claim that 1 is an approximate eigenvalue of $(\pi\otimes\overline{\pi})(\mu)$. Indeed, by assumption, the operator $(\pi\otimes\overline{\pi})(\mu)$ on $L^2(\mathcal{H})$ has a spectral value c with $|c| = 1$. Since $(\pi\otimes\overline{\pi})(\mu)$ is a contraction, there exists, by Lemma 14, a sequence of operators T_n in $L^2(\mathcal{H})$ with $\|T_n\|_2 = 1$ such that

$$\lim_n \|(\pi\otimes\overline{\pi})(\mu)(T_n) - cT_n\|_2 = 0,$$

that is,

$$\lim_n \left\|\int_G \pi(x)T_n\pi(x^{-1})d\mu(x) - cT_n\right\|_2 = 0,$$

or, equivalently,

$$\lim_n \int_G \langle\pi(x)T_n\pi(x^{-1}), T_n\rangle d\mu(x) = \lim_n \langle(\pi\otimes\overline{\pi})(\mu)(T_n), T_n\rangle = c.$$

Using Lemma 15 and Cauchy-Schwarz inequality, we have on the one hand

$$\left|\int_G \langle \pi(x)T_n\pi(x^{-1}), T_n\rangle d\mu(x)\right| \leq \int_G |\langle \pi(x)T_n\pi(x^{-1}), T_n\rangle| d\mu(x)$$

$$\leq \int_G \langle \pi(x)|T_n|\pi(x^{-1}), |T_n|\rangle^{1/2} \langle \pi(x)|T_n^*|\pi(x^{-1}), |T_n^*|\rangle^{1/2} d\mu(x)$$

$$\leq \left(\int_G \langle \pi(x)|T_n|\pi(x^{-1}), |T_n|\rangle d\mu(x)\right)^{1/2} \left(\int_G \langle \pi(x)|T_n^*|\pi(x^{-1}), |T_n^*|\rangle d\mu(x)\right)^{1/2}.$$

On the other hand, we have

$$\int_G \langle \pi(x)|T_n|\pi(x^{-1}), |T_n|\rangle d\mu(x) \leq \int_G \|\pi(x)T_n\pi(x^{-1})\|_2 \|T_n\|_2 d\mu(x) = 1$$

as well as

$$\int_G \langle \pi(x)|T_n^*|\pi(x^{-1}), |T_n^*|\rangle d\mu(x) \leq \int_G \|\pi(x)T_n^*\pi(x^{-1})\|_2 \|T_n^*\|_2 d\mu(x) = 1.$$

Since

$$\lim_n \left|\int_G \langle \pi(x)T_n\pi(x^{-1}), T_n\rangle d\mu(x)\right| = 1,$$

it follows that

$$\lim_n \langle (\pi\otimes\overline{\pi})(\mu)(|T_n|), |T_n|\rangle = \lim_n \int_G \langle \pi(x)|T_n|\pi(x^{-1}), |T_n|\rangle d\mu(x) = 1,$$

that is,

$$\lim_n \|(\pi\otimes\overline{\pi})(\mu)(|T_n|) - |T_n|\|_2 = 0.$$

Since $\||T_n|\|_2 = 1$, this shows that $(\pi\otimes\overline{\pi})(\mu)$ has 1 as approximate eigenvalue. This proves the second step.

- *Third step:* There exists a state M on $\mathcal{L}(\mathcal{H})$ which is invariant under a dense subgoup D of G. Indeed, by the second step, there exists a sequence $(T_n)_n$ of Hilbert-Schmidt operators on $\mathcal{H}$ with $\|T_n\|_2 = 1$ such that

$$\lim_n \int_G \langle \pi(x)T_n\pi(x^{-1}), T_n\rangle d\mu(x) = 1,$$

It follows that there exists a subsequence, still denoted by $(T_n)_n$, such that

$$\lim_n \langle \pi(x)T_n\pi(x^{-1}), T_n\rangle = 1$$

and therefore

$$(*) \qquad \lim_n \|\pi(x)T_n\pi(x^{-1}) - T_n\|_2 = 0$$

for μ-almost every x in G. The set of all x for which $(*)$ holds is clearly a measurable subgroup D of G. Since $\mu(D) = 1$, the support of μ is contained in the closure $\overline{D}$ of D. By assumption, the support of μ generates a dense subgroup of G. Hence, D is dense in G.

