INFINITE CHARACTERS ON $GL_n(\mathbf{Q})$, ON $\mathrm{SL}_n(\mathbf{Z})$, AND ON GROUPS ACTING ON TREES

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Abstract. Answering a question of J. Rosenberg from [Ros–89], we construct the first examples of infinite characters on $\mathrm{GL}_n(\mathbf{K})$ for a global field $\mathbf{K}$ and $n \geq 2$. The case $n = 2$ is deduced from the following more general result. Let G a non amenable countable subgroup acting on locally finite tree X. Assume either that the stabilizer in G of every vertex of X is finite or that the closure of the image of G in $\mathrm{Aut}(X)$ is not amenable. We show that G has uncountably many infinite dimensional irreducible unitary representations $(\pi, \mathcal{H})$ of G which are traceable, that is, such that the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\pi(G)$ contains the algebra of the compact operators on $\mathcal{H}$. In the case $n \geq 3$, we prove the existence of infinitely many characters for $G = \mathrm{GL}_n(R)$, where $n \geq 3$ and R is an integral domain such that G is not amenable. In particular, the group $\mathrm{SL}_n(\mathbf{Z})$ has infinitely many such characters for $n \geq 2$.

1. Introduction

Let G be a countable discrete group and $\widehat{G}$ the unitary dual of G, that is, the set of equivalence classes of irreducible unitary representations of G. The space $\widehat{G}$, equipped with a natural Borel structure, is a standard Borel space exactly when G is virtually abelian, by results of Glimm and Thoma (see [Gli–61] and [Tho–68]). So, unless G is virtually abelian (in which case the representation theory of G is well understood), a description of $\widehat{G}$ is hopeless or useless. There are at least two other dual objects of G, which seem to be more accessible than $\widehat{G}$:

- **Thoma's dual space** $E(G)$, that is, the set of indecomposable positive definite central functions on G;
- the space $\mathrm{Char}(G)$ of **characters** of G, that is, the space of lower semi-continuous semi-finite (not necessarily finite) traces t on the **maximal C^*-algebra** $C^*(G)$ of G (see Subsection 2.1)

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which satisfies the following extremality condition: every lower semi-continuous semi-finite trace on $C^*(G)$ dominated by t on the ideal of definition of t is proportional to t.

The space $\mathrm{Char}(G)$ parametrizes the quasi-equivalence classes of factorial representations of $C^*(G)$ which are **traceable** ; recall that a unitary representation π is factorial if the von Neumann algebra $\mathcal{M}$ generated by $\pi(G)$ is a factor and that a factorial representation π is traceable if there exists a faithful normal (not necessarily finite) trace τ on $\mathcal{M}$ and a positive element $x \in C^*(G)$ such that $0 < \tau(\pi(x)) < +\infty$. If this is the case, then $t = \tau \circ \pi$ belongs to $\mathrm{Char}(G)$. Conversely, every element of $\mathrm{Char}(G)$ is obtained in this way. Traceable representations are also called *normal* representations.

Two traceable factorial representations π_1 and π_2 are quasi-equivalent if there exists an isomorphism $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\Phi(\pi_1(g)) = \pi_2(g)$ for all $g \in G$, where $\mathcal{M}_i$ is the factor generated by $\pi_i(G)$.

Observe that an *irreducible* unitary representation $(\pi, \mathcal{H})$ of G is traceable if and only if $\pi(C^*(G))$ contains the algebra of compact operators on $\mathcal{H}$. The character associated to such a representation is given by the usual trace on $\mathcal{B}(\mathcal{H})$ and so does not belong to $E(G)$ whenever $\mathcal{H}$ is infinite dimensional; in this case, the character is said to be of type I_∞, in accordance with the type classification of von Neumann algebras. Observe also that two irreducible traceable representations of a group G are quasi-equivalent if and only if they are unitarily equivalent.

Thoma's dual space $E(G)$ is a subspace of $\mathrm{Char}(G)$ and classifies the quasi-equivalence classes of the factorial representations π of $C^*(G)$ for which the factor $\mathcal{M}$ generated by $\pi(G)$ is *finite*, that is, such that the trace τ on $\mathcal{M}$ takes only finite values (for more detail on all of this, see Chapters 6 and 17 in [Dix–77]).

