SPECTRAL GAP PROPERTY AND STRONG ERGODICITY FOR GROUPS OF AFFINE TRANSFORMATIONS OF SOLENOIDS

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Abstract. Let X be a solenoid, that is, a compact finite dimensional connected abelian group with normalized Haar measure μ and let $\Gamma \to \mathrm{Aff}(X)$ be an action of a countable discrete group Γ by continuous affine transformations of X. We show that the probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ does not have the spectral gap property if and only if there exists a $p_{\mathrm{a}}(\Gamma)$-invariant proper subsolenoid Y of X such that the image of Γ in $\mathrm{Aff}(X/Y)$ is a virtually solvable group, where $p_{\mathrm{a}} : \mathrm{Aff}(X) \to \mathrm{Aut}(X)$ is the canonical projection. When Γ is finitely generated or when X is the a-adic solenoid for an integer $a \geq 1$, the subsolenoid Y can be chosen so that the image Γ in $\mathrm{Aff}(X/Y)$ is a virtually abelian group. In particular, an action $\Gamma \curvearrowright (X, \mu)$ by affine transformations on a solenoid X has the spectral gap property if and only if $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic.

1. Introduction

Let X be a compact group and $\mathrm{Aut}(X)$ the group of continuous automorphisms of X. Denote by

$$\mathrm{Aff}(X) := \mathrm{Aut}(X) \ltimes X$$

the group of affine transformations of X, that is, of maps of the form

$$X \to X, \quad x \mapsto x_0\theta(x)$$

for some $\theta \in \mathrm{Aut}(X)$ and $x_0 \in X$. Let μ be the normalized Haar measure of X. By translation invariance and uniqueness of the Haar measure, every transformation in $\mathrm{Aff}(X)$ preserves μ.

Given a group Γ and a homomorphism $\Gamma \to \mathrm{Aff}(X)$, one has therefore a measure preserving action $\Gamma \curvearrowright (X, \mu)$. The study of the ergodicity of such actions is a classical theme going back to Halmos [Halm43] and Kaplansky [Kapl49], both for the case where $\Gamma = \mathbf{Z}$ is generated

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by a single automorphism of X. For a characterization of the ergodicity of an action $\Gamma \curvearrowright (X,\mu)$ by automorphisms on an arbitrary compact group, see [KiSc89, Lemma 2.2]. The following elementary proposition provides a neat characterization of the ergodicity for actions by affine transformations in the case where X is moreover abelian and connected (for the proof, see Subsection 3.3 below).

Proposition 1. *Let X be a compact connected abelian group and $\Gamma \subset \mathrm{Aff}(X)$ a countable group. Let $p_{\mathrm{a}} : \mathrm{Aff}(X) \to \mathrm{Aut}(X)$ denote the canonical projection. The following properties are equivalent:*

(i) *The action $\Gamma \curvearrowright (X,\mu)$ is not ergodic.*
(ii) *There exists a $p_{\mathrm{a}}(\Gamma)$-invariant* proper *and connected closed subgroup Y of X such that the image of Γ in $\mathrm{Aff}(X/Y)$ is a finite group.*

Our main concern in this article is the spectral gap property for the action $\Gamma \curvearrowright (X,\mu)$. Let π_X denote the corresponding Koopman representation of Γ on $L^2(X,\mu)$. Recall that $\Gamma \curvearrowright (X,\mu)$ is ergodic if and only if there is no non-zero invariant vector in the $\pi_X(\Gamma)$-invariant subspace $L^2_0(X,\mu) = (\mathbf{C}\mathbf{1}_X)^\perp$ of functions with zero mean. The action $\Gamma \curvearrowright (X,\mu)$ is said to have the **spectral gap property** (or has a spectral gap) if there are not even almost invariant vectors in $L^2_0(X,\mu)$, that is, there is no sequence of unit vectors f_n in $L^2_0(X,\mu)$ such that $\lim_n \|\pi_X(\gamma)f_n - f_n\| = 0$ for all $\gamma \in \Gamma$.

Group actions on general probability spaces with the spectral gap property have an amazing range of applications ranging from geometry and group theory to operator algebras and graph theory (for an account on this property, see [Bekk16]).

Given a specific non abelian compact group X, there is in general no known characterization of the countable subgroups Γ of $\mathrm{Aff}(X)$ such that $\Gamma \curvearrowright (X,\mu)$ has the spectral gap property: indeed, it is usually a difficult problem to even find subgroups Γ of X for which the action Γ by translations on X has a spectral gap (for a recent result in the case $X = SU(d)$, see [BoGa12]).

We characterize below (Theorem 2) actions by affine transformations $\Gamma \curvearrowright (X,\mu)$ with the spectral gap property for a solenoid X, in the same spirit as the ergodicity characterization from Proposition 1. This result (as well as Theorem 5 below) generalizes Theorem 5 in [BeGu15], where an analogous characterization was given for the case of a torus $X = \mathbf{T}^d$ (see also [FuSh99, Theorem 6.5] for a partial result).

Recall that a **solenoid** X is a finite-dimensional, connected, compact abelian group (see [HeRo63, §25]). Examples of solenoids of dimension $d \geq 1$ include the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ as well as the p-adic solenoid $\mathbf{S}_p^d$

for where p is a prime integer (see Appendix to Chapter I in [Robe00]) or, more generally, the a-adic solenoid $\mathbf{S}_a^d$ for a positive integer a (see below). In some sense the largest d-dimensional solenoid is provided by the solenoid $\mathbf{A}^d/\mathbf{Q}^d$, where $\mathbf{A}$ is the ring of adèles over $\mathbf{Q}$ (see Subsection 2.3).

Here is our main result. Recall that, given a group property $\mathcal{P}$, a group has virtually $\mathcal{P}$ if it has a finite index subgroup with the property $\mathcal{P}$. A subsolenoid of a solenoid X is a closed and connected subgroup of X.

Theorem 2. *Let X be a solenoid with normalized Haar measure μ and Γ a countable subgroup of* $\mathrm{Aff}(X)$*. Let $p_{\mathrm{a}} : \mathrm{Aff}(X) \to \mathrm{Aut}(X)$ denote the canonical projection. The following properties are equivalent:*

(i) *The action $\Gamma \curvearrowright (X, \mu)$ does not have the spectral gap property.*
(ii) *The action $p_{\mathrm{a}}(\Gamma) \curvearrowright (X, \mu)$ does not have the spectral gap property.*
(iii) *There exists a $p_{\mathrm{a}}(\Gamma)$-invariant* proper subsolenoid *Y of X such that the image of Γ in* $\mathrm{Aff}(X/Y)$ *is an amenable group.*
(iv) *There exists a $p_{\mathrm{a}}(\Gamma)$-invariant* proper subsolenoid *Y of X such that the image of Γ in* $\mathrm{Aff}(X/Y)$ *is a virtually solvable group.*

The proof of Theorem 2 is an extension to the adelic setting of the methods from [BeGu15] and is based on the consideration of appropriate invariant means on finite dimensional vector spaces over local fields and the associated invariant measures on the corresponding projective spaces.