Consider now the sequence of states M_n on $\mathcal{L}(\mathcal{H})$ defined by

$$M_n(T) = \langle TT_n, T_n\rangle, \qquad T \in \mathcal{L}(\mathcal{H}).$$

Let M be a weak *-limit point of $(M_n)_n$ in the dual space of $\mathcal{L}(\mathcal{H})$. Using $(*)$ above, it is readily verified that M is a state on $\mathcal{L}(\mathcal{H})$ which is $\mathrm{Ad}(D)$-invariant.

• *Fourth step:* the representation $\pi \otimes \overline{\pi}$ almost has invariant vectors. Indeed, consider the restriction M_0 of M to the subalgebra $\mathcal{X}(\mathcal{H})$ of all G-continuous operators on $\mathcal{H}$. Then M_0 is an $\mathrm{Ad}(D)$-invariant state on $\mathcal{X}(\mathcal{H})$. Since D is dense in G, it follows from the continuity of the action of G on $\mathcal{X}(\mathcal{H})$ that M_0 is $\mathrm{Ad}(G)$-invariant. Theorem 13 shows then that $\pi \otimes \overline{\pi}$ almost has invariant vectors. This concludes the proof of the main part of Theorem 1.

To show the last part of Theorem 1, observe first that μ is strongly adapted if and only if the probability measure $\nu = \check{\mu} * \mu$ is adapted, where the measure $\check{\mu}$ is defined by

$$\check{\mu}(E) = \mu(E^{-1})$$

for every Borel subset E of G. Observe also $\pi(\nu) = \pi(\mu)^*\pi(\mu)$ is a self-adjoint positive operator with norm 1 and that $\|\pi(\nu)\| = \|\pi(\mu)\|^2$.

Assume now that $\|\pi(\mu)\| = 1$. Then $\|\pi(\nu)\| = 1$ and hence 1 is a spectral value of $\pi(\nu)$. The proof above applied to ν in place of μ shows that $\pi \otimes \overline{\pi}$ almost has invariant vectors. ■

3 Proof of Proposition 8

Let G be a separable locally compact group acting on the measure space X equipped with a quasi-invariant σ-finite measure ν. Let λ_X be the corresponding unitary representation on $L^2(X, \nu)$. To show that (i) implies (ii), assume that the action of G on (X, ν) is co-amenable. Thus, there exists a G-invariant mean on $L^\infty(X, \nu)$. Then, by standard arguments, there exists a sequence $(f_n)_n$ of positive measurable functions in $L^1(X)$ with $\|f_n\|_1 = 1$ such that $\lim_n \|\lambda_X(g) f_n - f_n\|_1 = 0$ uniformly on compact subsets of G. Set $\xi_n = \sqrt{f_n}$. Then $\|\xi_n\|_2 = 1$ and $\lim_n \|\lambda_X(g)\xi_n - \xi_n\|_2 = 0$ uniformly on compact subsets of G, so that λ_X almost has invariant vectors.

The fact that (ii) applies (iii) is obvious. To show that (iii) implies (i), assume that the representation $\lambda_X \otimes \overline{\lambda}_X$ almost has invariant vectors. It follows from Theorem 13 that there exists an $\mathrm{Ad}(G)$-invariant state M on the algebra $\mathcal{L}(L^2(X, \nu))$. Observe that the algebra $L^\infty(X, \nu)$ can be embedded as a subalgebra of $\mathcal{L}(L^2(X, \nu))$: to every $\varphi \in L^\infty(X, \nu)$ corresponds the operator T_φ acting on $L^2(X, \nu)$ by pointwise mutiplication by φ. For $g \in G$ and $\varphi \in L^\infty(X, \nu)$, we have the relation

$$(**) \qquad \lambda_X(g) T_\varphi \lambda_X(g^{-1}) = T_{{}_g\varphi}$$

where ${}_g\varphi(x) = \varphi(g^{-1}x)$.

Define now a mean M_0 on $L^\infty(X, \nu)$ by

$$M_0(\varphi) = M(T_\varphi), \qquad \forall \varphi \in L^\infty(X, \nu).$$

Relation $(**)$ above shows that M_0 is G-invariant. Hence, the action of G on (X, ν) is co-amenable.

4 Proof of Corollary 12

Before we proceed with the proof of Corollary 12, several remarks are in order. In the following lemma, we will use and prove only the fact that (i) implies (ii). The fact that (ii) implies (i), which was shown in [JoRT94, Theorem 3.6] (see also [Guiv80, Proposition 1]), is actually the main motivation for the notion of an identity excluding group. The proof of this implication uses the arguments which appeared in the Third Step of the proof of Theorem 1.

