Université de Rennes 1
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Master 2

Théorie Ergodique et Systèmes Dynamiques
Exercise sheet 3

1 - (Ergodicity of an induced system) - Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$, and consider the induced system $(A, \mathcal{A}, \mu_A, T_A)$, as in Exercise 7 of Sheet 1. Assume that T is ergodic. Show that T_A is ergodic. (Hint: consider the partition of A by the sets $A_n = \{x \in A \mid n_A(x) = n\}$, where $n_A(x) = \min\{n \geq 1 | \ T^n x \in A\}$.)

2 - (Invertible extension) - Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. Define a new system $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ as follows:

- $\widetilde{X} = \{(x_n)_{n \in \mathbf{Z}} \ : \ x_n \in X, \ x_{n+1} = Tx_n, \quad \forall \ n \in \mathbf{Z}\}$;
- $\widetilde{T} : \widetilde{X} \to \widetilde{X}, \qquad \widetilde{T}((x_n)_{n \in \mathbf{Z}}) = (x_{n+1})_{n \in \mathbf{Z}}$;
- $\widetilde{\mathcal{B}}$ is the σ-algebra generated by the sets $p_n^{-1}(A)$ with $A \in \mathcal{B}$, where $p_n : \widetilde{X} \to X$ is given by $p_n((x_k)_{k \in \mathbf{Z}}) = x_n$ for $n \in \mathbf{Z}$;
- $\widetilde{\mu}$ is the probability measure on $\widetilde{\mathcal{B}}$ defined by $\widetilde{\mu}(p_n^{-1}(A)) = \mu(A)$.

(i) Show that $\widetilde{T}$ preserves $\widetilde{\mu}$, that $\widetilde{T}$ is invertible, and that $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is an extension of the system $(X, \mathcal{B}, \mu, T)$.
(ii) Let $(Y, \mathcal{A}, \nu, S)$ be an extension of the system $(X, \mathcal{B}, \mu, T)$ with S invertible. Show that $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is a factor of $(Y, \mathcal{A}, \nu, S)$.
(iii) Let $X_2 = \mathbf{S}^1$ and $T_2 : X \to X, x \mapsto 2x$. Show that $\widetilde{X}_2$ is an abelian compact group
(iv) $(*)$ Let $\mathbf{Z}[1/2]$ be the subring of $\mathbf{R}$ generated by $\mathbf{Z}$ and $1/2$. Let $\mathbf{Q}_2$ be the field of 2-adic numbers and $\mathbf{Z}_2$ the subring of 2-adic integers.

Show that the image of the injective homomorphism

$$\delta : \mathbf{Z}[1/2] \to \mathbf{R} \times \mathbf{Q}_2, \ r \mapsto (r, r)$$

is a discrete subgroup of $\mathbf{R} \times \mathbf{Q}_2$. Deduce from this that $\widetilde{X}_2$ can be identified with the compact abelian group $(\mathbf{R} \times \mathbf{Q}_2)/\delta(\mathbf{Z}[1/2])$. Describe $\widetilde{T}_2$ in this identification.

3 - (Normal numbers) Let $x \in [0,1]$ with unique binary expansion $0, x_1 x_1, x_3 \dots$ with $x_n \in \{0,1\}$. Then x is said to be normal (in base 2) if

$$\lim_{N \to \infty} \frac{1}{N} \mathrm{Card}\, \{1 \leq n \leq N \mid x_n = 0\} = 1/2.$$

Show that Lebesgue almost every $x \in [0,1]$ is normal.

4 - (Non hyperbolic torus automorphisms) For $d \geq 1$, let $X = \mathbf{R}^d/\mathbf{Z}^d$ be the d-dimensional torus, equipped with the Lebesgue measure λ. For $A \in GL_d(\mathbf{Z})$, let $T_A : X \to X$ be the associated automorphism.
(i) Assume that T_A is ergodic and that $d \leq 3$. Show that A has no eigenvalue of absolute value 1.

Let $A \in SL_4(\mathbf{Z})$ be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 8 & -6 & 8 \end{pmatrix}.$$

(ii) Show that A has two eigenvalues with of absolute value 1.
(iii) Show that T_A is ergodic.
(*Hint:* show that the characteristic polynomial of A is irreducible over $\mathbf{Q}$.)

5 - (Weak mixing and non mixing of irrational rotations) Let $X = [0,1]$ be equipped with the Lebesgue measure λ. Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and $R_\alpha : [0,1] \to [0,1]$, given by $R_\alpha(x) = \{x + \alpha\}$.
Show that R_α is not mixing.

6 - (Mixing of $\times 2$-map) Let $X = [0,1]$ be equipped with the Lebesgue measure λ and $T_2 : X \to X$ defined by par $T_2 x = \{2x\}$. Show that T_2 is mixing.

7 - (Mixing of Bernoulli shift) Let (Σ^+, μ, σ) be the one-sided Bernoulli shift, where $\Sigma^+ = \{0,1\}^{\mathbf{N}}$ is equipped with the measure $\mu = \nu^{\otimes \mathbf{N}}$ defined on the σ-algebra generated by the cylinders, ν is the probability $(1/2, 1/2)$ on $\{0,1\}$ and σ is the shift $\sigma((a_i)_{i \in \mathbf{N}}) = (a_{i+1})_{i \in \mathbf{N}}$.
Show that σ is mixing.