Remark 3. Theorem 2 can be sharpened in the case where Γ is a *finitely generated* subgroup $\mathrm{Aff}(X)$: the subsolenoid Y in (iv) can be chosen so that the image of $p_{\mathrm{a}}(\Gamma)$ in $\mathrm{Aut}(X/Y)$ is virtually abelian (see Remark 13).

The spectral gap property is related to another strengthening of ergodicity. Recall that the action of a countable group Γ by measure preserving transformations on a probability space (X, μ) is **strongly ergodic** (see [Schm81]) if every sequence $(A_n)_n$ of measurable subsets of X which is asymptotically invariant (that is, which is such that $\lim_n \mu(\gamma A_n \triangle A_n) = 0$ for all $\gamma \in \Gamma$) is trivial (that is, $\lim_n \mu(A_n)(1 - \mu(A_n)) = 0$). As is easily seen, the spectral gap property implies strong ergodicity (the converse implication does not hold in general; see Example 2.7 in [Schm81]). Moreover, no ergodic measure preserving action of an *amenable* group on a non atomic probability space is strongly ergodic, by [Schm81, Theorem 2.4]. The following corollary is therefore a direct consequence of Theorem 2.

Corollary 4. *Let X be a solenoid and $\Gamma \subset \mathrm{Aff}(X)$ a countable group. The following properties are equivalent:*

(i) *The action $\Gamma \curvearrowright (X, \mu)$ has the spectral gap property.*
(ii) *The action $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic.*

It is worth mentioning that the equivalence of (i) and (ii) in Corollary 4 holds for actions by *translations* on a general compact group X (see Proposition 3.1 in [AbEl12]).

We can prove improve Theorem 2 in the case of a-adic solenoids. Let a be a square free positive integer, that is, $a = p_1 \cdots p_r$ is a product of different primes p_i. Then

$$\mathbf{A}_a^d := \mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d,$$

is a locally compact ring, where $\mathbf{Q}_p$ is the field of p-adic numbers. Let $\mathbf{Z}[1/a] = \mathbf{Z}[1/p_1, \cdots, 1/p_r]$ denote the subring of $\mathbf{Q}$ generated by 1 and $1/a$. Through the diagonal embedding

$$\mathbf{Z}[1/a]^d \to \mathbf{A}_a^d, \qquad b \mapsto (b, b, \cdots, b),$$

we may identify $\mathbf{Z}[1/a]^d$ with a discrete and cocompact subring of $\mathbf{A}_a^d$. The **a-adic solenoid** is defined as the quotient

$$\mathbf{S}_a = \mathbf{A}_a^d / \mathbf{Z}[1/a]^d.$$

(see Chap. II, §10 in [HeRo63]). Moreover, $\mathrm{Aut}(\mathbf{A}_a^d)$ is canonically isomorphic to $GL_d(\mathbf{R}) \times GL_d(\mathbf{Q}_{p_1}) \times \cdots \times GL_d(\mathbf{Q}_{p_r})$ and so $\mathrm{Aut}(\mathbf{S}_a^d)$ can be identified with $GL_d(\mathbf{Z}[1/a])$.

For a subset S of $GL_d(\mathbf{K})$ for a field $\mathbf{K}$, we denote by $S^t = \{g^t \mid g \in S\}$ the set of transposed matrices from S.

Theorem 5. *Let $a \geq 1$ be a square free integer. Let Γ be a subgroup of $\mathrm{Aff}(\mathbf{S}_a^d) \cong GL_d(\mathbf{Z}[1/a]) \ltimes \mathbf{S}_a^d$. The following properties are equivalent:*

(i) *The action $\Gamma \curvearrowright (\mathbf{S}_a^d, \mu)$ does not have the spectral gap property.*
(ii) *There exists a non zero linear subspace W of $\mathbf{Q}^d$ which is invariant under $p_{\mathrm{a}}(\Gamma)^t \subset GL_d(\mathbf{Q})$ and such that the image of $p_{\mathrm{a}}(\Gamma)^t$ in $GL(W)$ is a virtually abelian group.*

Examples of group actions on solenoids with the spectral gap property are provided by the following immediate consequence of Theorem 5.

Corollary 6. *For a square free integer $a \geq 1$, let Γ be subgroup of $GL_d(\mathbf{Z}[1/a])$. Assume that Γ is not virtually abelian and that Γ acts irreducibly on $\mathbf{Q}^d$. Then the action of Γ by automorphisms of $\mathbf{S}_a^d$ has the spectral gap property.*

Remark 7. Corollary 6 generalizes Theorem 6.8 in [FuSh99] in which the same result is proved under the stronger assumption that Γ acts irreducibly on $\mathbf{R}^d$.

This paper is organized as follows. In Section 1, we establish and recall some preliminary facts which are necessary to the proofs of our results. Section 2 is devoted to the proofs of Theorem 2, Theorem 5, and Proposition 1.

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2. Some preliminary results

2.1. **Reduction to the case of automorphisms.** Let X be a compact abelian group with normalized Haar measure μ and Γ a countable subgroup of $\mathrm{Aff}(X)$. The aim of this subsection is to reduce the study of the spectral gap property for $\Gamma \curvearrowright (X,\mu)$ to that of the action $p_{\mathrm{a}}(\Gamma) \curvearrowright (X,\mu)$, where $p_{\mathrm{a}} : \mathrm{Aff}(X) \to \mathrm{Aut}(X)$ is the canonical projection.