Lemma 16 *Let G be a second countable, locally compact group. Let $(\pi, \mathcal{H})$ be a unitary representation of G. The following properties are equivalent:*

(i) There exists a dense subgroup D of G such that the restriction $\pi|_D$ of π to D almost has invariant vectors;

(ii) there exists an adapted probability measure μ on the Borel subsets of G such that 1 is an approximate eigenvalue of $\pi(\mu)$.

Proof To show that (i) implies (ii), let $(U_n)_{n\in\mathbb{N}}$ be a basis for the topology of G. Since D is dense, we can find, for every n, and element $g_n \in D \cap U_n$. Clearly, the sequence $(g_n)_n$ is dense in G. Set

$$\mu = \sum_{n\in\mathbb{N}} 2^{-n}\delta_{g_n}.$$

Then μ is an adapted probability measure on G. Since $\pi|_D$ almost has invariant vectors, there exists a sequence $(\xi)_i$ of unit vectors in $\mathcal{H}$ such that, for every $n \in \mathbb{N}$,

$$\lim_i \|\pi(g_n)\xi_i - \xi_i\| = 0.$$

It follows that 1 is an approximate eigenvalue of $\pi(\mu)$. Indeed, for a given $\varepsilon > 0$, choose N such that $\sum_{n>N} 2^{-n} \leq \varepsilon$. Let i_0 be such that, for all $n \leq N$,

$$\|\pi(g_n)\xi_i - \xi_i\| \leq \varepsilon/N \qquad \text{for all} \quad i \geq i_0.$$

Then, for all $i \geq i_0$, we have

$$\begin{aligned}
\|\pi(\mu)\xi_i - \xi_i\| &\leq \sum_{n\geq 1} \|\pi(g_n)\xi_i - \xi_i\| \\
&\leq \sum_{n\leq N} \|\pi(g_n)\xi_i - \xi_i\| + 2\varepsilon \\
&\leq 3\varepsilon. \quad \blacksquare
\end{aligned}$$

Remark 17 An irreducible unitary representation $(\pi, \mathcal{H})$ of a locally compact group G is said to be CCR if, for every $f \in L^1(G)$, the operator $\pi(f)$ is a compact operator on $\mathcal{H}$. If this is the case, then π is -up to unitary equivalence- the unique irreducible unitary representation of G which is weakly contained in π (see [Dixm69, Corollaire 4.1.11]). In particular, a CCR representation $\pi \neq 1_G$ does not almost have invariant vectors. This shows that Part (i) of Corollary 12 is a consequence of Corollary 3.

It is well-known that all irreducible unitary representations of a semisimple real Lie group with finite centre are CCR (see [Dixm69, Théorème 15.5.6]).

We now give the proof of Corollary 12. In view of the previous remark and of Lemma 16, it is clear that Part (i) and Part (ii) follow from Corollary 3 and Corollary 10, respectively. It remains to prove Part (iii).

So, let $G = \mathbb{G}(\mathbb{K})$ be the group of $\mathbb{K}$-rational points of a $\mathbb{K}$-isotropic simple algebraic group $\mathbb{G}$ over a local field $\mathbb{K}$. Let π be an irreducible unitary representation of G. We claim that $r_{\text{spec}}(\pi(\mu)) < 1$ for every adapted probability measure μ on G.

Indeed, it is known that there is a real number p in $[2, \infty)$ such that all the matrix coefficients of π lie in $L^p(G)$ (see [BoWa80, Chap. XI, 3.6 Proposition]). Hence, for some integer N, the tensor power $\pi^{\otimes N} \otimes \overline{\pi}^{\otimes N}$ is contained in an infinite multiple of the regular representation λ_G. It follows that

$$r_{\text{spec}}((\pi^{\otimes N} \otimes \overline{\pi}^{\otimes N})(\mu)) \leq r_{\text{spec}}(\lambda_G(\mu)).$$

On the other hand, the argument used in the First Step of the proof of Theorem 1 shows that

$$r_{\text{spec}}((\pi)(\mu)) \leq \left(r_{\text{spec}}((\pi^{\otimes N} \otimes \overline{\pi}^{\otimes N})(\mu))\right)^{1/2N}.$$

Hence $r_{\text{spec}}((\pi)(\mu)) \leq r_{\text{spec}}(\lambda_G(\mu))^{1/2N}$. As G is not compact, G is not amenable. Therefore, $r_{\text{spec}}(\lambda_G(\mu)) < 1$, by Corollary 6. This proves the claim.

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