Thoma's dual space $E(G)$ was determined for several examples of countable groups G, among them $G = GL_n(\mathbf{K})$ or $G = SL_n(\mathbf{K})$ for an infinite field $\mathbf{K}$ and $n \geq 2$ ([Kiri–65]; see also [PeT–16]), and $G = SL_n(\mathbf{Z})$ for $n \geq 3$ ([Bek–07]); a procedure is given in [How–77, Proposition 3] to compute $E(G)$ when G is a nilpotent finitely generated group.

The space $\mathrm{Char}(G)$ has been described for some *amenable* groups G:

- when G is nilpotent, we have $E(G) = \mathrm{Char}(G)$ (see [CaM–84, Theorem 2.1]);
- the space $\mathrm{Char}(G)$ is determined in [Guic–63] for the Baumslag-Solitar group $BS(1, 2)$ and in [VeK–91] for the infinite symmetric group;

- for $G = GL_n(\mathbf{K})$ and $n \geq 2$, it is shown in [Ros–89] that $E(G) = \mathrm{Char}(G)$ in the case where $\mathbf{K}$ an algebraic extension of a finite field. (Observe that $GL_n(\mathbf{K})$ is amenable if and only if $\mathbf{K}$ an algebraic extension of a finite field; see Proposition 9 in [HoR–89] or Proposition 11 below.)

J. Rosenberg asked in [Ros–89, Remark after Théorème 1] whether there exists an *infinite* character on $G = \mathrm{GL}_n(\mathbf{K})$, that is, whether $\mathrm{Char}(G) \neq E(G)$, for a field $\mathbf{K}$ which is not an algebraic extension of a finite field. We will show below that the answer to this question is positive, by exhibiting as far we know the first examples of such characters. The case where $n = 2$ and $\mathbf{K}$ is a global field (see below) will be deduced from a general result concerning groups acting on trees, which we now state.

Recall that a graph X is locally finite if every vertex on X has only finitely many neighbours. In this case, the group $\mathrm{Aut}(X)$ of automorphisms of X, equipped with the topology of pointwise convergence, is a locally compact group for which the vertex stabilizers are compact. Concerning the notion of weakly equivalent representations, see Chapters 3 and 18 in [Dix–77] (see also Section 2.1).

Theorem 1. *Let X be a tree and G a countable subgroup acting on X. Assume that*

(a) *either G is not amenable and the stabilizer in G of every vertex of X is finite, or*
(b) *X is locally finite and the closure of the image of G in $\mathrm{Aut}(X)$ is not amenable.*

There exists an uncountable family $(\pi_t)_t$ of irreducible unitary representations of G with the following properties: π_t is infinite dimensional, is traceable and is not weakly equivalent to $\pi_{t'}$ for $t' \neq t$.

Recall that a global field is a finite extension of either the field $\mathbf{Q}$ of rational numbers or of the field $\mathbf{F}_p(T)$ of rational functions in T over the finite field $\mathbf{F}_p$ (see Chapter III in [Wei–67]).

Corollary 2. *Let G be either*

(i) $\mathrm{GL}_2(\mathbf{K})$ *or* $\mathrm{SL}_2(\mathbf{K})$ *for a global field* $\mathbf{K}$, *or*
(ii) $\mathrm{SL}_2(\mathbf{Z})$, *or*
(iii) *F_n, the free non abelian group over $n \in \{2, \ldots, +\infty\}$ generators.*

There exists an uncountable family $(\pi_t)_t$ of unitary representations of G with the properties from Theorem 1; moreover, in case $G = F_n$, the representations π_t are all faithful.

Turning to the case $n \geq 3$, we prove a result for $G = \mathrm{GL}_n(R)$ or $G = \mathrm{SL}_n(R)$, valid for every integral domain R such that G is not amenable.