Let $\widehat{X}$ be the Pontrjagin dual group of X, which is a discrete group. The group $\mathrm{Aut}(X)$ acts by duality on $\widehat{X}$, giving rise to a *right* action $\widehat{X} \times \mathrm{Aut}(X) \to \widehat{X}$ given by

$$\chi^\theta(x) = \chi(\theta(x)) \qquad \text{for all} \quad \theta \in \mathrm{Aut}(X), \chi \in \widehat{X}, x \in X.$$

The Fourier transform $\mathcal{F} : L^2(X,\mu) \to \ell^2(\widehat{X})$, given by

$$(\mathcal{F}f)(\chi) = \int_X f(x)\overline{\chi}(x)d\mu(x) \qquad \text{for all} \quad f \in L^2(X,\mu), \chi \in \widehat{X},$$

is a Hilbert space isomorphism. The Koopman representation of $\mathrm{Aff}(X)$ on $L^2(X,\mu)$ corresponds under $\mathcal{F}$ to the unitary representation π_X of $\mathrm{Aff}(X)$ on $\ell^2(\widehat{X})$ given by

$$(*) \qquad \pi_X(\gamma)(\xi)(\chi) = \chi(x)\xi(\chi^\theta) \qquad \text{for all} \quad \xi \in \ell^2(\widehat{X}), \chi \in \widehat{X},$$

for $\gamma = (\theta, x)$ in $\mathrm{Aff}(X) = \mathrm{Aut}(X) \ltimes X$. Observe that $L^2_0(X,\mu)$ corresponds under $\mathcal{F}$ to the subspace $\ell^2(\widehat{X} \setminus \{\mathbf{1}_X\})$ of $\ell^2(\widehat{X})$.

Proposition 8. *Let X be a compact abelian group with normalized Haar measure μ and let Γ be a countable subgroup of $\mathrm{Aff}(X)$ such that the action $\Gamma \curvearrowright (X,\mu)$ does not have the spectral gap property. Then the action $p_{\mathrm{a}}(\Gamma) \curvearrowright (X,\mu)$ does not have the spectral gap property.*

Proof. We realize the Koopman representation π_X on $\ell^2(\widehat{X})$ as above. Since $\Gamma \curvearrowright (X,\mu)$ does not have the spectral gap property, there exists a sequence $(\xi_n)_{n\geq 1}$ of unit vectors in $\ell^2(\widehat{X}\setminus\{\mathbf{1}_X\})$ such that $\lim_n \|\pi_X(\gamma)\xi_n - \xi_n\| = 0$, that is, by Formula $(*)$,

$$\lim_n \sum_{\chi\in\widehat{X}} |\chi(x)\xi_n(\chi^\theta) - \xi_n(\chi)|^2 = 0,$$

for every $\gamma = (\theta, x) \in \Gamma$.

For $n \geq 1$, set $\eta_n = |\xi_n|$. Then η_n is a unit vector in $\ell^2(\widehat{X}\setminus\{\mathbf{1}_X\})$ and, for every $\gamma = (\theta, x) \in \Gamma$, we have

$$\begin{aligned}
\|\pi_X(\theta)\eta_n - \eta_n\|^2 &= \sum_{\chi\in\widehat{X}} \big||\xi_n(\chi^\theta)| - |\xi_n(\chi)|\big|^2 \\
&= \sum_{\chi\in\widehat{X}} \big||\chi(x)\xi_n(\chi^\theta)| - |\xi_n(\chi)|\big|^2 \\
&\leq \sum_{\chi\in\widehat{X}} |\chi(x)\xi_n(\chi^\theta) - \xi_n(\chi)|^2 \\
&= \|\pi_X(\gamma)\xi_n - \xi_n\|^2.
\end{aligned}$$

Hence, (η_n) is a sequence of almost $\pi_X(p_{\mathrm{a}}(\Gamma))$-invariant and so $p_{\mathrm{a}}(\Gamma) \curvearrowright (X,\mu)$ does not have the spectral gap property.

□

2.2. **Invariant means, invariant measures, and linear actions.** Let X be a locally compact topological space. A **mean** on X is positive linear functional M on the space $C^b(X)$ of continuous bounded functions on X such that $M(\mathbf{1}_X) = 1$. If Y is another locally compact topological space and $\Phi : X \to Y$ a continuous map, the pushforward $\Phi_*(M)$ of M by Φ is the mean on Y given by $\Phi_*(M)(f) = M(f\circ\Phi)$ for $f \in C^b(Y)$.

Let Γ be a group and $\Gamma \curvearrowright X$ an action of Γ by homeomorphisms of X. A Γ-invariant mean on X is a mean on M which is invariant for the induced action of Γ on $C^b(X)$. The following lemma is well-known and easy to prove.

Lemma 9. *Let X, Y be respectively a locally compact space and a compact space. Let $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ be actions of the group Γ by homeomorphisms of X and Y. Let $\Phi : X \to Y$ a continuous Γ-equivariant map. Assume that there exists M be an invariant mean on X. Then $\Phi_*(M)$ is given by integration against a Γ-invariant probability measure μ on Y.*

Proof. Since $\Phi_*(M)$ is a positive linear functional on $C(Y)$ and since Y is compact, there exists by the Riesz representation theorem a probability measure μ on Y such that

$$\int_Y f(x)d\mu(x) = \Phi_*(M)(f) \qquad \text{for all} \quad f \in C(Y).$$

The measure μ is Γ-invariant, since $\Phi_*(M)$ is Γ-invariant. □

Let $\mathbf{k}$ be a local field (that is, a non discrete locally compact field) and V a finite dimensional vector space over $\mathbf{k}$. Then V is a locally compact vector space and $GL(V)$ is a locally compact group, for the topology inherited from $\mathbf{k}$. This is the only topology on $GL(V)$ we will consider in the sequel (with the exception of the proof of Lemma 12).

Every subgroup Γ of $GL(V)$ acts by homeomorphisms on the projective space $\mathbf{P}(V)$. A crucial tool for our proof of Theorems 2 and 5 is the consideration of Γ-invariant probability measures on $\mathbf{P}(V)$, a theme which goes back to the proof of the Borel density theorem in [Furs76]. The following proposition summarizes the main consequences, as we will use them, of the existence of such a measure. Variants of this proposition appeared already at several places (see for instance [BeGu15], [Corn04], [FuSh99]), but not exactly in the form we need; so, we will briefly indicate its proof.

For a group G, we denote by $[G, G]$ the commutator subgroup of G.

Proposition 10. *Let V be a finite dimensional vector space over a local field $\mathbf{k}$ and G a closed subgroup of $GL(V)$. Assume that there exists a G-invariant probability measure on the Borel subsets of $\mathbf{P}(V)$ which is not supported on a proper projective subspace. Then there exists a subgroup G_0 of G of finite index such that $[G_0, G_0]$ is relatively compact in $GL(V)$. In particular, the locally compact group G is amenable.*

Proof. Let ν be G-invariant probability Borel measure on $\mathbf{P}(V)$. As in the proof of Lemma 11 in [BeGu15] or of Theorem 6.5.i in [FuSh99], there exists finitely many subspaces $V_1, \dots, V_r$ of V and a subgroup G_0 of finite index in G with the following properties:

- ν is supported by the union of the projective subspaces corresponding to the V_i's;
- G_0 stabilizes V_i for every $i \in \{1, \dots, r\}$;
- the image of G_0 in $PGL(V_i)$ is relatively compact for every $i \in \{1, \dots, r\}$.

