

ON THE SPECTRAL THEORY OF GROUPS OF AUTOMORPHISMS OF S -ADIC NILMANIFOLDS

BACHIR BEKKA AND YVES GUIVARC'H

ABSTRACT. Let $S = \{p_1, \dots, p_r, \infty\}$ for prime integers $p_1, \dots, p_r$. Let X be an S -adic compact nilmanifold, equipped with the unique translation invariant probability measure μ . We characterize the countable groups Γ of automorphisms of X for which the Koopman representation κ on $L^2(X, \mu)$ has a spectral gap. More specifically, let Y be the maximal quotient solenoid of X (thus, Y is a finite-dimensional, connected, compact abelian group). We show that κ does not have a spectral gap if and only if there exists a Γ -invariant proper subsolenoid of Y on which Γ acts a virtually abelian group,

1. INTRODUCTION

Let Γ be a countable group acting measurably on a probability space (X, μ) by measure preserving transformations. Let $\kappa = \kappa_X$ denote the corresponding Koopman representation of Γ , that is, the unitary representation of Γ on $L^2(X, \mu)$ given by

$$\kappa(\gamma)\xi(x) = \xi(\gamma^{-1}x) \quad \text{for all } \xi \in L^2(X, \mu), x \in X, \gamma \in \Gamma.$$

We say that the action $\Gamma \curvearrowright (X, \mu)$ of Γ on (X, μ) has a **spectral gap** if the restriction κ_0 of κ to the Γ -invariant subspace

$$L_0^2(X, \mu) = \{\xi \in L^2(X, \mu) : \int_X \xi(x) d\mu(x) = 0\}$$

does not weakly contain the trivial representation 1_Γ ; equivalently, if κ_0 does not have almost invariant vectors, that is, there is **no** sequence $(\xi_n)_n$ of unit vectors in $L_0^2(X, \mu)$ such that

$$\lim_n \|\kappa_0(\gamma)\xi_n - \xi_n\| = 0 \quad \text{for all } \gamma \in \Gamma.$$

The existence of a spectral gap admits the following useful quantitative version. Let ν be a probability measure on Γ and $\kappa_0(\nu)$ the convolution operator defined on $L_0^2(X, \mu)$ by

$$\kappa_0(\nu)\xi = \sum_{\gamma \in \Gamma} \nu(\gamma)\kappa_0(\gamma)\xi \quad \text{for all } \xi \in L_0^2(X, \mu).$$

2020 *Mathematics Subject Classification.* 37A05, 22F30, 60B15, 60G50.

Observe that we have $\|\kappa_0(\nu)\| \leq 1$ and hence $r(\kappa_0(\nu)) \leq 1$ for the spectral radius $r(\kappa_0(\nu))$ of $\kappa_0(\nu)$. Assume that ν is aperiodic, that is, the support of ν is not contained in the coset of a proper subgroup of Γ . Then the action of Γ on X has a spectral gap if and only if $r(\kappa_0(\nu)) < 1$ and this is equivalent to $\|\kappa_0(\nu)\| < 1$; for more details, see the survey [Bekk16].

In this paper, we will be concerned with the case where X is an S -adic nilmanifold, to be introduced below, and Γ is a subgroup of automorphisms of X .

Fix a finite set $\{p_1, \dots, p_r\}$ of integer primes and set $S = \{p_1, \dots, p_r, \infty\}$. For an integer $d \geq 1$, the product

$$\mathbf{Q}_S := \prod_{p \in S} \mathbf{Q}_p = \mathbf{Q}_\infty \times \mathbf{Q}_{p_1} \times \dots \times \mathbf{Q}_{p_r}$$

is a locally compact ring, where $\mathbf{Q}_\infty = \mathbf{R}$ and $\mathbf{Q}_p$ is the field of p -adic numbers for a prime p . Let $\mathbf{Z}[1/S] = \mathbf{Z}[1/p_1, \dots, 1/p_r]$ denote the subring of $\mathbf{Q}$ generated by 1 and $\{1/p_1, \dots, 1/p_r\}$. Through the diagonal embedding

$$\mathbf{Z}[1/S] \rightarrow \mathbf{Q}_S, \quad b \mapsto (b, \dots, b),$$

we may identify $\mathbf{Z}[1/S]$ with a discrete and cocompact subring of $\mathbf{Q}_S$.

If $\mathbf{G}$ is a linear algebraic group defined over $\mathbf{Q}$, we denote by $\mathbf{G}(R)$ the group of elements of $\mathbf{G}$ with coefficients in R and determinant invertible in R , for every subring R of an overfield of $\mathbf{Q}$.

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$, that is, $\mathbf{U}$ is an algebraic subgroup of the group of $n \times n$ upper triangular unipotent matrices for some $n \geq 1$. The group $\mathbf{U}(\mathbf{Q}_S)$ is a locally compact group and $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$ is a cocompact lattice in $\mathbf{U}(\mathbf{Q}_S)$. The corresponding S -adic compact nilmanifold

$$\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S])$$

will be equipped with the unique translation-invariant probability measure μ on its Borel subsets.

For $p \in S$, let $\text{Aut}(\mathbf{U}(\mathbf{Q}_p))$ be the group of continuous automorphisms of $\mathbf{U}(\mathbf{Q}_p)$. Set

$$\text{Aut}(\mathbf{U}(\mathbf{Q}_S)) := \prod_{p \in S} \text{Aut}(\mathbf{U}(\mathbf{Q}_p))$$

and denote by $\text{Aut}(\mathbf{Nil}_S)$ the subgroup

$$\{g \in \text{Aut}(\mathbf{U}(\mathbf{Q}_S)) \mid g(\Lambda) = \Lambda\}.$$

Every $g \in \text{Aut}(\mathbf{Nil}_S)$ acts on $\mathbf{Nil}_S$ preserving the probability measure μ .

The abelian quotient group

$$\overline{\mathbf{U}(\mathbf{Q}_S)} := \mathbf{U}(\mathbf{Q}_S) / [\mathbf{U}(\mathbf{Q}_S), \mathbf{U}(\mathbf{Q}_S)]$$

can be identified with $\mathbf{Q}_S^d$ for some $d \geq 1$ and the image Δ of $\mathbf{U}(\mathbf{Z}[1/S])$ in $\overline{\mathbf{U}(\mathbf{Q}_S)}$ is a cocompact and discrete subgroup of $\overline{\mathbf{U}(\mathbf{Q}_S)}$; so,

$$\mathbf{Sol}_S := \overline{\mathbf{U}(\mathbf{Q}_S)} / \Delta$$

is a solenoid (that is, is a finite-dimensional, connected, compact abelian group; see [HeRo63, §25]). We refer to $\mathbf{Sol}_S$ as the S -adic solenoid attached to the S -adic nilmanifold $\mathbf{Nil}_S$. We equip $\mathbf{Sol}_S$ with the probability measure ν which is the image of μ under the canonical projection $\mathbf{Nil}_S \rightarrow \mathbf{Sol}_S$.

Observe that $\text{Aut}(\mathbf{Q}_S^d)$ is canonically isomorphic to $\prod_{s \in S} GL_d(\mathbf{Q}_s)$ and that $\text{Aut}(\mathbf{Sol}_S)$ can be identified with the subgroup $GL_d(\mathbf{Z}[1/S])$. The group $\text{Aut}(\mathbf{Nil}_S)$ acts naturally by automorphisms of $\mathbf{Sol}_S$; we denote by

$$p_S : \text{Aut}(\mathbf{Nil}_S) \rightarrow GL_d(\mathbf{Z}[1/S]) \subset GL_d(\mathbf{Q})$$

the corresponding representation.

Theorem 1. *Let $\mathbf{U}$ be an algebraic unipotent group defined over $\mathbf{Q}$ and $S = \{p_1, \dots, p_r, \infty\}$, where $p_1, \dots, p_r$ are integer primes. Let $\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S) / \mathbf{U}(\mathbf{Z}[1/S])$ be the associated S -adic nilmanifold and let $\mathbf{Sol}_S$ be the corresponding S -adic solenoid, equipped with respectively the probability measures μ and ν as above. Let Γ be a countable subgroup of $\text{Aut}(\mathbf{Nil}_S)$. The following properties are equivalent:*

- (i) *The action $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ has a spectral gap.*
- (ii) *The action $p_S(\Gamma) \curvearrowright (\mathbf{Sol}_S, \nu)$ has a spectral gap, where $p_S : \text{Aut}(\mathbf{Nil}_S) \rightarrow GL_d(\mathbf{Z}[1/S])$ is the canonical homomorphism.*

Actions with spectral gap of groups of automorphisms (or more generally groups of affine transformations) of the S -adic solenoid $\mathbf{Sol}_S$ have been completely characterized in [BeFr20, Theorem 5]. The following result is an immediate consequence of this characterization and of Theorem 1. For a subset T of $GL_d(\mathbf{K})$ for a field $\mathbf{K}$, we denote by $T^t = \{g^t \mid g \in T\}$ the set of transposed matrices from T .

Corollary 2. *With the notation as in Theorem 1, the following properties are equivalent:*

- (i) *The action of Γ on the S -adic nilmanifold $\mathbf{Nil}_S$ does not have a spectral gap.*
- (ii) *There exists a non-zero linear subspace W of $\mathbf{Q}^d$ which is invariant under $p_S(\Gamma)^t$ and such that the image of $p_S(\Gamma)^t$ in $GL(W)$ is a virtually abelian group.*

Here is an immediate consequence of Corollary 2.

Corollary 3. *With the notation as in Corollary 1, assume that the linear representation of $p_S(\Gamma)^t$ in $\mathbf{Q}^d$ is irreducible. Then the action $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ has a spectral gap.*

Recall that the action of a countable group Γ by measure preserving transformations on a probability space (X, μ) is **strongly ergodic** (see [Schm81]) if every sequence $(B_n)_n$ of measurable subsets of X which is asymptotically invariant (that is, which is such that $\lim_n \mu(\gamma B_n \Delta B_n) = 0$ for all $\gamma \in \Gamma$) is trivial (that is, $\lim_n \mu(B_n)(1 - \mu(B_n)) = 0$). It is straightforward to check that the spectral gap property implies strong ergodicity and it is known that the converse does not hold in general.