Theorem 3. *Let R be a countable unital commutative ring which is an integral domain; in case the characteristic of R is positive, assume that the field of fractions of R is not an algebraic extension of its prime field. For $n \geq 3$, let $G = \mathrm{GL}_n(R)$ or $G = \mathrm{SL}_n(R)$. There exists an infinite dimensional irreducible unitary representation of G which is traceable.*

In the case where R is a field or the ring of integers, we can even produce infinitely many non equivalent representations as in Theorem 3.

Corollary 4.
- (i) *For $n \geq 3$, let $G = GL_n(\mathbf{K})$ for a countable field $\mathbf{K}$ which is not an algebraic extension of a finite field. There exists an uncountable family $(\pi_t)_t$ of pairwise non equivalent infinite dimensional irreducible unitary representations of G which are traceable. Moreover, the representations π_t all have a trivial central character, that is, the π_t's are representations of* $\mathrm{PGL}_n(\mathbf{K})$.
- (ii) *Let $G = SL_n(\mathbf{Z})$ for $n \geq 3$. There exists an infinite family of pairwise non equivalent infinite dimensional irreducible unitary representations of G which are traceable.*

The methods of proofs of Theorem 1 and Theorem 3 are quite different in nature:

- the proof of Theorem 1 is based on properties of a remarkable family of unitary representations of groups acting on trees constructed in [JuV–84] and used to show their K-theoretic amenability, a notion which originated from [Cun–83] in the case of free groups;
- the traceable representations we construct in Theorem 3 are induced representations from suitable subgroups. The case $n \geq 4$ uses the existence of appropriate subgroups of $\mathrm{GL}_n(R)$ with Kazhdan's Property (T).

Remark 5. For a group G as in Theorem 1 or Theorem 1, our results show that the set $\mathrm{Char}(G)$ contains characters of type I_∞.

For, say, $G = \mathrm{GL}_n(\mathbf{Q})$, we do not know whether $\mathrm{Char}(G)$ contains characters of type II_∞, that is, characters for which the corresponding factorial representation generates a factor of type II_∞.

This paper is organized as follows. In Section 2, we establish some preliminary facts which are necessary to the proofs of our results. Section 3 is devoted to the proofs of Theorem 1 and Corollary 2; Theorem 3 and Corollary 4 are proved in Section 4.

2. Some preliminary results

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2.1. **C^*-algebras.** Let G be a countable group. Recall that a unitary representation of G is a homomorphism $\pi : G \to U(\mathcal{H})$ from G to the unitary group of a complex separable Hilbert space $\mathcal{H}$. From now on, we will simply write **representation** of G instead of "unitary representation of G".

Every representation $(\pi, \mathcal{H})$ of G extends naturally to a $*$-representation, denoted again by π, of the group algebra $\mathbf{C}[G]$ by bounded operators on $\mathcal{H}$.

Recall that the maximal C^*-algebra $C^*(G)$ of G is the completion of $\mathbf{C}[G]$ of G with respect to the norm

$$f \mapsto \sup_{\pi \in \mathrm{Rep}(G)} \|\pi(f)\|,$$

where $\mathrm{Rep}(G)$ denotes the set of representations $(\pi, \mathcal{H})$ of G in a separable Hilbert space $\mathcal{H}$.

We can view G as subset of $\mathbf{C}[G]$ and hence as a subset of $C^*(G)$. The C^*-algebra $C^*(G)$ has the following universal property: every representation $(\pi, \mathcal{H})$ of G extends to a unique representation (that is, $*$-homomorphism) $\pi : C^*(G) \to \mathcal{B}(\mathcal{H})$. The correspondence $G \to C^*(G)$ is functorial: every homomorphism $\varphi : G_1 \to G_2$ between two countable groups G_1 and G_2 extends to a unique morphism

$$\varphi_* : C^*(G_1) \to C^*(G_2)$$

of C^*-algebras. In particular, given a subgroup H of a group G, the injection map $i : H \to G$ extends to a morphism $i_* : C^*(H) \to C^*(G)$; the map i_* is injective and so $C^*(H)$ can be viewed naturally as a subalgebra of $C^*(G)$: indeed, this follows from the fact that every representation σ of H occurs as subrepresentation of the restriction to H of some representation π of G (one may take as π the induced representation $\mathrm{Ind}_H^G \sigma$, as shown below in Proposition 9).