Since the image of the commutator subgroup $[G_0, G_0]$ in $GL(V_i)$ is contained $SL(V_i)$, it follows that the image of $[G_0, G_0]$ in $GL(V_i)$ is relatively compact for every $i \in \{1, \dots, r\}$. As ν is not supported on a proper projective subspace, the linear span of $V_1 \cup \cdots \cup V_r$ coincides with V. This implies that $[G_0, G_0]$ is relatively compact in $GL(V)$. Therefore, G_0 and hence G is amenable. $\square$

A further ingredient we will need is the following result which is Proposition 9 and Lemma 10 in [BeGu15]; observe that, although only the case $\mathbf{k} = \mathbf{R}$ is considered there, the arguments for the proof are valid without change for any local field $\mathbf{k}$.

Proposition 11. *Let V be a finite dimensional vector space over a local field $\mathbf{k}$ and G a subgroup of $GL(V)$. There exists a largest G-invariant linear subspace $V(G)$ of V such that the closure of the image of G in $GL(V(G))$ is an amenable locally compact group. Moreover, we have $\overline{V}(G) = \{0\}$ for $\overline{V} = V/V(G)$.* $\square$

We will also need the following (probably well-known) lemma, for which we could not find a reference. Recall that a group is *linear* if it can be embedded as a subgroup of $GL_n(\mathbf{k})$ for some field $\mathbf{k}$.

Lemma 12. *Let Γ be a linear group. Assume that Γ is finite-by-abelian (that is, Γ is a finite extension of an abelian group). Then Γ is virtually abelian (that is, Γ is abelian-by-finite).*

Proof. We may assume that Γ is a subgroup of $GL_n(\mathbf{k})$ for an algebraically closed field $\mathbf{k}$.

By assumption, there exists a finite normal subgroup of Γ containing $[\Gamma, \Gamma]$. In particular, $[\Gamma, \Gamma]$ is finite.

Let $G \subset GL_n(\mathbf{k})$ be the closure of Γ in the Zariski topology. Since $[\Gamma, \Gamma]$ is finite, $[\Gamma, \Gamma]$ is a Zariski closed subgroup of G. It follows that $[G, G] = [\Gamma, \Gamma]$ and hence that $[G, G]$ is finite. In particular, $[G^0, G^0]$ is finite, where G^0 is the Zariski connected component of G. However, $[G^0, G^0]$ is connected (see Proposition 17.2 in [Hump75]). Therefore, $[G^0, G^0] = \{e\}$, that is, G^0 is abelian. Let $\Gamma^0 = \Gamma \cap G^0$. Then Γ^0 is a subgroup of finite index in Γ and Γ^0 is abelian. $\square$

Observe that the previous lemma does not hold for non linear groups: let V be an infinite dimensional vector space over a finite field $\mathbf{F}$ of characteristic different from 2 and $\omega : V \times V \to \mathbf{F}$ a symplectic form on V. Let Γ be the associated "Heisenberg group", that is, $\Gamma = V \times \mathbf{F}$ with the law $(v, \lambda)(w, \beta) = (v + w, \lambda + \beta + \omega(v, w))$. Then Γ is finite-by-abelian but not virtually abelian.

2.3. **The dual group of a solenoid and the ring of adèles.** Solenoids are characterized in terms of their Pontrjagin dual groups as follows. Recall that the rank (also called Prüfer rank) of an abelian group A

is the cardinality of a maximal linearly independent subset of A. A compact abelian group X is a solenoid if and only if $\widehat{X}$ is a finite rank, torsion-free, abelian group; when this is the case, the topological dimension of X coincides with the rank of $\widehat{X}$ (see Theorem (23.18) in [HeRo63]).

Let X be a solenoid. Let $d \geq 1$ be the rank of $\widehat{X}$. Since $\widehat{X}$ is torsion-free,

$$V_{\mathbf{Q}} := \widehat{X} \otimes_{\mathbf{Z}} \mathbf{Q}$$

is a $\mathbf{Q}$-vector space of dimension d and we may (and will) view $\widehat{X}$ as a subgroup of $V_{\mathbf{Q}}$ via the embedding

$$\widehat{X} \to V_{\mathbf{Q}}, \qquad \chi \mapsto \chi \otimes 1.$$

(Since, obviously, every subgroup of $\mathbf{Q}^d$ is torsion-free abelian group of finite rank, we see that the solenoids are exactly the dual groups of subgroups of $\mathbf{Q}^d$ for some $d \geq 1$.)

We will need to further embed $\widehat{X}$ in vector spaces over various local fields. Let $\mathcal{P}$ be the set of primes of $\mathbf{N}$. Recall that, for every $p \in \mathcal{P}$, the additive group of the field $\mathbf{Q}_p$ of p-adic numbers is a locally compact group containing the subring $\mathbf{Z}_p$ of p-adic integers as compact open subgroup. The ring $\mathbf{A}$ of adèles of $\mathbf{Q}$ is the restricted product $\mathbf{A} = \mathbf{R} \times \prod_{p\in\mathcal{P}}(\mathbf{Q}_p, \mathbf{Z}_p)$ relative to the subgroups $\mathbf{Z}_p$; thus,

$$\mathbf{A} = \left\{ (a_\infty, a_2, a_3, \cdots) \in \mathbf{R} \times \prod_{p\in\mathcal{P}} \mathbf{Q}_p \mid a_p \in \mathbf{Z}_p \text{ for almost every } p \in \mathcal{P} \right\}.$$

The field $\mathbf{Q}$ can be viewed as discrete and cocompact subring of the locally compact ring $\mathbf{A}$ via the diagonal embedding

$$\mathbf{Q} \to \mathbf{A}, \qquad q \mapsto (q, q, \dots).$$

Set $\mathbf{Q}_\infty := \mathbf{R}$ and for $p \in \mathcal{P} \cup \{\infty\}$, set

$$V_p = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_p.$$

Then V_p is a d-dimensional vector space over $\mathbf{Q}_p$ and $V_{\mathbf{Q}}$ can be viewed as a subspace of V_p for every $p \in \mathcal{P} \cup \{\infty\}$.