The following corollary is a direct consequence of Theorem 1 (compare with Corollary 2 in [BeGu15]).

Corollary 4. *With the notation as in 1, the following properties are equivalent:*

- (i) *The action $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ has the spectral gap property.*
- (ii) *The action $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$ is strongly ergodic.*

Theorem 1 generalizes our previous work [BeGu15], where we treated the real case (that is, the case $S = \infty$). This requires an extension of our methods to the S -adic setting, which is a non straightforward task; more specifically, we had to establish the following four main tools for our proof:

- a canonical decomposition of the Koopman representation of Γ in $L^2(\mathbf{Nil}_S)$ as a direct sum of certain representations of Γ induced from stabilizers of representations of $\mathbf{U}(\mathbf{Q}_S)$; this fact is valid more generally for compact homogeneous spaces (see Proposition 9);
- a result of Howe and Moore [HoMo79] about the decay of matrix coefficients of algebraic groups (see Proposition 11);
- the fact that the irreducible representations of $\mathbf{U}(\mathbf{Q}_S)$ appearing in the decomposition of $L^2(\mathbf{Nil}_S)$ are rational, in the sense that the Kirillov data associated to each one of them are defined over $\mathbf{Q}$ (see Proposition 13);
- a characterization (see Lemma 12) of the projective kernel of the extension of an irreducible representation of $\mathbf{U}(\mathbf{Q}_p)$ to its stabilizer in $\text{Aut}(\mathbf{U}(\mathbf{Q}_p))$.

Another tool we constantly use is a generalized version of Herz's majoration principle (see Lemma 7).

Given a probability measure ν on Γ , our approach does not seem to provide quantitative estimates for the operator norm of the convolution

operator $\kappa_0(\nu)$ acting on $L_0^2(\mathbf{Nil}_S, \mu)$ for a general unipotent group $\mathbf{U}$. However, using known bounds for the so-called metaplectic representation of the symplectic group $Sp_{2n}(\mathbf{Q}_p)$, we give such estimates in the case of S -adic Heisenberg nilmanifolds (see Section 11).

Corollary 5. *For an integer $n \geq 1$, let $\mathbf{U} = \mathbf{H}_{2n+1}$ be the $(2n+1)$ -dimensional Heisenberg group and $\mathbf{Nil}_S = \mathbf{H}_{2n+1}(\mathbf{Q}_S)/\mathbf{H}_{2n+1}(\mathbf{Z}[1/S])$. Let ν be a probability measure on the subgroup $Sp_{2n}(\mathbf{Z}[1/S])$ of $\text{Aut}(\mathbf{Nil}_S)$. Then*

$$\|\kappa_0(\nu)\| \leq \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where κ_1 is the restriction of κ_0 to $L_0^2(\mathbf{Sol}_S)$ and λ_Γ is the regular representation of the group Γ generated by the support of ν . In particular, in the case $n = 1$ and ν is aperiodic, the action of Γ on $\mathbf{Nil}_S$ has a spectral gap if and only if Γ is non-amenable.

2. EXTENSION OF REPRESENTATIONS

Let G be a locally compact group which we assume to be second countable. We will need the notion of a projective representation. Recall that a mapping $\pi : G \rightarrow U(\mathcal{H})$ from G to the unitary group of the Hilbert space $\mathcal{H}$ is a **projective representation** of G if the following holds:

- $\pi(e) = I$,
- for all $g_1, g_2 \in G$, there exists $c(g_1, g_2) \in \mathbf{C}$ such that

$$\pi(g_1 g_2) = c(g_1, g_2) \pi(g_1) \pi(g_2),$$

- the function $g \mapsto \langle \pi(g)\xi, \eta \rangle$ is measurable for all $\xi, \eta \in \mathcal{H}$.

The mapping $c : G \times G \rightarrow \mathbf{S}^1$ is a 2-cocycle with values in the unit circle $\mathbf{S}^1$. Every projective unitary representation of G can be lifted to an ordinary unitary representation of a central extension of G (for all this, see [Mack76] or [Mack58]).

Let N be a closed normal subgroup of G . Let π be an irreducible unitary representation of N on a Hilbert space $\mathcal{H}$. Consider the stabilizer

$$G_\pi = \{g \in G \mid \pi^g \text{ is equivalent to } \pi\}$$

of π in G for the natural action of G on the unitary dual $\widehat{N}$ given by $\pi^g(n) = \pi(g^{-1}ng)$. Then G_π is a closed subgroup of G containing N . The following lemma is a well-known part of Mackey's theory of unitary representations of group extensions.

Lemma 6. *Let π be an irreducible unitary representation of N on the Hilbert space $\mathcal{H}$. There exists a projective unitary representation $\tilde{\pi}$ of G_π on $\mathcal{H}$ which extends π . Moreover, $\tilde{\pi}$ is unique, up to scalars: any*

other projective unitary representation $\tilde{\pi}'$ of G_π extending π is the form $\tilde{\pi}' = \lambda \tilde{\pi}$ for a measurable function $\lambda : G_\pi \rightarrow \mathbf{S}^1$.

Proof For every $g \in G_\pi$, there exists a unitary operator $\tilde{\pi}(g)$ on $\mathcal{H}$ such that

$$\pi(g(n)) = \tilde{\pi}(g)\pi(n)\tilde{\pi}(g)^{-1} \quad \text{for all } n \in N.$$

One can choose $\tilde{\pi}(g)$ such that $g \mapsto \tilde{\pi}(g)$ is a projective unitary representation of G_π which extends π (see Theorem 8.2 in [Mack58]). The uniqueness of π follows from the irreducibility of π and Schur's lemma. ■

3. A WEAK CONTAINMENT RESULT FOR INDUCED REPRESENTATIONS

Let G be a locally compact group with Haar measure μ_G . Recall that a unitary representation $(\rho, \mathcal{K})$ of G is weakly contained in another unitary representation $(\pi, \mathcal{H})$ of G , if every matrix coefficient

$$g \mapsto \langle \rho(g)\eta \mid \eta \rangle \quad \text{for } \eta \in \mathcal{K}$$

of π is the limit, uniformly over compact subsets of G , of a finite sum of matrix coefficients of ρ . Equivalently, if $\|\rho(f)\| \leq \|\pi(f)\|$ for every $f \in C_c(G)$, where $C_c(G)$ is the space of continuous functions with compact support on G and where the operator $\pi(f) \in \mathcal{B}(\mathcal{H})$ is defined by the integral

$$\pi(f)\xi = \int_G f(g)\pi(g)\xi d\mu_G(g) \quad \text{for all } \xi \in \mathcal{H}.$$

The trivial representation 1_G is weakly contained in π if and only if there exists, for every compact subset Q of G and every $\varepsilon > 0$, a unit vector $\xi \in \mathcal{H}$ which is (Q, ε) -invariant, that is such that

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| \leq \varepsilon.$$

Let H be a closed subgroup of G . We will always assume that the coset space $H \backslash G$ admits a non-zero G -invariant (possibly infinite) measure on its Borel subsets. Let $(\sigma, \mathcal{K})$ be a unitary representation of H . We will use the following model for the induced representation $\pi := \text{Ind}_H^G \sigma$. Choose a Borel fundamental domain $X \subset G$ for the action of G on $H \backslash G$. For $x \in X$ and $g \in G$, let $x \cdot g \in X$ and $c(x, g) \in H$ be defined by

$$xg = c(x, g)(x \cdot g).$$

There exists a non-zero G -invariant measure on X for the action $(x, g) \mapsto x \cdot g$ of G on X . The Hilbert space of π is the space $L^2(X, \mathcal{K}, \mu)$ of all

square-integrable measurable mappings $\xi : X \rightarrow \mathcal{K}$ and the action of G on $L^2(X, \mathcal{K}, \mu)$ is given by

$$(\pi(g)\xi)(x) = \sigma(c(x, g))(\xi(x \cdot g)), \quad g \in G, \xi \in L^2(X, \mathcal{K}, \mu), x \in X.$$

Observe that, in the case where σ is the trivial representation 1_H , the induced representation $\text{Ind}_H^G 1_H$ is equivalent to **quasi-regular representation** $\lambda_{H \setminus G}$, that is the natural representation of G on $L^2(H \setminus G, \mu)$ given by right translations.

We will use several times the following elementary but crucial lemma, which can be viewed as a generalization of Herz's majoration principle (see Proposition 17 in [BeGu15]).