The following simple lemma will be one of our tools in order to show that $\pi(C^*(G))$ contains a non-zero compact operator for a representation π of G.

Let $\mathcal{A}$ be a C^*-algebra. Recall that a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ weakly contains another representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ if

$$\|\rho(a)\| \leq \|\pi(a)\| \qquad \text{for all} \quad a \in \mathcal{A},$$

or, equivalently, $\ker \pi \subset \ker \rho$ (see Chapter 3 in [Dix–77]). Two representations π and ρ are weakly equivalent if π weakly contains ρ and ρ weakly contains π, that is, if $\ker \pi = \ker \rho$.

Lemma 6. *Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a representation of $\mathcal{A}$. Assume that $\mathcal{H}$ contains a non-zero finite dimensional $\pi(\mathcal{A})$-invariant subspace $\mathcal{K}$ and that the restriction π_1 of π to $\mathcal{K}$ is not weakly contained in the restriction π_0 of π to the orthogonal complement $\mathcal{K}^\perp$. Then $\pi(\mathcal{A})$ contains a non-zero compact operator.*

Proof. The ideal $\ker \pi_0$ is not contained in $\ker \pi_1$, since π_1 is not weakly contained in π_0. Hence, there exists $a \in \mathcal{A}$ with $\pi_0(a) = 0$ and $\pi_1(a) \neq 0$. Then $\pi(a) = \pi_1(a)$ has a finite dimensional range and is non-zero. □

Knowing that a representation of $\mathcal{A}$ contains in its image a non-zero compact operator, the following lemma enables us to construct an *irreducible* representation of $\mathcal{A}$ with the same property.

Lemma 7. *Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a representation of $\mathcal{A}$ in a separable Hilbert space $\mathcal{H}$ Let $a \in \mathcal{A}$ be such that $\pi(a)$ is a non-zero compact operator. Then there exists an irreducible subrepresentation σ of π such that $\sigma(a)$ is a compact operator and such that $\|\sigma(a)\| = \|\pi(a)\|$.*

Proof. We can decompose π as a direct integral $\int_\Omega^\oplus \pi_\omega d\mu(\omega)$ of irreducible representations π_ω; thus, we can find a probabilility measure μ on a standard Borel space Ω, a measurable field $\omega \to \pi_\omega$ of irreducible representations of $\mathcal{A}$ in a measurable field $\omega \to \mathcal{H}_\omega$ of separable Hilbert spaces on Ω, and a Hilbert space isomorphism $U : \mathcal{H} \to \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$ such that

$$U\pi(x)U^{-1} = \int_\Omega^\oplus \pi_\omega(x) d\mu(\omega). \qquad \text{for all} \quad x \in \mathcal{A}$$

(see [Dix–77, §8.5]). Without loss of generality, we will identify π with $\int_\Omega^\oplus \pi_\omega d\mu(\omega)$.

Let $a \in \mathcal{A}$ be such that $\pi(a)$ is a non-zero compact operator. Since $\|\sigma(a^*a)\| = \|\sigma(a)\|^2$ for every representation σ of $\mathcal{A}$, upon replacing a by a^*a, we can assume that a is a positive element of $\mathcal{A}$. So $\pi(a)$ is a positive selfadjoint compact operator on $\mathcal{H}$ with $\pi(a) \neq 0$.

There exists an orthonormal basis $(F_n)_{n\geq 1}$ of $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$ consisting of eigenvectors of $\pi(a)$, with corresponding eigenvalues $(\lambda_n)_{n\geq 1}$, counted with multiplicities. For every $\omega \in \Omega$ and every $n \geq 1$, we have

$$(*) \qquad \pi_\omega(a)(F_n(\omega)) = \lambda_n F_n(\omega).$$

Let $n_0 \geq 1$ be such that $\lambda_{n_0} = \max\{\lambda_n \mid n \geq 1\}$. Then $\|\pi(a)\| = \lambda_{n_0}$. Set