Fix a basis $\mathcal{B}$ of $V_{\mathbf{Q}}$ over $\mathbf{Q}$ contained in $\widehat{X}$. Then $\mathcal{B}$ is a basis of V_p over $\mathbf{Q}_p$ for every $p \in \mathcal{P} \cup \{\infty\}$. For $p \in \mathcal{P}$, let $\mathcal{B}_p$ be the $\mathbf{Z}_p$-module generated by $\mathcal{B}$ in V_p. The restricted product

$$V_{\mathbf{A}} = V_\infty \times \prod_{p\in\mathcal{P}} (V_p, \mathcal{B}_p)$$

is a locally compact ring and $V_{\mathbf{Q}}$ embeds diagonally as a discrete and cocompact subgroup of $V_{\mathbf{A}}$ (for all this, see Chap. IV in [Weil67]).

As a result of this discussion, we can view $\widehat{X}$ as a subgroup of $V_{\mathbf{Q}}$ which is itself a discrete and cocompact subgroup of $V_{\mathbf{A}}$. Since the dual group of $V_{\mathbf{Q}} \cong \mathbf{Q}^d$ may be identified with $\mathbf{A}^d/\mathbf{Q}^d$ (see Subsection 3.2), observe that X is a quotient of the full d-dimensional solenoid $\mathbf{A}^d/\mathbf{Q}^d$.

We discuss now the automorphisms of $\widehat{X}$. Every $\theta \in \mathrm{Aut}(\widehat{X})$ extends, in a unique way, to an automorphism $\widetilde{\theta}$ of $V_{\mathbf{Q}}$ defined by

$$\widetilde{\theta}(\chi \otimes n/m) = \theta(n\chi) \otimes (1/m) \qquad \text{for all} \quad \chi \in \widehat{X},\, n/m \in \mathbf{Q}.$$

Therefore, we may identify $\mathrm{Aut}(\widehat{X})$ with a subgroup $GL(V_{\mathbf{Q}})$. So, $\mathrm{Aut}(\widehat{X})$ embeds diagonally as a *discrete* subgroup of the locally compact group $GL(V_{\mathbf{A}}) \cong GL_d(\mathbf{A})$, which is also the restricted product

$$GL(V_{\mathbf{A}}) = GL(V_\infty) \times \prod_{p \in \mathcal{P}} (GL(V_p), GL(\mathcal{B}_p)).$$

Let $p_1, \cdots, p_r$ be different primes and $a = p_1 \dots p_2$. Let $\mathrm{Aut}(\widehat{X})_{\mathbf{Z}[1/a]}$ be the subgroup of all $\theta \in \mathrm{Aut}(\widehat{X})$ such that

$$\theta(\mathcal{B}_p) = \mathcal{B}_p \qquad \text{for all} \quad p \in \mathcal{P} \setminus \{p_1, \cdots, p_r\}.$$

Then $\mathrm{Aut}(\widehat{X})_{\mathbf{Z}[1/a]}$ may be identified with a subgroup of $GL_d(\mathbf{Z}[1/a])$ and embeds diagonally as a *discrete* subgroup of the locally compact group

$$GL(V_\infty) \times GL(V_{p_1}) \times \cdots \times GL(V_{p_r}).$$

2.4. **Dual of the a-adic solenoid.** Set $X := \mathbf{S}_a^d$ for a square free integer $a = p_1 \dots p_r$. Recall that $X = \mathbf{A}_a^d/\mathbf{Z}[1/a]^d$, with $\mathbf{Z}[1/a]^d$ diagonally embedded in the locally compact ring

$$\mathbf{A}_a^d := \mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d.$$

We identify $\widehat{\mathbf{R}}$ with $\mathbf{R}$ via the map $\mathbf{R} \to \widehat{\mathbf{R}}, y \mapsto e_y$ given by $e_y(x) = e^{2\pi i x y}$ and $\widehat{\mathbf{Q}_p}$ with $\mathbf{Q}_p$ via the map $\mathbf{Q}_p \to \widehat{\mathbf{Q}_p}, y \mapsto \chi_y$ given by $\chi_y(x) = \exp(2\pi i \{xy\})$, where $\{x\} = \sum_{j=m}^{-1} a_j p^j$ denotes the "fractional part" of a p-adic number $x = \sum_{j=m}^{\infty} a_j p^j$ for integers $m \in \mathbf{Z}$ and $a_j \in \{0, \dots, p-1\}$ (see [BeHV08, Section D.4]).

Then $\widehat{\mathbf{A}_a^d}$ is identified with $\mathbf{A}_a^d$ and $\widehat{X}$ with the annihilator of $\mathbf{Z}[1/a]^d$ in $\mathbf{A}_a^d$, that is, with $\mathbf{Z}[1/a]^d$ embedded in $\mathbf{A}_a^d$ via the map

$$\mathbf{Z}[1/a]^d \to \mathbf{A}_a^d, \qquad b \mapsto (b, -b \cdots, -b).$$

Under this identification, the dual action of the automorphism group

$$\mathrm{Aut}(\mathbf{A}_d^d) \cong GL_d(\mathbf{R}) \times GL(\mathbf{Q}_{p_1}) \times \cdots \times GL(\mathbf{Q}_{p_r})$$

on $\widehat{\mathbf{A}_a^d}$ corresponds to the right action on $\mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d$ given by

$$((g_\infty, g_1, \cdots, g_r), (a_\infty, a_1, \cdots, a_r)) \mapsto (g_\infty^t a_\infty, g_1^t a_1, \cdots, g_r^t a_r),$$

where $(g, a) \mapsto ga$ is the usual (left) linear action of $GL_d(\mathbf{k})$ on $\mathbf{k}^d$ for a field $\mathbf{k}$.

3. Proofs of Theorem 2, Theorem 5, and Proposition 1

3.1. **Proof of Theorem 2.** Proposition 8 shows that (i) implies (ii). The fact that (iii) implies (i) follows a general result : a measure preserving action of a countable amenable group on a non atomic probability (Y, ν) never has the spectral gap property (see [JuRo79, Theorem 2.4] or [Schm81, (2.4) Theorem]). Since Γ, which is isomorphic to a subgroup of $GL_d(\mathbf{Q})$, is a linear subgroup over a field of characteristic zero, (iii) implies (iv) by one part of Tits' alternative theorem ([Tits72]). As (iii) is an obvious consequence of (iv), it remains to show that (ii) implies (iii). We will proceed in several steps.