Lemma 7. *Let $(H_i)_{i \in I}$ be a family of closed subgroups of G such that $H_i \setminus G$ admits a non-zero G -invariant measure. Let $(\sigma_i, \mathcal{K}_i)$ be a unitary representation of H_i . Assume that 1_G is weakly contained in the direct sum $\oplus_{i \in I} \text{Ind}_{H_i}^G \sigma_i$. Then 1_G is weakly contained in $\oplus_{i \in I} \lambda_{H_i \setminus G}$.*

Proof Let Q be a compact subset of G and $\varepsilon > 0$. For every $i \in I$, let $X_i \subset G$ be a Borel fundamental domain for the action of G on $H_i \setminus G$ and μ_i a non-zero G -invariant measure on X_i . There exists a family of vectors $\xi_i \in L^2(X_i, \mathcal{K}_i, \mu_i)$ such that $\sum_i \|\xi_i\|^2 = 1$ and

$$\sup_{g \in Q} \sum_i \|\text{Ind}_{H_i}^G \sigma_i(g) \xi_i - \xi_i\|^2 \leq \varepsilon,$$

Define φ_i in $L^2(X_i, \mu_i)$ by $\varphi_i(x) = \|\xi_i(x)\|$. Then $\sum_i \|\varphi_i\|^2 = 1$ and, denoting by $(x, g) \mapsto x \cdot_i g$ the action of G on X_i and by $c_i : X_i \times G \rightarrow H_i$ the associated map as above, we have

$$\begin{aligned} \|(\text{Ind}_{H_i}^G \sigma_i(g)) \xi_i - \xi_i\|^2 &= \int_{X_i} \|\sigma_i(c_i(x, g))(\xi_i(x \cdot_i g)) - \xi_i(x)\|^2 d\mu_i(x) \\ &\geq \int_{X_i} \|\|\sigma_i(c_i(x, g))(\xi_i(x \cdot_i g))\| - \|\xi_i(x)\|\|^2 d\mu_i(x) \\ &= \int_{X_i} \|\|\xi_i(x \cdot_i g)\| - \|\xi_i(x)\|\|^2 d\mu_i(x) \\ &= \int_{X_i} |\varphi_i(x \cdot_i g) - \varphi_i(x)|^2 d\mu_i(x) \\ &= \|\lambda_{H_i \setminus G}(g) \varphi_i - \varphi_i\|^2, \end{aligned}$$

for every $g \in G$, and the claim follows. ■

4. DECAY OF MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS

We recall a few general facts about decay of matrix coefficients of unitary representations. Recall that the projective kernel of a (genuine or projective) representation π of the locally compact group G is the closed normal subgroup P_π of G consisting of the elements $g \in G$ such that $\pi(g)$ is a scalar multiple of the identity operator, that is such that $\pi(g) = \lambda_\pi(g)I$ for some $\lambda_\pi(g) \in \mathbf{S}^1$.

Observe also that, for $\xi, \eta \in \mathcal{H}$, the absolute value of the matrix coefficient

$$C_{\xi, \eta}^\pi : g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is constant on cosets modulo P_π . For a real number p with $1 \leq p < +\infty$, the representation π is said to be **strongly L^p modulo P_π** , if there is dense subspace $D \subset \mathcal{H}$ such that $|C_{\xi, \eta}^\pi| \in L^p(G/P_\pi)$ for all $\xi, \eta \in D$.

Proposition 8. *Assume that the unitary representation π of the locally compact group G is strongly L^p modulo P_π for $1 \leq p < +\infty$. Let k be an integer $k \geq p/2$. Then the tensor power $\pi^{\otimes k}$ is contained in an infinite multiple of $\text{Ind}_{P_\pi}^G \lambda_\pi^k$, where λ_π is the unitary character of P_π associated to π .*

Proof Observe $\sigma := \pi^{\otimes k}$ is square-integrable modulo P_π for every integer $k \geq p/2$. It follows (see Proposition 4.2 in [HoMo79] or Proposition 1.2.3 in Chapter V of [HoTa92]) that σ is contained in an infinite multiple of $\text{Ind}_{P_\sigma}^G \lambda_\sigma = \text{Ind}_{P_\pi}^G \lambda_\pi^k$. ■

5. THE KOOPMAN REPRESENTATION OF THE AUTOMORPHISM GROUP OF A HOMOGENEOUS SPACE

We establish a decomposition result for the Koopman representation of a group of automorphisms of an S -adic compact nilmanifold. We will state the result in the general context of a compact homogeneous space.

Let G be a locally compact group and Λ a lattice in G . We assume that Λ is cocompact in G . The homogeneous space $X := G/\Lambda$ carries a probability measure μ on the Borel subsets X which is invariant by translations with elements from G . Every element from

$$\text{Aut}(X) := \{\gamma \in \text{Aut}(G) \mid \gamma(\Lambda) \subset \Lambda\}$$

induces a Borel isomorphism of X , which leaves μ invariant, as follows from the uniqueness of μ .

Given a subgroup Γ of $\text{Aut}(X)$, the following crucial proposition gives a decomposition of the associated Koopman Γ on $L^2(X, \mu)$ as direct sum of certain induced representations of Γ .

Proposition 9. *Let G be a locally compact group and Λ a cocompact lattice in G , and let Γ be a countable subgroup of $\text{Aut}(X)$ for $X := G/\Lambda$. Let κ denote the Koopman representation of Γ associated to the action $\Gamma \curvearrowright X$. There exists a family $(\pi_i)_{i \in I}$ of irreducible unitary representations of G such that κ is equivalent to a direct sum*

$$\bigoplus_{i \in I} \text{Ind}_{\Gamma_i}^{\Gamma} (\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\tilde{\pi}_i$ is an irreducible projective representation of the stabilizer G_i of π_i in $\text{Aut}(G) \ltimes G$ extending π_i , and where W_i is a finite dimensional projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.

Proof We extend κ to a unitary representation, again denoted by κ , of $\Gamma \ltimes G$ on $L^2(X, \mu)$ given by

$$\kappa(\gamma, g)\xi(x) = \xi(\gamma^{-1}(gx)) \quad \text{for all } \gamma \in \Gamma, g \in G, \xi \in L^2(X, \mu), x \in X.$$

Identifying Γ and G with subgroups of $\Gamma \ltimes G$, we have

$$(*) \quad \kappa(\gamma^{-1})\kappa(g)\kappa(\gamma) = \kappa(\gamma^{-1}(g)) \quad \text{for all } \gamma \in \Gamma, g \in G.$$

Since Λ is cocompact in G , we can consider the decomposition of $L^2(X, \mu)$ into G -isotypical components: we have (see Theorem in Chap. I, §3 of [GGPS69])

$$L^2(X, \mu) = \bigoplus_{\pi \in \Sigma} \mathcal{H}_{\pi},$$

where Σ is a certain set of pairwise non-equivalent irreducible unitary representations of G ; for every $\pi \in \Sigma$, the space $\mathcal{H}_{\pi}$ is the union of the closed $\kappa(G)$ -invariant subspaces $\mathcal{K}$ of $\mathcal{H} := L^2(X, \mu)$ for which the corresponding representation of G in $\mathcal{K}$ is equivalent to π ; moreover, the multiplicity of every π is finite, that is, every $\mathcal{H}_{\pi}$ is a direct sum of finitely many irreducible unitary representations of G .

Let γ be a fixed automorphism in Γ . Let κ^{γ} be the conjugate representation of κ by γ , that is, $\kappa^{\gamma}(g) = \kappa(\gamma g \gamma^{-1})$ for all $g \in \Gamma \ltimes G$. On the one hand, for every $\pi \in \Sigma$, the isotypical component of $\kappa^{\gamma}|_G$ corresponding to π is $\mathcal{H}_{\pi^{\gamma^{-1}}}$. On the other hand, relation $(*)$ shows that $\kappa(\gamma)$ is a unitary equivalence between $\kappa|_G$ and $\kappa^{\gamma}|_G$. It follows that

$$\kappa(\gamma)(\mathcal{H}_{\pi}) = \mathcal{H}_{\pi^{\gamma}} \quad \text{for all } \gamma \in \Gamma;$$

so, Γ permutes the $\mathcal{H}_{\pi}$'s among themselves according to its action on $\widehat{G}$.

Write $\Sigma = \bigcup_{i \in I} \Sigma_i$, where the Σ_i 's are the Γ -orbits in Σ , and set

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{\pi \in \Sigma_i} \mathcal{H}_{\pi}.$$

Every $\mathcal{H}_{\Sigma_i}$ is invariant under $\Gamma \ltimes G$ and we have an orthogonal decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i}.$$

Fix $i \in I$. Choose a representation π_i in Σ_i and set $\mathcal{H}_i = \mathcal{H}_{\pi_i}$. Let Γ_i denote the stabilizer of π_i in Γ . The space $\mathcal{H}_i$ is invariant under Γ_i . Let V_i be the corresponding representation of Γ_i on $\mathcal{H}_i$.

Choose a set S_i of representatives for the cosets in

$$\Gamma/\Gamma_i = (\Gamma \ltimes G)/(\Gamma_i \ltimes G)$$

with $e \in S_i$. Then $\Sigma_i = \{\pi_i^s : s \in S_i\}$ and the Hilbert space $\mathcal{H}_{\Sigma_i}$ is the sum of mutually orthogonal spaces:

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{s \in S_i} \mathcal{H}_i^s.$$

Moreover, $\mathcal{H}_i^s$ is the image under $\kappa(s)$ of $\mathcal{H}_i$ for every $s \in S_i$. This means that the restriction κ_i of κ to $\mathcal{H}_{\Sigma_i}$ of the Koopman representation κ of Γ is equivalent to the induced representation $\text{Ind}_{\Gamma_i}^{\Gamma} V_i$.

Since every $\mathcal{H}_i$ is a direct sum of finitely many irreducible unitary representations of G , we can assume that $\mathcal{H}_i$ is the tensor product

$$\mathcal{H}_i = \mathcal{K}_i \otimes \mathcal{L}_i$$

of the Hilbert space $\mathcal{K}_i$ of π_i with a finite dimensional Hilbert space $\mathcal{L}_i$, in such a way that

$$(**) \quad V_i(g) = \pi_i(g) \otimes I_{\mathcal{L}_i} \quad \text{for all } g \in G.$$

Let $\gamma \in \Gamma_i$. By (*) and (**) above, we have

$$(***) \quad V_i(\gamma) (\pi_i(g) \otimes I_{\mathcal{L}_i}) V_i(\gamma)^{-1} = \pi_i(\gamma g \gamma^{-1}) \otimes I_{\mathcal{L}_i}$$

for all $g \in G$. On the other hand, let G_i be the stabilizer of π_i in $\text{Aut}(G) \ltimes G$; then π_i extends to an irreducible projective representation $\tilde{\pi}_i$ of G_i (see Section 2). Since

$$\tilde{\pi}_i(\gamma) \pi_i(g) \tilde{\pi}_i(\gamma^{-1}) = \pi_i(\gamma g \gamma^{-1}) \quad \text{for all } g \in G,$$

it follows from (***) that $(\tilde{\pi}_i(\gamma^{-1}) \otimes I_{\mathcal{L}_i}) V_i(\gamma)$ commutes with $\pi_i(g) \otimes I_{\mathcal{L}_i}$ for all $g \in G_i$. As π_i is irreducible, there exists a unitary operator $W_i(\gamma)$ on $\mathcal{L}_i$ such that

$$V_i(\gamma) = \tilde{\pi}_i(\gamma) \otimes W_i(\gamma).$$

It is clear that W_i is a projective unitary representation of $\Gamma_i \ltimes G$, since V_i is a unitary representation of $\Gamma_i \ltimes G$. ■