$$\Omega_0 = \{\omega \in \Omega \mid F_{n_0}(\omega) \neq 0.\}.$$

Since $F_{n_0} \neq 0$, we have $\mu(\Omega_0) > 0$. We claim that Ω_0 is a finite subset of Ω consisting of atoms of μ. Indeed, assume by contradiction that is not the case. Then there exists an infinite sequence $(A_k)_k$ of pairwise disjoint Borel subsets of Ω_0 with $\mu(A_k) > 0$. Observe that $\mathbf{1}_{A_k} F_{n_0}$ is a non-zero vector in $\mathcal{H}$ and that $\langle \mathbf{1}_{A_k} F_{n_0}, \mid \mathbf{1}_{A_l} F_{n_0}\rangle = 0$ for every $k \neq l$. Moreover, we have

$$\begin{aligned}
\pi(a)(\mathbf{1}_{A_k} F_{n_0}) &= \int_{A_k}^\oplus \pi_\omega(a)(F_{n_0}(\omega)) d\mu(\omega) \\
&= \lambda_{n_0} \int_{A_k}^\oplus F_{n_0}(\omega) d\mu(\omega) \\
&= \lambda_{n_0} \mathbf{1}_{A_k} F_{n_0}.
\end{aligned}$$

Since $\pi(a)$ is a compact operator and $\lambda_{n_0} \neq 0$, this is a contradiction.

Let $\omega_0 \in \Omega_0$ be such that $\mu(\{\omega_0\}) > 0$. We claim that the linear span of $\{F_n(\omega_0) \mid n \geq 1\}$ is dense in $\mathcal{H}_{\omega_0}$. Indeed, let $v \in \mathcal{H}_{\omega_0}$ be such that

$$\langle v \mid F_n(\omega_0)\rangle = 0 \qquad \text{for all} \quad n \geq 1.$$

Let $F = \mathbf{1}_{\omega_0} \otimes v \in \mathcal{H}$ be defined by $F(\omega_0) = v$ and $F(\omega) = 0$ for $\omega \neq \omega_0$. Then $\langle F \mid F_n\rangle = 0$ for all $n \geq 1$. Hence, $F = 0$, that is, $v = 0$, since $(F_n)_{n\geq 1}$ is a basis of $\mathcal{H}$.

By $(*)$, $F_n(\omega_0)$ is an eigenvector of $\pi_{\omega_0}(a)$ with eigenvalue λ_n for every $n \geq 1$ such that $F_n(\omega_0) \neq 0$. Since $\{F_n(\omega_0) \mid n \geq 1\}$ is a total subset of $\mathcal{H}_{\omega_0}$, it follows that there exists a basis of $\mathcal{H}_{\omega_0}$ consisting of eigenvectors of $\pi_{\omega_0}(a)$. As

$$\lim_{n\to\infty} \lambda_n = 0$$

(in case the sequence $(\lambda_n)_{n\geq 1}$ is infinite), it follows that $\pi_{\omega_0}(a)$ is a compact operator on $\mathcal{H}_{\omega_0}$. Moreover, we have

$$\|\pi_{\omega_0}(a)\| = \max\{\lambda_n \mid n \geq 1\} = \lambda_{n_0} = \|\pi(a)\|.$$

Finally, an equivalence between π_{ω_0} and a subrepresentation of π is provided by the unitary linear map

$$\mathcal{H}_{\omega_0} \to \mathcal{H},\ v \mapsto \mathbf{1}_{\omega_0} \otimes v.$$

□

2.2. **Induced representations of groups.** In the sequel, we will often consider group representations which are induced representations. Let G be a countable group, H a subgroup of G and $(\sigma, \mathcal{K})$ a representation of H. Recall that the induced representation $\mathrm{Ind}_H^G \sigma$ of G may be realized as follows. Let $\mathcal{H}$ be the Hilbert space of maps $f : G \to \mathcal{K}$ with the following properties

(i) $f(hx) = \sigma(h)f(x)$ for all $x \in G, h \in H$;
(ii) $\sum_{x \in H\backslash G} \|f(x)\|^2 < \infty$. (Observe that $\|f(x)\|$ only depends on the coset of x in $H\backslash G$.)