• *First step.* Assume that there exists $p_{\mathrm{a}}(\Gamma)$-invariant proper subsolenoid Y of X such that the image Δ of $p_{\mathrm{a}}(\Gamma)$ in $\mathrm{Aut}(X/Y)$ is amenable. We claim that the image of Γ in $\mathrm{Aff}(X/Y)$ is amenable.

Indeed, the image of Γ in $\mathrm{Aff}(X/Y)$ is a subgroup of $\Delta \ltimes (X/Y)$. Since X/Y is abelian, $\Delta \ltimes (X/Y)$ is amenable (as discrete group) and the claim follows.

In view of the first step, we may and will assume in the sequel that $\Gamma \subset \mathrm{Aut}(X)$. By duality, we can also view Γ as a subgroup of $\mathrm{Aut}(\widehat{X})$. In the sequel, we write 0 for the neutral element in $\widehat{X}$ instead of $\mathbf{1}_X$.

• *Second step.* We claim that there exists a Γ-invariant mean on $\widehat{X} \setminus \{0\}$.

Indeed, since $\Gamma \curvearrowright (X, \mu)$ does not have a spectral gap, this follows by a standard argument: there exists a sequence $(\xi_n)_{n\geq 1}$ of unit vectors in $\ell^2(\widehat{X} \setminus \{0_X\})$ such that

$$\lim_n \|\pi_X(\gamma)\xi_n - \xi_n\|_2 = 0 \qquad \text{for all} \quad \gamma \in \Gamma,$$

for the associated Koopman representation π_X (see Proof of Proposition 8). Then $\eta_n := |\xi_n|^2$ is a unit vector in $\ell^1(\widehat{X} \setminus \{0\})$ and

$$\lim_n \|\pi_X(\gamma)\eta_n - \eta_n\|_1 = 0 \qquad \text{for all} \quad \gamma \in \Gamma.$$

Any weak*-limit of $(\eta_n)_n$ in the dual space of $\ell^\infty(\widehat{X}\setminus\{0\})$ is a Γ-invariant mean on $\widehat{X} \setminus \{0\}$.

Let d be the rank of $\widehat{X}$. As in Subsection 2.3, we embed $\widehat{X}$ in the d-dimensional $\mathbf{Q}$-vector space $V_{\mathbf{Q}} = \widehat{X} \otimes_{\mathbf{Z}} \mathbf{Q}$ as well as in the d-dimensional $\mathbf{Q}_p$-vector spaces $V_p = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_p$ for $p \in \mathcal{P} \cup \{\infty\}$, where $\mathcal{P}$ is the set of primes and where $\mathbf{Q}_\infty = \mathbf{R}$. Accordingly, we identify $\mathrm{Aut}(\widehat{X})$ with a subgroup of $GL(V_{\mathbf{Q}})$.

We fix a Γ-invariant mean M on $\widehat{X} \setminus \{0\}$, which we view as mean on $\widehat{X}$ and write $M(A)$ instead of $M(\mathbf{1}_A)$ for a subset A of $\widehat{X}$.

- *Third step.* Let $p \in \mathcal{P} \cup \{\infty\}$. We claim that

$$M(\widehat{X} \cap V_p(\Gamma)) = 1,$$

where $V_p(\Gamma)$ is the Γ-invariant linear subspace of V_p defined in Proposition 11.

The proof of the claim is similar to the proof of Proposition 13 in [BeGu15]; for the convenience of the reader, we repeat the main arguments. Assume, by contradiction, that $M(\widehat{X} \cap V_p(\Gamma)) < 1$. We therefore have

$$t := M(\widehat{X} \setminus V_p(\Gamma)) > 0.$$

Then a Γ-invariant mean M_1 is defined on $\widehat{X} \setminus V_p(\Gamma)$ by

$$M_1(A) = \frac{1}{t} M(A) \qquad \text{for all} \quad A \subset \widehat{X} \setminus V_p(\Gamma).$$

Consider the quotient vector space $\overline{V_p} = V_p/V_p(\Gamma)$ with the induced Γ-action. The image of $\widehat{X} \setminus V_p(\Gamma)$ under the canonical projection $j : V_p \to \overline{V_p}$ does not contain $\{0\}$. So, $\overline{M_1} := j_*(M_1)$ is a Γ-invariant mean on $\overline{V_p} \setminus \{0\}$. By Lemma 9, the pushforward of M_1 on the projective $\mathbf{P}(\overline{V_p})$ defines a Γ-invariant probability measure ν on $\mathbf{P}(\overline{V_p})$.

Let $\overline{W}$ be the linear span of the inverse image of $\mathrm{supp}(\nu)$ in $\overline{V_p}$. Then $\overline{W} \neq \{0\}$ and ν is not supported on a proper projective subspace in $\mathbf{P}(\overline{W})$. Proposition 10 shows that the closure of the image of Γ in $GL(\overline{W})$ is an amenable group. It follows that $\overline{V_p}(\Gamma) \neq \{0\}$. This contradicts Proposition 11.

Let $\mathcal{P} = \{p_1, p_2, p_3, \cdots\}$ be an enumeration of the set $\mathcal{P}$ of prime integers.

- *Fourth step.* We claim that, for every $n \in \mathbf{N}$, we have

$$\widehat{X} \cap V_\infty(\Gamma) \cap \bigcap_{i=1}^{n} V_{p_i}(\Gamma) \neq \{0\}.$$

Indeed, by the third step, we have $M(\widehat{X} \setminus \{0\} \cap V_p(\Gamma)) = 1$ for every $p \in \{p_1, \dots, p_r\} \cup \{\infty\}$. By finite-additivity of M, it follows that

$$M\left(\widehat{X} \setminus \{0\} \cap V_\infty(\Gamma) \cap \bigcap_{i=1}^{n} V_{p_i}(\Gamma)\right) = 1;$$

this proves the claim in particular.

Fixing a basis $\mathcal{B}$ of $V_{\mathbf{Q}}$ over $\mathbf{Q}$ contained in $\widehat{X}$, and denoting by $\mathcal{B}_p$ the $\mathbf{Z}_p$-module generated by $\mathcal{B}$ in V_p for $p \in \mathcal{P}$, we consider the locally compact group $GL(V_{\mathbf{A}})$, which is the restricted product of the $GL(V_p)$'s with respect to the compact groups $GL(\mathcal{B}_p)$'s (see Subsection 2.3).