6. UNITARY DUAL OF SOLENOIDS

Let p be either a prime integer or $p = \infty$. Define an element e_p in the unitary dual group $\widehat{\mathbf{Q}_p}$ of the additive group of $\mathbf{Q}_p$ (recall that $\mathbf{Q}_\infty = \mathbf{R}$) by $e_p(x) = e^{2\pi i x}$ if $p = \infty$ and $e_p(x) = \exp(2\pi i \{x\})$ otherwise, where $\{x\} = \sum_{j=m}^{-1} a_j p^j$ denotes the “fractional part” of a p -adic number $x = \sum_{j=m}^{\infty} a_j p^j$ for integers $m \in \mathbf{Z}$ and $a_j \in \{0, \dots, p-1\}$. Observe that $\text{Ker}(e_p) = \mathbf{Z}$ in case $p = \infty$ and that $\text{Ker}(e_p) = \mathbf{Z}_p$ in case p is a prime integer, where $\mathbf{Z}_p$ is the ring of p -adic integers. The map

$$\mathbf{Q}_p \rightarrow \widehat{\mathbf{Q}_p}, \quad y \mapsto (x \mapsto e_p(xy))$$

is an isomorphism of topological groups (see [BeHV08, Section D.4]).

Fix an integer $d \geq 1$. Then $\widehat{\mathbf{Q}_p^d}$ will be identified with $\mathbf{Q}_p^d$ by means of the map

$$\mathbf{Q}_p^d \rightarrow \widehat{\mathbf{Q}_p^d}, \quad y \mapsto x \mapsto e_p(x \cdot y),$$

where $x \cdot y = \sum_{i=1}^d x_i y_i$ for $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbf{Q}_p^d$.

Let $S = \{p_1, \dots, p_r, \infty\}$, where $p_1, \dots, p_r$ are integer primes. For an integer $d \geq 1$, consider the S -adic solenoid

$$\mathbf{Sol}_S = \mathbf{Q}_S^d / \mathbf{Z}[1/S]^d,$$

where $\mathbf{Z}[1/S]^d = \mathbf{Z}[1/p_1, \dots, 1/p_r]^d$ is embedded diagonally in $\mathbf{Q}_S = \prod_{p \in S} \mathbf{Q}_p$.

Then $\widehat{\mathbf{Sol}_S}$ is identified with the annihilator of $\mathbf{Z}[1/S]^d$ in $\mathbf{Q}_S^d$, that is, with $\mathbf{Z}[1/S]^d$ embedded in $\mathbf{Q}_S^d$ via the map

$$\mathbf{Z}[1/S]^d \rightarrow \mathbf{Q}_S^d, \quad b \mapsto (b, -b, \dots, -b).$$

Under this identification, the dual action of the automorphism group

$$\text{Aut}(\mathbf{Q}_S^d) \cong GL_d(\mathbf{R}) \times GL(\mathbf{Q}_{p_1}) \times \dots \times GL(\mathbf{Q}_{p_r})$$

on $\widehat{\mathbf{Q}_S^d}$ corresponds to the right action on $\mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \dots \times \mathbf{Q}_{p_r}^d$ given by

$$((g_\infty, g_1, \dots, g_r), (a_\infty, a_1, \dots, a_r)) \mapsto (g_\infty^t a_\infty, g_1^t a_1, \dots, g_r^t a_r),$$

where $(g, a) \mapsto ga$ is the usual (left) linear action of $GL_d(\mathbf{k})$ on $\mathbf{k}^d$ for a field $\mathbf{k}$.

7. UNITARY REPRESENTATIONS OF UNIPOTENT GROUPS

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$. The Lie algebra $\mathfrak{u}$ is defined over $\mathbf{Q}$ and the exponential map $\exp : \mathfrak{u} \rightarrow U$ is a bijective morphism of algebraic varieties.

Let p be either a prime integer or $p = \infty$. The irreducible unitary representations of $U_p := \mathbf{U}(\mathbf{Q}_p)$ are parametrized by Kirillov's theory as follows.

The Lie algebra of U_p is $\mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$, where $\mathfrak{u}(\mathbf{Q})$ is the Lie algebra over $\mathbf{Q}$ consisting of the $\mathbf{Q}$ -points in $\mathfrak{u}$.

Fix an element f in the dual space $\mathfrak{u}_p^* = \mathcal{H}om_{\mathbf{Q}_p}(\mathfrak{u}_p, \mathbf{Q}_p)$ of $\mathfrak{u}_p$. There exists a polarization $\mathfrak{m}$ for f , that is, a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{u}_p$ such that $f([\mathfrak{m}, \mathfrak{m}]) = 0$ and which is of maximal dimension. The induced representation $\text{Ind}_M^{U_p} \chi_f$ is irreducible, where $M = \exp(\mathfrak{m})$ and χ_f is the unitary character of M defined by

$$\chi_f(\exp X) = e_p(f(X)) \quad \text{for all } X \in \mathfrak{m},$$

where $e_p \in \widehat{\mathbf{Q}_p}$ is as in Section 6. The unitary equivalence class of $\text{Ind}_M^{U_p} \chi_f$ only depends on the co-adjoint orbit $\text{Ad}^*(U_p)f$ of f . The map

$$\mathfrak{u}_p^* / \text{Ad}^*(U_p) \rightarrow \widehat{U_p}, \quad \mathcal{O} \mapsto \pi_{\mathcal{O}}$$

called the Kirillov map, from the orbit space $\mathfrak{u}_p^* / \text{Ad}^*(U_p)$ of the co-adjoint representation to the unitary dual $\widehat{U_p}$ of U_p , is a bijection. In particular, U_p is a so-called type I locally compact group. For all of this, see [Kiri62] or [CoGr89] in the case $p = \infty$ and [Moor65] in the case of a prime integer p .

The group $\text{Aut}(U_p)$ of continuous automorphisms of U can be identified with the group of $\mathbf{Q}_p$ -points of the algebraic group $\text{Aut}(\mathfrak{u})$ of automorphisms of the Lie algebra $\mathfrak{u}$ of $\mathbf{U}$. Notice also that the natural action of $\text{Aut}(U_p)$ on $\mathfrak{u}_p$ as well as its dual action on $\mathfrak{u}_p^*$ are algebraic.

Let $\pi \in \widehat{U_p}$ with corresponding Kirillov orbit $\mathcal{O}_{\pi}$ and $g \in \text{Aut}(U_p)$. Then $g(\mathcal{O}_{\pi})$ is the Kirillov orbit associated to the conjugate representation π^g .

Lemma 10. *Let π be an irreducible unitary representation of U_p . The stabilizer G_{π} of π in $\text{Aut}(U_p)$ is an algebraic subgroup of $\text{Aut}(U_p)$.*

Proof Let $\mathcal{O}_{\pi} \subset \mathfrak{u}_p^*$ be the Kirillov orbit corresponding to π . Then G_{π} is the set of $g \in \text{Aut}(U_p)$ such that $g(\mathcal{O}_{\pi}) = \mathcal{O}_{\pi}$. As $\mathcal{O}_{\pi}$ is an algebraic subvariety of $\mathfrak{u}_p^*$, the claim follows. ■

8. DECAY OF MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS OF S -ADIC GROUPS

Let p be an integer prime or $p = \infty$ and let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}_p$. Set $U_p := \mathbf{U}(\mathbf{Q}_p)$.

Let π be an irreducible unitary representation of U_p . Recall (see Lemma 10) that the stabilizer G_{π} of π in $\text{Aut}(U_p)$ is an algebraic

subgroup of $\text{Aut}(U_p)$. Recall also (see Lemma 6) that π extends to a projective representation of G_π . The following result was proved in Proposition 22 of [BeGu15] in the case where $p = \infty$, using arguments from [HoMo79]. The proof in the case where p is a prime integer is along similar lines and will be omitted.

Proposition 11. *Let π be an irreducible unitary representation of U_p and let $\tilde{\pi}$ be a projective unitary representation of G_π which extends π . There exists a real number $r \geq 1$, only depending on the dimension of G_π , such that $\tilde{\pi}$ is strongly L^r modulo its projective kernel.*

We will need later a precise description of the projective kernel of a representation $\tilde{\pi}$ as above.

Lemma 12. *Let π be an irreducible unitary representation of U_p and $\tilde{\pi}$ a projective unitary representation of G_π which extends π . Let $\mathcal{O}_\pi \subset \mathfrak{u}_p^*$ be the corresponding Kirillov orbit of π . For $g \in \text{Aut}(U_p)$, the following properties are equivalent:*

- (i) g belongs to the projective kernel $P_{\tilde{\pi}}$ of $\tilde{\pi}$;
- (ii) for every $u \in U_p$, we have

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \exp(\text{Ker}(f)).$$

Proof We can assume that $\pi = \text{Ind}_M^{U_p} \chi_{f_0}$, for $f_0 \in \mathcal{O}_\pi$, and $M = \exp \mathfrak{m}$ for a polarization $\mathfrak{m}$ of f_0 .