For $p \in \mathcal{P} \cup \{\infty\}$, let G_p denote the closure of the image of Γ in $GL(V_p(\Gamma))$. Set

$$G := (G_\infty \times \prod_{p \in \mathcal{P}} G_p) \cap GL(V_{\mathbf{A}}).$$

• *Fifth step.* We claim that G is a closed amenable subgroup of $GL(V_{\mathbf{A}})$.

Indeed, for every $n \geq 1$, set

$$H_n := G_\infty \times \prod_{i=1}^{n} G_{p_i} \times K_n,$$

where K_n is the compact group $\prod_{i>n} (G_{p_i} \cap GL(\mathcal{B}_{p_i})$. Then $(H_n)_n$ is an increasing sequence of open subgroups of G and $G = \bigcup_{n \geq 1} H_n$. Clearly, every H_n is a closed subgroup of $GL(V_{\mathbf{A}})$. Hence, G is a locally compact and therefore a closed subgroup of $GL(V_{\mathbf{A}})$.

To show that G is amenable, it suffices to show that every H_n is amenable (see [BeHV08, Proposition G.2.2]). This is indeed the case, since every G_p is amenable by definition of $V_p(\Gamma)$ and since K_n is compact.

For every $n \in \mathbf{N}$, denote by W^n the $\mathbf{Q}$-linear span of

$$\widehat{X} \cap V_\infty(\Gamma) \cap \bigcap_{i=1}^{n} V_{p_i}(\Gamma).$$

• *Sixth step.* We claim that there exists $N \in \mathbf{N}$ such that $W^n = W^N$ for every $n \geq N$.

Indeed, $(W^n)_{n \geq 1}$ is a decreasing sequence of linear subspaces of $V_{\mathbf{Q}}$. By the fourth step, we have $\dim_{\mathbf{Q}} W^n > 0$ for every $n \geq 1$. Hence, there exists $N \in \mathbf{N}$ such that $\dim_{\mathbf{Q}} W^n = \dim_{\mathbf{Q}} W^N$ for every $n \geq N$ and the claim is proved.

Set $W := W^N$ and observe that W is Γ-invariant.

• *Seventh step.* We claim that the image of Γ in $\mathrm{Aut}(\widehat{X} \cap W)$ is amenable.

Indeed, W is a subspace of $V_{\mathbf{Q}}$ and is contained in every $V_p(\Gamma)$. On the one hand, under the diagonal embedding, $G \cap GL(W)$ is a discrete subgroup of G, since the neighbourhood

$$U \times \prod_{p \in P} (G_p \cap GL(\mathcal{B}_p))$$

of e in G has trivial intersection with $GL(W)$, for a sufficiently small neighbourhood U of e in G_∞. On the other hand, $G \cap GL(W)$ is amenable, by the fifth step. It follows that the image $\widetilde{\Gamma} \subset G \cap GL(W)$ of Γ in $GL(W)$ is amenable. The image of Γ in $\mathrm{Aut}(\widehat{X} \cap W)$ is a quotient of $\widetilde{\Gamma}$ and is therefore amenable.

Let

$$Y := (\widehat{X} \cap W)^\perp = \left\{ x \in X \mid \chi(x) = 1 \qquad \text{for all} \quad \chi \in \widehat{X} \cap W \right\}$$

be the annihilator in X of the subgroup $\widehat{X} \cap W$ of $\widehat{X}$.

• *Eighth step.* We claim Y is a Γ-invariant proper subsolenoid of X and that the image of Γ in $\mathrm{Aut}(X/Y)$ is amenable.

Indeed, Y is clearly a closed Γ-invariant subgroup of X and $Y \neq X$ since $\widehat{X} \cap W$ is non trivial, by the fourth step. Moreover, the dual group $\widehat{Y}$ of Y, which is isomorphic to $\widehat{X}/(\widehat{X} \cap W)$, is torsion free: if $\chi \in \widehat{X}$ is such that $n\chi \in W$ for some integer $n \geq 1$, then $\chi \in W$, since W is a $\mathbf{Q}$-linear subspace. As, obviously, $\widehat{Y}$ has finite rank, it follows that the compact group Y is a solenoid.

By the seventh step, the image of Γ in $\mathrm{Aut}(\widehat{X} \cap W)$ is amenable. Since, $\mathrm{Aut}(X/Y)$ is isomorphic to $\mathrm{Aut}(\widehat{X} \cap W)$ by duality, it follows that the image of Γ in $\mathrm{Aut}(X/Y)$ is amenable. $\square$

3.2. **Proof of Theorem 5.** We only have to show that (i) implies (ii). Set $X := \mathbf{S}_a^d$ for $a = p_1 \dots p_r$ and let Γ be a subgroup of $\mathrm{Aff}(\mathbf{S}_a^d)$. As in the proof of Theorem 2, we may assume that $\Gamma \subset \mathrm{Aut}(X)$.

Recall from Subsection 2.4 that we may identify $\widehat{X}$ with the discrete subring $\mathbf{Z}[1/a]^d$ of

$$\mathbf{A}_a^d = \mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d$$

and $\mathrm{Aut}(\widehat{X})$ with the discrete subgroup $GL_d(\mathbf{Z}[1/a])$ of $GL_d(\mathbf{A}_a)$, with the dual action of $\gamma \in \mathrm{Aut}(X)$ on $\mathbf{A}_a^d$ given by matrix transpose.

As in the proof of Theorem 2, there exists a Γ-invariant mean M on $\widehat{X} \setminus \{0\}$. Let W be a non zero $\mathbf{Q}$-linear subspace of $V_{\mathbf{Q}} = \widehat{X} \otimes_{\mathbf{Z}} \mathbf{Q}$ of minimal dimension with $M(W) = 1$. Then W is Γ-invariant, by Γ-invariance of M. We claim that the image of Γ in $GL(W)$ is virtually abelian.

Indeed, fix $p \in \{p_1, \dots, p_r\} \cup \{\infty\}$. Set $W_p = W \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and let G_p be the closure of the image of Γ in $GL(W_p)$. Let μ_p be the G_p-invariant probability measure on $\mathbf{P}(W_p)$ which is the pushforward of M under the map $W \setminus \{0\} \to \mathbf{P}(W_p)$. Then μ_p is not supported on a proper projective subspace of W_p : if W' is a $\mathbf{Q}_p$-linear subspace of W_p with $\mu_p([W']) = 1$, where $[W']$ is the image of W' in $\mathbf{P}(W_p)$, then $M(W' \cap W) = 1$ and hence $W' \cap W = W$, by minimality of W; so $W' = W \otimes_{\mathbf{Q}} \mathbf{Q}_p = W_p$.