Let $g \in \text{Aut}(U_p)$. If g is in the stabilizer G_π of π in $\text{Aut}(U_p)$, recall that (see Proof of Lemma 6) we have

$$\pi(g(u)) = \tilde{\pi}(g)\pi(u)\tilde{\pi}(g^{-1}) \quad \text{for all } u \in U_p.$$

Since π is irreducible, it follows from Schur's lemma that $g \in P_{\tilde{\pi}}$ if and only if

$$\pi(g(u)) = \pi(u) \quad \text{for all } u \in U_p$$

that is,

$$g(u)u^{-1} \in \text{Ker}(\pi) \quad \text{for all } u \in U_p.$$

Now, we have (see [BeGu15, Lemma 18])

$$\text{Ker}(\pi) = \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f)$$

and so, $g \in P_{\tilde{\pi}}$ if and only if

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p.$$

Let $g \in P_{\tilde{\pi}}$. Denote by $X \mapsto g(X)$ the automorphism of $\mathfrak{u}_p$ corresponding to g . Let $u = \exp(X)$ for $X \in \mathfrak{u}_p$ and $f \in \mathcal{O}_{\pi}$. Set $u_t = \exp(tX)$. By the Campbell Hausdorff formula, there exists $Y_1, \dots, Y_r \in \mathfrak{u}_p$ such that

$$g(u_t)(u_t)^{-1} = \exp(tY_1 + t^2Y_2 + \dots + t^rY_r),$$

for every $t \in \mathbf{Q}_p$; Since

$$(*) \quad 1 = \chi_f(g(u_t)(u_t)^{-1}) = e_p(f(tY_1 + t^2Y_2 + \dots + t^rY_r)),$$

it follows that the polynomial

$$t \mapsto Q(t) = tf(Y_1) + t^2f(Y_2) + \dots + t^rf(Y_r)$$

takes its values in $\mathbf{Z}$ in case $p = \infty$ and in $\mathbf{Z}_p$ (and so Q has bounded image) otherwise. This clearly implies that $Q(t) = 0$ for all $t \in \mathbf{Q}_p$; in particular, we have

$$\log(g(u)u^{-1}) = Y_1 + Y_2 + \dots + Y_r \in \text{Ker}(f).$$

This shows that (i) implies (ii).

Conversely, assume that (ii) holds. Then clearly

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_{\pi}} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p$$

and so $g \in P_{\tilde{\pi}}$. ■

9. DECOMPOSITION OF THE KOOPMAN REPRESENTATION FOR A NILMANIFOLD

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$. Let $S = \{p_1, \dots, p_r, \infty\}$, where $p_1, \dots, p_r$ are integer primes. Set

$$U := \mathbf{U}(\mathbf{Q}_S) = \prod_{p \in S} U_p.$$

Since U is a type I group, the unitary dual $\widehat{U}$ of U can be identified with the cartesian product $\prod_{p \in S} \widehat{U}_p$ via the map

$$\prod_{p \in S} \widehat{U}_p \rightarrow \widehat{U}, \quad (\pi_p)_{p \in S} \mapsto \otimes_{p \in S} \pi_p,$$

where $\otimes_{p \in S} \pi_p = \pi_{\infty} \otimes \pi_{p_1} \otimes \dots \otimes \pi_{p_r}$ is the tensor product of the π_p 's.

Let $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$ and consider the corresponding S -**adic compact nilmanifold**

$$\mathbf{Nil}_S := U/\Lambda,$$

equipped with the unique U -invariant probability measure μ on its Borel subsets.

The associated S -**adic solenoid** is

$$\mathbf{Sol}_S = \overline{U}/\overline{\Lambda},$$

where $\overline{U} := U/[U, U]$ is the quotient of U by its closed commutator subgroup $[U, U]$ and where $\overline{\Lambda}$ is the image of $\mathbf{U}(\mathbf{Z}[1/S])$ in $\overline{U}$.

Set

$$\mathrm{Aut}(U) := \prod_{p \in S} \mathrm{Aut}(\mathbf{U}(\mathbf{Q}_p))$$

and denote by $\mathrm{Aut}(\mathbf{Nil}_S)$ the subgroup of all $g \in \mathrm{Aut}(U)$ with $g(\Lambda) = \Lambda$. Observe that $\mathrm{Aut}(\mathbf{Nil}_S)$ is a discrete subgroup of $\mathrm{Aut}(U)$, where every $\mathrm{Aut}(U_p)$ is endowed with its natural (locally compact) topology and $\mathrm{Aut}(U)$ with the product topology.

Let Γ be a subgroup of $\mathrm{Aut}(\mathbf{Nil}_S)$. Let κ be the Koopman representation of $\Gamma \ltimes U$ on $L^2(\mathbf{Nil}_S)$ associated to the action $\Gamma \ltimes U \curvearrowright \mathbf{Nil}_S$. By Proposition 9, there exists a family $(\pi_i)_{i \in I}$ of irreducible representations of U , such that κ is equivalent to

$$\bigoplus_{i \in I} \mathrm{Ind}_{\Gamma_i \ltimes U}^{\Gamma \ltimes U} (\tilde{\pi}_i \otimes W_i),$$

where $\tilde{\pi}_i$ is an irreducible projective representation $\tilde{\pi}_i$ of the stabilizer G_i of π_i in $\mathrm{Aut}(U) \ltimes U$ extending π_i , and where W_i is a projective unitary representation of $G_i \cap (\Gamma \ltimes U)$.

Fix $i \in I$. We have $\pi_i = \otimes_{p \in S} \pi_{i,p}$ for irreducible representations $\pi_{i,p}$ of U_p .

We will need the following more precise description of π_i . Recall that $\mathfrak{u}$ is the Lie algebra of $\mathbf{U}$ and that $\mathfrak{u}(\mathbf{Q})$ denotes the Lie algebra over $\mathbf{Q}$ consisting of the $\mathbf{Q}$ -points in $\mathfrak{u}$.

Let $\mathfrak{u}^*(\mathbf{Q})$ be the set of $\mathbf{Q}$ -rational points in the dual space $\mathfrak{u}^*$; so, $\mathfrak{u}^*(\mathbf{Q})$ is the subspace of $f \in \mathfrak{u}^*$ with $f(X) \in \mathbf{Q}$ for all $X \in \mathfrak{u}(\mathbf{Q})$. Observe that, for $f \in \mathfrak{u}^*(\mathbf{Q})$, we have $f(X) \in \mathbf{Q}_p$ for all $X \in \mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}_p)$.

A polarization for $f \in \mathfrak{u}^*(\mathbf{Q})$ is a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{u}(\mathbf{Q})$ such that $f([\mathfrak{m}, \mathfrak{m}]) = 0$ and which is of maximal dimension with this property.

Proposition 13. *Let $\pi_i = \otimes_{p \in S} \pi_{i,p}$ be one of the irreducible representations of $U = \mathbf{U}(\mathbf{Q}_S)$ appearing in the decomposition $L^2(\mathbf{Nil}_S)$ as above. There exist $f_i \in \mathfrak{u}^*(\mathbf{Q})$ and a polarization $\mathfrak{m}_i \subset \mathfrak{u}(\mathbf{Q})$ for f_i with the following property: for every $p \in S$, the representation $\pi_{i,p}$ is equivalent to $\mathrm{Ind}_{M_{i,p}}^U \chi_{f_i}$, where*

- $M_{i,p} = \exp(\mathfrak{m}_{i,p})$ for $\mathfrak{m}_{i,p} = \mathfrak{m}_i \otimes_{\mathbf{Q}} \mathbf{Q}_p$,
- χ_{f_i} is the unitary character of $M_{i,p}$ given by $\chi_{f_i}(\exp X) = e_p(f_i(X))$, for all $X \in \mathfrak{m}_{i,p}$, with $e_p \in \widehat{\mathbf{Q}_p}$ as in Section 6.

Proof The same result is proved in Theorem 11 in [Moor65] (see also Theorem 1.2 in [Fox89]) for the Koopman representation of $\mathbf{U}(\mathbf{A})$ in $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$, where $\mathbf{A}$ is the ring of adeles of $\mathbf{Q}$. We could check that the proof, which proceeds by induction of the dimension of $\mathbf{U}$, carries over to the Koopman representation on $L^2(\mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S]))$, with the appropriate changes. We prefer to deduce our claim from the result for $\mathbf{U}(\mathbf{A})$, as follows.

It is well-known (see [Weil74]) that

$$\mathbf{A} = \left(\mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p \right) + \mathbf{Q}$$

and that

$$\left(\mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p \right) \cap \mathbf{Q} = \mathbf{Z}[1/S].$$

This gives rise to a well defined projection $\varphi : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{Q}_S/\mathbf{Z}[1/S]$ given by

$$\varphi((a_S, (a_p)_{p \notin S}) + \mathbf{Q}) = a_S + \mathbf{Z}[1/S] \quad \text{for all } a_S \in \mathbf{Q}_S, (a_p)_{p \notin S} \in \prod_{p \notin \mathcal{P}} \mathbf{Z}_p;$$

so the fiber over a point $a_S + \mathbf{Z}[1/S] \in \mathbf{Q}_S/\mathbf{Z}[1/S]$ is

$$\varphi^{-1}(a_S + \mathbf{Z}[1/S]) = \{(a_S, (a_p)_{p \notin S}) + \mathbf{Q} \mid a_p \in \mathbf{Z}_p \text{ for all } p\}.$$

This induces an identification of $\mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S]) = \mathbf{Nil}_S$ with the double coset space $K_S \backslash \mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})$, where K_S is the compact subgroup

$$K_S = \prod_{p \notin S} \mathbf{U}(\mathbf{Z}_p)$$

of $\mathbf{U}(\mathbf{A})$. Observe that this identification is equivariant under translation by elements from $\mathbf{U}(\mathbf{Q}_S)$. In this way, we can view $L^2(\mathbf{Nil}_S)$ as the $\mathbf{U}(\mathbf{Q}_S)$ -invariant subspace $L^2(K_S \backslash \mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$ of $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$.

Choose a system T of representatives for the $\text{Ad}^*(\mathbf{U}(\mathbf{Q}))$ -orbits in $\mathfrak{u}^*(\mathbf{Q})$. By [Moor65, Theorem 11], for every $f \in T$, we can find a polarization $\mathfrak{m}_f \subset \mathfrak{u}(\mathbf{Q})$ for f with the following property: setting

$$\mathfrak{m}_f(\mathbf{A}) = \mathfrak{m}_f \otimes_{\mathbf{Q}} \mathbf{A},$$

we have a decomposition

$$L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})) = \bigoplus_{f \in T} \mathcal{H}_f$$

into irreducible $\mathbf{U}(\mathbf{A})$ -invariant subspaces $\mathcal{H}_f$ such that the representation π_f of $\mathbf{U}(\mathbf{A})$ in $\mathcal{H}_f$ is equivalent to $\text{Ind}_{M_f(\mathbf{A})}^{\mathbf{U}(\mathbf{A})} \chi_f$, where

$$M_f(\mathbf{A}) = \exp(\mathfrak{m}_f(\mathbf{A}))$$

and $\chi_{f,\mathbf{A}}$ is the unitary character of $M_f(\mathbf{A})$ given by

$$\chi_{f,\mathbf{A}}(\exp X) = e(f(X)), \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{A});$$

here, e is the unitary character of $\mathbf{A}$ defined by

$$e((a_p)_p) = \prod_{p \in \mathcal{P} \cup \{\infty\}} e_p(a_p) \quad \text{for all } (a_p)_p \in \mathbf{A},$$

where $\mathcal{P}$ is the set of integer primes.

We have

$$L^2(K_S \backslash \mathbf{U}(\mathbf{A}) / \mathbf{U}(\mathbf{Q})) = \bigoplus_{f \in T} \mathcal{H}_f^{K_S},$$

where $\mathcal{H}_f^{K_S}$ is the space of K_S -fixed vectors in $\mathcal{H}_f$. It is clear that the representation of $\mathbf{U}(\mathbf{Q}_S)$ in $\mathcal{H}_f^{K_S}$ is equivalent to

$$\text{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} (\otimes_{p \in S} \chi_{f,p}) = \otimes_{p \in S} \left(\text{Ind}_{M_f(\mathbf{Q}_p)}^{\mathbf{U}(\mathbf{Q}_p)} \chi_{f,p} \right),$$

where $\chi_{f,p}$ is the unitary character of $M_f(\mathbf{Q}_p)$ given by

$$\chi_{f,p}(\exp X) = e_p(f(X)), \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{Q}_p).$$

Since $M_f(\mathbf{Q}_p)$ is a polarization for f , each of the $\mathbf{U}(\mathbf{Q}_p)$ -representations $\text{Ind}_{M_f(\mathbf{Q}_p)}^{\mathbf{U}(\mathbf{Q}_p)} \chi_{f,p}$ and, hence, each of the $\mathbf{U}(\mathbf{Q}_S)$ -representations

$$\text{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} (\otimes_{p \in S} \chi_{f,p})$$

is irreducible. This proves the claim. $\blacksquare$

We establish another crucial fact about the representations π_i 's in the following proposition.

Proposition 14. *With the notation of Proposition 13, let $\mathcal{O}_{\mathbf{Q}}(f_i)$ be the co-adjoint orbit of f_i under $\mathbf{U}(\mathbf{Q})$ and set*

$$\mathfrak{k}_{i,p} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_p(f),$$

where $\mathfrak{k}_p(f)$ is the kernel of f in $\mathfrak{u}_p$. Let $K_{i,p} = \exp(\mathfrak{k}_{i,p})$ and $K_i = \prod_{p \in S} K_{i,p}$.

- (i) K_i is a closed normal subgroup of U and $K_i \cap \Lambda = K_i \cap \mathbf{U}(\mathbf{Z}[1/S])$ is a lattice in K_i .

- (ii) Let $P_{\tilde{\pi}_i}$ be the projective kernel of the extension $\tilde{\pi}_i$ of π_i to the stabilizer G_i of π_i in $\text{Aut}(U) \ltimes U$. For $g \in G_i$, we have $g \in P_{\tilde{\pi}_i}$ if and only if $g(u) \in uK_i$ for every $u \in U$.

Proof (i) Let

$$\mathfrak{k}_{i,\mathbf{Q}} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_{\mathbf{Q}}(f),$$

where $\mathfrak{k}_{\mathbf{Q}}(f)$ is the kernel of f in $\mathfrak{u}(\mathbf{Q})$. Observe that $\mathfrak{k}_{i,\mathbf{Q}}$ is an ideal in $\mathfrak{u}(\mathbf{Q})$, since it is $\text{Ad}(\mathbf{U}(\mathbf{Q}))$ -invariant. So, we have

$$\mathfrak{k}_{i,\mathbf{Q}} = \mathfrak{k}_i(\mathbf{Q})$$

for an ideal $\mathfrak{k}_i$ in $\mathfrak{u}$. Since $f \in \mathfrak{u}^*(\mathbf{Q})$ for $f \in \mathcal{O}_{\mathbf{Q}}(f_i)$, we have

$$\mathfrak{k}_{i,p}(f) = \mathfrak{k}_{i,\mathbf{Q}}(f) \otimes_{\mathbf{Q}} \mathbf{Q}_p$$

and hence

$$\mathfrak{k}_{i,p} = \mathfrak{k}_i(\mathbf{Q}_p).$$

Let $\mathbf{K}_i = \exp(\mathfrak{k}_i)$. Then $\mathbf{K}_i$ is a normal algebraic $\mathbf{Q}$ -subgroup of $\mathbf{U}$ and we have $K_{i,p} = \mathbf{K}_i(\mathbf{Q}_p)$ for every p ; so,

$$K_i = \prod_{s \in S} \mathbf{K}_i(\mathbf{Q}_p) = \mathbf{K}_i(\mathbf{Q}_S)$$

and $K_i \cap \Lambda = \mathbf{K}_i(\mathbf{Z}[1/S])$ is a lattice in K_i . This proves Item (i).

To prove Item (ii), observe that

$$P_{\tilde{\pi}_i} = \prod_{p \in S} P_{i,p},$$

where $P_{i,p}$ is the projective kernel of $\tilde{\pi}_{i,p}$.

Fix $p \in S$ and let $g \in G_i$. By Lemma 12, $g \in P_{i,p}$ if and only if $g(u) \in uK_{i,p}$ for every $u \in U_p = \mathbf{U}(\mathbf{Q}_p)$. This finishes the proof. ■

10. PROOF OF THEOREM 1

Let $\mathbf{U}$ be a linear algebraic unipotent group defined over $\mathbf{Q}$ and $S = \{p_1, \dots, p_r, \infty\}$, where $p_1, \dots, p_r$ are integer primes. Set $U := \mathbf{U}(\mathbf{Q}_S)$ and $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$. Let $\mathbf{Nil}_S = U/\Lambda$ and $\mathbf{Sol}_S$ be the S -adic nilmanifold and the associated S -adic solenoid as in Section 9. Denote by μ the translation invariant probability measure on $\mathbf{Nil}_S$ and let ν be the image of μ under the canonical projection $\varphi : \mathbf{Nil}_S \rightarrow \mathbf{Sol}_S$. We identify $L^2(\mathbf{Sol}_S) = L^2(\mathbf{Sol}_S, \nu)$ with the closed $\text{Aut}(\mathbf{Nil}_S)$ -invariant subspace

$$\{f \circ \varphi \mid f \in L^2(\mathbf{Sol}_S)\}$$

of $L^2(\mathbf{Nil}_S) = L^2(\mathbf{Nil}_S, \mu)$. We have an orthogonal decomposition into $\text{Aut}(\mathbf{Nil}_S)$ -invariant subspaces

$$L^2(\mathbf{Nil}_S) = \mathbf{C}1_{\mathbf{Nil}_S} \oplus L_0^2(\mathbf{Sol}_S) \oplus \mathcal{H},$$

where

$$L_0^2(\mathbf{Sol}_S) = \{f \in L^2(\mathbf{Sol}_S) \mid \int_{\mathbf{Nil}_S} f d\mu = 0\}$$

and where $\mathcal{H}$ is the orthogonal complement of $L^2(\mathbf{Sol}_S)$ in $L^2(\mathbf{Nil}_S)$.

Let Γ be a subgroup of $\text{Aut}(\mathbf{Nil}_S)$. Let κ be the Koopman representation of Γ on $L^2(\mathbf{Nil}_S)$ and denote by κ_1 and κ_2 the restrictions of κ to respectively $L_0^2(\mathbf{Sol}_S)$ and $\mathcal{H}$.

Let Σ_1 be a set of representatives for the Γ -orbits in $\widehat{\mathbf{Sol}_S} \setminus \{1_{\mathbf{Sol}_S}\}$. We have

$$\kappa_1 \cong \bigoplus_{\chi \in \Sigma_1} \lambda_{\Gamma/\Gamma_\chi},$$

where Γ_χ is the stabilizer of χ in Γ and $\lambda_{\Gamma/\Gamma_\chi}$ is the quasi-regular representation of Γ on $\ell^2(\Gamma/\Gamma_\chi)$.

By Proposition 9, there exists a family $(\pi_i)_{i \in I}$ of irreducible representations of U , such that κ_2 is equivalent to a direct sum

$$\bigoplus_{i \in I} \text{Ind}_{\Gamma_i}^\Gamma(\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\tilde{\pi}_i$ is an irreducible projective representation of the stabilizer G_i of π_i in $\text{Aut}(U)$ and where W_i is a projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.

Proposition 15. *For $i \in I$, let $\tilde{\pi}_i$ be the (projective) representation of G_i and let Γ_i be as above. There exists a real number $r \geq 1$ such that $\tilde{\pi}_i|_{\Gamma_i}$ is strongly L^r modulo $P_{\tilde{\pi}_i} \cap \Gamma_i$, where $P_{\tilde{\pi}_i}$ is the projective kernel of $\tilde{\pi}_i$.*

Proof By Proposition 11, there exists a real number $r \geq 1$ such that the representation $\tilde{\pi}_i$ of the algebraic group G_i is strongly L^r modulo $P_{\tilde{\pi}_i}$. In order to show that $\tilde{\pi}_i|_{\Gamma_i}$ is strongly L^r modulo $P_{\tilde{\pi}_i} \cap \Gamma_i$, it suffices to show that $\Gamma_i P_{\tilde{\pi}_i}$ is closed in G_i (compare with the proof of Proposition 6.2 in [HoMo79]).