By Proposition 10, there exists therefore a finite index subgroup H_p of G_p with a relatively compact commutator subgroup $[H_p, H_p]$.

Set $G = G_\infty \times \prod_{i=1}^r G_{p_i}$. As in the proof of Theorem 2, the image $\widetilde{\Gamma}$ of Γ in G is discrete. Then $\widetilde{\Gamma_0} := \widetilde{\Gamma} \cap \prod_{i=1}^r H_{p_i}$ is a subgroup of finite index in $\widetilde{\Gamma}$ and its commutator $[\widetilde{\Gamma_0}, \widetilde{\Gamma_0}]$ is finite. Since $\widetilde{\Gamma_0} \subset GL(W)$ is linear, it follows therefore from Lemma 12 that $\widetilde{\Gamma_0}$ and hence $\widetilde{\Gamma}$ is virtually abelian. This concludes the proof of Theorem 5. $\square$

Remark 13. Let X be as in Theorem 2 and let Γ be finitely generated subgroup of $\mathrm{Aut}(X)$. We claim that there exists finitely many different primes $p_1, \cdots, p_r$ such that Γ is contained in the subgroup $\mathrm{Aut}(\widehat{X})_{\mathbf{Z}[1/a]}$ defined in Subsection 2.3, where $a = p_1 \dots p_r$.

Indeed, let $\gamma_1, \dots, \gamma_n$ be a set generators of Γ. Let $\mathcal{B}$ be a basis of $V_{\mathbf{Q}} = \widehat{X} \otimes_{\mathbf{Z}} \mathbf{Q}$ over $\mathbf{Q}$ contained in $\widehat{X}$. Then every γ_i leaves invariant the $\mathbf{Z}_p$-module $\mathcal{B}_p$ generated by $\mathcal{B}$ in $V_p = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_p$ for almost every prime p and the claim follows.

Assume that Item (i) in Theorem 2 holds for the action of Γ on X. The proof of Theorem 5 shows that there exists a Γ-invariant subspace W of $V_{\mathbf{Q}}$ such that the image of Γ in $GL(W)$ is virtually abelian. Then $Y = (\widehat{X} \cap W)^\perp$ is a subsolenoid in X and the image of Γ in $\mathrm{Aut}(X/Y)$ is virtually abelian.

3.3. **Proof of Proposition 1.** Let Γ be a subgroup of $\mathrm{Aff}(X)$.

Assume that there exists a proper closed subgroup Y such that the image $\overline{\Gamma}$ of Γ in $\mathrm{Aff}(X/Y)$ is finite. Since X is compact and connected, $\overline{X} = X/Y$ is a non trivial compact connected group. It is easy to see that there exist two $\overline{\Gamma}$-invariant non empty open subsets of $\overline{X}$ which are disjoint. The preimages U and V of these sets in X are Γ-invariant non empty open subsets and are disjoint. Since the support of the Haar measure μ of X coincides with X, we have $\mu(U) \neq 0$ and $\mu(V) \neq 0$. Hence, $\Gamma \curvearrowright (X, \mu)$ is not ergodic.

Conversely, assume that $\Gamma \curvearrowright (X, \mu)$ is not ergodic. Since X is connected, $\widehat{X}$ is torsion free. As in the previous sections, we view $\widehat{X}$ as subgroup of the (possibly infinite dimensional) $\mathbf{Q}$-vector space $V_{\mathbf{Q}} = \widehat{X} \otimes_{\mathbf{Z}} \mathbf{Q}$. We realize the associated Koopman representation π_X of Γ in $\ell^2(\widehat{X})$ as in Subsection 2.1. By non ergodicity of the action, there exists a non-zero Γ-invariant vector $\xi \in \ell^2(\widehat{X} \setminus \{0\})$. Thus, we have (see Formula $(*)$ from Subsection 2.1)

$$(**) \qquad \chi(x)\xi(\chi^\theta) = \xi(\chi) \qquad \text{for all} \quad \xi \in \ell^2(\widehat{X}), \chi \in \widehat{X},$$

for all $(\theta, x) \in \Gamma$.

Set $\eta := |\xi|$. Then $\eta \neq 0$ and Formula $(**)$ shows that η is $p_{\mathrm{a}}(\Gamma)$-invariant. Let $\chi_0 \in \widehat{X} \setminus \{0\}$ be such that $\eta(\chi_0) \neq 0$. Since $\eta \in \ell^2(\widehat{X})$ and since $\eta \neq 0$, it follows that the $p_{\mathrm{a}}(\Gamma)$-orbit is finite.

Let W be the linear span of the $p_{\mathrm{a}}(\Gamma)$-orbit of χ_0 in $V_{\mathbf{Q}}$ and let $Y := (\widehat{X} \cap W)^{\perp}$ be the annihilator of $\widehat{X} \cap W$ in X. Then Y is a $p_{\mathrm{a}}(\Gamma)$-invariant closed subgroup of X and $Y \neq X$ since $\chi_0 \neq 0$.

Moreover, $\widehat{Y} \cong \widehat{X}/(\widehat{X} \cap W)$ is torsion free and hence Y is connected: if $\chi \in \widehat{X}$ is such that $n\chi \in W$ for some integer $n \geq 1$, then $\chi \in W$, since W is a $\mathbf{Q}$-linear subspace.

We claim that the image of Γ in $\mathrm{Aff}(X/Y)$ is a finite group. Indeed, since the $p_{\mathrm{a}}(\Gamma)$-orbit of χ_0 is finite, we can find a normal subgroup Λ in $p_{\mathrm{a}}(\Gamma)$ of finite index which fixes χ_0. Set

$$\Gamma_0 := p_{\mathrm{a}}^{-1}(\Lambda) \cap \Gamma.$$

Then Γ_0 is a normal subgroup of finite index in Γ.

Let $\gamma = (\theta, x) \in \Gamma_0$. Formula $(**)$ shows that $\chi(x) = 1$ for every χ in the $p_{\mathrm{a}}(\Gamma)$-orbit of χ_0 and hence for every $\chi \in \widehat{X} \cap W$. Using Formula $(*)$ from Subsection 2.1, it follows that Γ_0 acts trivially under the Koopman representation $\pi_{X/Y}$ on $\ell^2(\widehat{X/Y}) = \ell^2(\widehat{X} \cap W)$ associated to the action of Γ on X/Y. So, the image of Γ_0 in $\mathrm{Aff}(X/Y)$ is trivial and therefore the image of Γ in $\mathrm{Aff}(X/Y)$ is finite.□

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