Let K_i be the closed G_i -invariant normal subgroup K_i of U as described in Proposition 14. Then $\bar{\Lambda} = K_i \Lambda / K_i$ is a lattice in the unipotent group $\bar{U} = U / K_i$. By Proposition 14.ii, $P_{\tilde{\pi}_i}$ coincides with the kernel of the natural homomorphism $\varphi : \text{Aut}(U) \rightarrow \text{Aut}(\bar{U})$. Hence, we have

$$\Gamma_i P_{\tilde{\pi}_i} = \varphi^{-1}(\varphi(\Gamma_i)).$$

Now, $\varphi(\Gamma_i)$ is a discrete (and hence closed) subgroup of $\text{Aut}(\overline{U})$, since $\varphi(\Gamma_i)$ preserves $\overline{\Lambda}$ (and so $\varphi(\Gamma_i) \subset \text{Aut}(\overline{U}/\overline{\Lambda})$). It follows from the continuity of φ that $\varphi^{-1}(\varphi(\Gamma_i))$ is closed in $\text{Aut}(U)$. ■

Proof of Theorem 1

We have to show that, if 1_Γ is weakly contained in κ_2 , then 1_Γ is weakly contained in κ_1 . It suffices to show that, if 1_Γ is weakly contained in κ_2 , then there exists a finite index subgroup H of Γ such that 1_H is weakly contained in $\kappa_1|_H$ (see Theorem 2 in [BeFr20]).

We proceed by induction on the integer

$$n(\Gamma) := \sum_{p \in S} \dim Z_{C_p}(\Gamma),$$

where $Z_{C_p}(\Gamma)$ is the Zariski closure of the projection of Γ in $GL_n(\mathbf{Q}_p)$.

If $n(\Gamma) = 0$, then Γ is finite and there is nothing to prove.

Assume that $n(\Gamma) \geq 1$ and that the claim above is proved for every countable subgroup H of $\text{Aut}(\mathbf{Nil}_S)$ with $n(H) < n(\Gamma)$.

Let $I_{\text{fin}} \subset I$ be the set of all $i \in I$ such that $\Gamma_i = G_i \cap \Gamma$ has finite index in Γ and set $I_\infty = I \setminus I_{\text{fin}}$. With $V_i = \tilde{\pi}_i|_{\Gamma_i} \otimes W_i$, set

$$\kappa_2^{\text{fin}} = \bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{\Gamma_i}^\Gamma V_i \quad \text{and} \quad \kappa_2^\infty = \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma V_i.$$

Two cases can occur.

- *First case:* 1_Γ is weakly contained in κ_2^∞ .

Observe that $n(\Gamma_i) < n(\Gamma)$ for $i \in I_\infty$. Indeed, otherwise $Z_{C_p}(\Gamma_i)$ and $Z_{C_p}(\Gamma)$ would have the same connected component C_p^0 for every $p \in S$, since $\Gamma_i \subset \Gamma$. Then

$$C^0 := \prod_{p \in S} C_p^0$$

would stabilize π_i and $\Gamma \cap C^0$ would therefore be contained in Γ_i . Since $\Gamma \cap C^0$ has finite index in Γ , this would be a contradiction to the fact that Γ_i has infinite index in Γ .

By restriction, 1_{Γ_i} is weakly contained in $\kappa_2|_{\Gamma_i}$ for every $i \in I$. Hence, by the induction hypothesis, 1_{Γ_i} is weakly contained in $\kappa_1|_{\Gamma_i}$ for every $i \in I_\infty$. Now, on the one hand, we have

$$\kappa_1|_{\Gamma_i} \cong \bigoplus_{\chi \in T_i} \lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i},$$

for a subset T_i of $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$. It follows that $\text{Ind}_{\Gamma_i}^\Gamma 1_{\Gamma_i} = \lambda_{\Gamma/\Gamma_i}$ is weakly contained in

$$\bigoplus_{\chi \in T_i} \text{Ind}_{\Gamma_i}^\Gamma (\lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i}) = \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i},$$

for every $i \in I_\infty$. On the other hand, since 1_Γ is weakly contained in

$$\kappa_2^\infty \cong \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma (\widetilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

Lemma 7 shows that 1_Γ is weakly contained in $\bigoplus_{i \in I_\infty} \lambda_{\Gamma/\Gamma_i}$. It follows that 1_Γ is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i}.$$

Hence, by Lemma 7 again, 1_Γ is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi}.$$

This shows that 1_Γ is weakly contained in κ_1 .

• *Second case:* 1_Γ is weakly contained in κ_2^{fin} .

By the Noetherian property of the Zariski topology, we can find finitely many indices $i_1, \dots, i_r$ in I_{fin} such that, for every $p \in S$, we have

$$Z_{C_p}(\Gamma_{i_1}) \cap \dots \cap Z_{C_p}(\Gamma_{i_r}) = \bigcap_{i \in I_{\text{fin}}} Z_{C_p}(\Gamma_i),$$

Set $H := \Gamma_{i_1} \cap \dots \cap \Gamma_{i_r}$. Observe that H has finite index in Γ . Moreover, it follows from Lemma 10 that $Z_{C_p}(\Gamma_{i_1}) \cap \dots \cap Z_{C_p}(\Gamma_{i_r})$ stabilizes $\pi_{i,p}$ for every $i \in I_{\text{fin}}$ and $p \in S$. Hence, H is contained in Γ_i for every $i \in I_{\text{fin}}$.

By Proposition 9, we have a decomposition of $\kappa_2^{\text{fin}}|_H$ into the direct sum

$$\bigoplus_{i \in I_{\text{fin}}} (\widetilde{\pi}_i \otimes W_i)|_H.$$

By Proposition 11 and Proposition 15, there exists a real number $r \geq 1$, which is independent of i , such that $(\widetilde{\pi}_i \otimes W_i)|_H$ is a strongly L^r representation of H modulo its projective kernel P_i . Observe that P_i is contained in the projective kernel $P_{\widetilde{\pi}_i}$ of $\widetilde{\pi}_i$, since $P_i = P_{\widetilde{\pi}_i} \cap H$. Hence (see Proposition 8), there exists an integer $k \geq 1$ such that $(\kappa_2^{\text{fin}}|_H)^{\otimes k}$ is contained in a multiple of the direct sum

$$\bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{P_i}^H \rho_i,$$

for representations ρ_i of P_i . Since 1_H is weakly contained in $\kappa_2^{\text{fin}}|_H$ and hence in $(\kappa_2^{\text{fin}}|_H)^{\otimes k}$, using Lemma 7, it follows that 1_H is weakly contained in

$$\bigoplus_{i \in I_{\text{fin}}} \lambda_{H/P_i}.$$

Let $i \in I$. We claim that P_i is contained in Γ_χ for some character χ from $\widehat{\mathbf{Sol}_S} \setminus \{1_{\mathbf{Sol}_S}\}$. Once proved, this will imply, again by Lemma 7, that $\bigoplus_{i \in I_{\text{fin}}} \lambda_{H/P_i}$ and, hence 1_H , is weakly contained in $\kappa_1|_H$. Since H has finite index in Γ , this will show that 1_Γ is weakly contained in κ_1 and conclude the proof.

To prove the claim, recall from Proposition 14 that there exists a closed normal subgroup K_i of U with the following properties: $K_i\Lambda/K_i$ is a lattice in the unipotent algebraic group U/K_i , K_i is invariant under P_{π_i} and P_{π_i} acts as the identity on U/K_i . Observe that $K_i \neq U$, since π_i is not trivial on U . We can find a non-trivial unitary character χ of U/K_i which is trivial on $K_i\Lambda/K_i$. Then χ lifts to a non-trivial unitary character which is fixed by P_{π_i} and hence by P_i . Observe that $\chi \in \widehat{\mathbf{Sol}_S}$, since χ is trivial on Λ . ■

11. AN EXAMPLE: THE S -ADIC HEISENBERG NILMANIFOLD

As an example, we study the spectral gap property for group of automorphisms of the S -adic Heisenberg nilmanifold, proving Corollary 5. We will give a quantitative estimate for the norm of associated convolution operators, as we did in [BeHe11] in the case of real Heisenberg nilmanifolds (that is, in the case $S = \{\infty\}$).

Let $\mathbf{K}$ be an algebraically closed field containing $\mathbf{Q}_p$ for $p = \infty$ and for all prime integers p . For an integer $n \geq 1$, consider the symplectic form β on $\mathbf{K}^{2n}$ given by

$$\beta((x, y), (x', y')) = (x, y)^t J (x', y') \quad \text{for all } (x, y), (x', y') \in \mathbf{K}^{2n},$$

where J is the $(2n \times 2n)$ -matrix

$$J = \begin{pmatrix} 0 & I \\ -I_n & 0 \end{pmatrix}.$$

The symplectic group

$$Sp_{2n} = \{g \in GL_{2n}(\mathbf{K}) \mid {}^t g J g = J\}$$

is an algebraic group defined over $\mathbf{Q}$.

The $(2n+1)$ -dimensional Heisenberg group is the unipotent algebraic group $\mathbf{H}$ defined over $\mathbf{Q}$, with underlying set $\mathbf{K}^{2n} \times \mathbf{K}$ and product

$$((x, y), s)((x', y'), t) = ((x + x', y + y'), s + t + \beta((x, y), (x', y'))),$$

for $(x, y), (x', y') \in \mathbf{K}^{2n}$, $s, t \in \mathbf{K}$.

The group Sp_{2n} acts by rational automorphisms of $\mathbf{H}$, given by

$$g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}, (x, y) \in \mathbf{K}^{2n}, t \in \mathbf{K}.$$

Let p be either an integer prime or $p = \infty$. Set $H_p = \mathbf{H}(\mathbf{Q}_p)$. The center Z of H_p is $\{(0, 0, t) \mid t \in \mathbf{Q}_p\}$. The unitary dual $\widehat{H_p}$ of H_p consists of the equivalence classes of the following representations:

- the unitary characters of the abelianized group H_p/Z ;
- for every $t \in \mathbf{Q}_p \setminus \{0\}$, the infinite dimensional representation π_t defined on $L^2(\mathbf{Q}_p^n)$ by the formula

$$\pi_t((a, b), s)\xi(x) = e_p(ts)e_p(\langle a, x - b \rangle)\xi(x - b)$$

for $((a, b), s) \in H_p$, $\xi \in L^2(\mathbf{Q}_p^n)$, and $x \in \mathbf{Q}_p^n$, where $e_p \in \widehat{\mathbf{Q}_p}$ is as in Section 6.

For $t \neq 0$, the representation π_t is, up to unitary equivalence, the unique irreducible unitary representation of H_p whose restriction to the centre Z is a multiple of the unitary character $s \mapsto e_p(ts)$.

For $g \in Sp_{2n}(\mathbf{Q}_p)$ and $t \in \mathbf{Q}_p \setminus \{0\}$, the representation π_t^g is unitary equivalent to π_t , since both representations have the same restriction to Z . This shows that $Sp_{2n}(\mathbf{Q}_p)$ stabilizes π_t . We denote the corresponding projective representation of $Sp_{2n}(\mathbf{Q}_p)$ by $\omega_t^{(p)}$. The representation $\omega_t^{(p)}$ has different names: it is called **metaplectic representation**, **Weil's representation**, or **oscillator representation**. The projective kernel of $\omega_t^{(p)}$ coincides with the (finite) center of $Sp_{2n}(\mathbf{Q}_p)$ and $\omega_t^{(p)}$ is strongly $L^{4n+2+\varepsilon}$ on $Sp_{2n}(\mathbf{Q}_p)$ for every $\varepsilon > 0$ (see Proposition 6.4 in [HoMo79] or Proposition 8.1 in [Howe82]).

Let $S = \{p_1, \dots, p_r, \infty\}$, where $p_1, \dots, p_r$ are integer primes. Set $H := \mathbf{H}(\mathbf{Q}_S)$ and

$$\Lambda := \mathbf{H}(\mathbf{Z}[1/S]) = \{((x, y), s) : x, y \in \mathbf{Z}^n[1/S], s \in \mathbf{Z}[1/S]\}.$$

Let $\mathbf{Nil}_S = H/\Lambda$; the associated S -adic solenoid is $\mathbf{Sol}_S = \mathbf{Q}_S^{2n}/\mathbf{Z}[1/S]^{2n}$. The group $Sp_{2n}(\mathbf{Z}[1/S])$ is a subgroup of $\text{Aut}(\mathbf{Nil}_S)$. The action of $Sp_{2n}(\mathbf{Z}[1/S])$ on $\mathbf{Sol}_S$ is induced by its representation by linear bijections on $\mathbf{Q}_S^{2n}$.

Let Γ be a subgroup of $Sp_{2n}(\mathbf{Z}[1/S])$. The Koopman representation κ of Γ on $L^2(\mathbf{Nil}_S)$ decomposes as

$$\kappa = \mathbf{1}_{\mathbf{Nil}_S} \oplus \kappa_1 \oplus \kappa_2,$$

where κ_1 is the restriction of κ to $L_0^2(\mathbf{Sol}_S)$ and κ_2 the restriction of κ to the orthogonal complement of $L^2(\mathbf{Sol}_S)$ in $L^2(\mathbf{Nil}_S)$. Since $Sp_{2n}(\mathbf{Q}_p)$ stabilizes every infinite dimensional representation of H_p , it follows

from Proposition 14 that there exists a subset $I \subset \mathbf{Q}$ such that κ_2 is equivalent to a direct sum

$$\bigoplus_{t \in I} \left(\bigotimes_{p \in S} (\omega_t^{(p)}|_{\Gamma} \otimes W_i) \right),$$

where W_i is a projective representation of Γ .

Let ν be a probability measure on Γ . We can give an estimate of the norm of $\kappa_2(\nu)$ as in [BeHe11] in the case of $S = \{\infty\}$. Indeed, by a general inequality (see Proposition 30 in [BeGu15]), we have

$$\|\kappa_2(\nu)\| \leq \|(\kappa_2 \otimes \overline{\kappa_2})^{\otimes k}(\nu)\|^{1/2k},$$

for every integer $k \geq 1$, where $\overline{\kappa_2}$ denotes the representation conjugate to κ_2 . Since $\omega_t^{(p)}$ is strongly $L^{4n+2+\varepsilon}$ on $Sp_{2n}(\mathbf{Q}_p)$ for any $t \in I$ and $p \in S$, Proposition 8 implies that $(\kappa_2 \otimes \overline{\kappa_2})^{\otimes(n+1)}$ is contained in an infinite multiple of the regular representation λ_{Γ} of Γ . Hence,

$$\|\kappa_2(\nu)\| \leq \|\lambda_{\Gamma}(\nu)\|^{1/2n+2}$$

and so,

$$\|\kappa_0(\nu)\| \leq \max\{\|\lambda_{\Gamma}(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where κ_0 is the restriction of κ to $L_0^2(\mathbf{Nil}_S)$.

Assume that ν is aperiodic and that the subgroup generated by the support of ν coincides with Γ . If Γ is not amenable then $\|\lambda_{\Gamma}(\nu)\| < 1$ by Kesten's theorem (see [BeHV08, Appendix G]); so, in this case, the action of Γ on $\mathbf{Nil}_S$ has a spectral gap if and only if $\|\kappa_1(\nu)\| < 1$, as stated in Theorem 1.

Observe that, if Γ is amenable, then the action of Γ on $\mathbf{Nil}_S$ or $\mathbf{Sol}_S$ does not have a spectral gap; indeed, by a general result (see [JuRo79, Theorem 2.4]), no action of a countable amenable group by measure preserving transformations on a non-atomic probability space has a spectral gap.

Let us look more closely to the case $n = 1$. We have $Sp_2(\mathbf{Z}[1/S]) = SL_2(\mathbf{Z}[1/S])$ and the stabilizer of every element in $\widehat{\mathbf{Sol}_S} \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$ is conjugate to the group of unipotent matrices in $SL_2(\mathbf{Z}[1/S])$ and hence amenable. This implies that κ_1 is weakly contained in λ_{Γ} (see the decomposition of κ_1 appearing before Proposition 15); so, we have

$$\|\kappa_1(\nu)\| < 1 \iff \Gamma \text{ is not amenable.}$$

As a consequence, we see that the action of Γ on $\mathbf{Nil}_S$ has a spectral gap if and only if Γ is not amenable.

REFERENCES

- [BeFr20] B. Bekka and C. Francini, Spectral gap property and strong ergodicity for groups of affine transformations of solenoids, *Ergodic Theory Dynam. Systems* **40** (2020), no.5, 1180–1193.
- [Bekk16] B. Bekka. Spectral rigidity of group actions on homogeneous spaces. In: Handbook of group actions. Vol. IV, 563–622, Adv. Lect. Math. (ALM), 41, Int. Press, Somerville, MA, 2018.
- [BeGu15] B. Bekka, Y. Guivarc’h. On the spectral theory of groups of affine transformations of compact nilmanifolds. *Ann. Sci. École Normale Supérieure* **48** (2015), 607–645.
- [BeHV08] B. Bekka, P. de la Harpe, A. Valette. *Kazhdan’s Property (T)*. Cambridge University Press 2008.
- [BeHe11] B. Bekka, J-R. Heu. Random products of automorphisms of Heisenberg nilmanifolds and Weil’s representation. *Ergod. Théor. Dynam. Sys.* **31**, 1277–1286 (2011).
- [Fox89] J. Fox. Adeles and the spectrum of compact nilmanifolds. *Pacific J. Math.* **140**, 233–250 (1989).
- [GGPS69] I.M. Gelfand, M.I. Graev, and I.I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W.B. Saunders Company, 1969
- [CoGr89] L. Corwin and F. Greenleaf. *Representations of nilpotent Lie groups and their applications*, Cambridge University Press 1989.
- [HeRo63] E. Hewitt, K. Ross. *Abstract harmonic analysis*, Volume I. Die Grundlehren der mathematischen Wissenschaften **115**, Springer-Verlag, New York, 1963.
- [Howe82] Howe, R. On a notion of rank for unitary representations of the classical groups. In: *Harmonic analysis and group representations*, 223–331, Liguori, Naples, 1982.
- [HoMo79] R. Howe, C. C. Moore. Asymptotic properties of unitary representations. *J. Funct. Anal.* **32**, 72–96 (1979).
- [HoTa92] R. Howe, E.C. Tan. *Non-abelian harmonic analysis*, Springer 1992.
- [JuRo79] A. del Junco, J. Rosenblatt. Counterexamples in ergodic theory. *Math. Ann.* **245**, 185–197 (1979).
- [Kiri62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. *Russian Math. Surveys* **17**, 53–104 (1962).
- [Mack58] G.W. Mackey. Unitary representations of group extensions I. *Acta Math.* **99**, 265–311 (1958).
- [Mack76] G.W. Mackey. *The theory of unitary group representations*, Chicago Lectures in Mathematics, The University of Chicago Press 1976.
- [Moor65] C. C. Moore. Decomposition of unitary representations defined by discrete subgroups of nilpotent Lie groups. *Ann. Math.* **82**, 146–182 (1965).
- [Schm81] K. Schmidt. Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions. *Ergodic Theory Dynamical Systems* **1** (1981), 223–236.
- [Weil74] A. Weil. *Basic number theory*, Springer-Verlag, New York-Berlin 1974.

BACHIR BEKKA, IRMAR, UMR-CNRS 6625 UNIVERSITÉ DE RENNES 1,
CAMPUS BEAULIEU, F-35042 RENNES CEDEX, FRANCE

Email address: `bachir.bekka@univ-rennes1.fr`

YVES GUIVARCH'H, IRMAR, UMR-CNRS 6625, UNIVERSITÉ DE RENNES 1,
CAMPUS BEAULIEU, F-35042 RENNES CEDEX, FRANCE

Email address: `yves.guivarch@univ-rennes1.fr`