The induced representation $\pi = \mathrm{Ind}_H^G \sigma$ is given on $\mathcal{H}$ by right translation:

$$(\pi(g)f)(x) = f(xg) \quad \text{for all } g \in G, \ f \in \mathcal{H} \text{ and } x \in G.$$

Recall that the **commensurator** of H in G is the subgroup, denoted by $\mathrm{Comm}_G(H)$, of the elements $g \in G$ such that $gHg^{-1} \cap H$ is of finite index in both H and $g^{-1}Hg$.

The following result appeared in [Mac–51] in the case where σ is of dimension 1 and was extended to its present form in [Kle–61] and [Cor–75].

Theorem 8. *Let G be a countable group and H a subgroup of G such that* $\mathrm{Comm}_G(H) = H$.

(i) *For every* finite dimensional *irreducible representation σ of H, the induced representation* $\mathrm{Ind}_H^G \sigma$ *is irreducible.*
(ii) *Let σ_1 and σ_2 be non equivalent* finite dimensional *irreducible representations of H. The representations* $\mathrm{Ind}_H^G \sigma_1$ *and* $\mathrm{Ind}_H^G \sigma_2$ *are non equivalent.*

We will need to decompose the restriction to a subgroup of an induced representation $\mathrm{Ind}_H^G \sigma$ as in Theorem 8. For $g \in G$, we denote by σ^g the representation of $g^{-1}Hg$ defined by $\sigma^g(x) = \sigma(gxg^{-1})$ for $x \in g^{-1}Hg$.

For the convenience of the reader, we give a short and elementary proof of the following special case of the far more general result [Mac–52, Theorem 12.1].

Proposition 9. *Let G be a countable group, H, L subgroups of G and $(\sigma, \mathcal{K})$ a representation of H. Let S be a system of representatives for the double coset space $H\backslash G/L$. The restriction $\pi|_L$ to L of the induced* *representation* $\pi = \mathrm{Ind}_H^G \sigma$ *is equivalent to the direct sum*

$$\bigoplus_{s\in S} \mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L})$$

Proof. Let $\mathcal{H}$ be the Hilbert space of π, as described above. For every $s \in S$, let $\mathcal{H}_s$ be the space of maps $f \in \mathcal{H}$ such that $f = 0$ outside the double coset HsL. We have an orthogonal L-invariant decomposition

$$\mathcal{H} = \bigoplus_{s\in S} \mathcal{H}_s.$$

Fix $s \in S$. The Hilbert space $\mathcal{H}'_s$ of $\mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L})$ consists of the maps $f : L \to \mathcal{K}$ such that

- $f(tx) = \sigma(sts^{-1})f(x)$ for all $t \in s^{-1}Hs \cap L, x \in L$;
- $\sum_{x\in s^{-1}Hs\cap L\backslash L} \|f(x)\|^2 < \infty$.

Define a linear map $U : \mathcal{H}_s \to \mathcal{H}'_s$ by

$$Uf(x) = f(sx) \qquad \text{for all} \quad f \in \mathcal{H}_s, x \in L.$$

Observe that, for $t \in s^{-1}Hs \cap L$, $x \in L$ and $f \in \mathcal{H}_s$, we have

$$Uf(tx) = f(stx) = f((sts^{-1})sx) = \sigma(sts^{-1})f(sx) = \sigma^s(t)Uf(x)$$

and that

$$\sum_{x\in s^{-1}Hs\cap L\backslash L} \|Uf(x)\|^2 = \sum_{x\in s^{-1}Hs\cap L\backslash L} \|f(sx)\|^2 = \sum_{y\in H\backslash G} \|f(y)\|^2 < \infty,$$

so that $Uf \in \mathcal{H}'_s$ and U is an isometry. It is easy to check that the map U is invertible, with inverse given by

$$U^{-1}f(y) = \begin{cases} \sigma(h)f(x) & \text{if } y = hsx \in HsL \\ 0 & \text{otherwise }, \end{cases}$$

for $f \in \mathcal{H}'_s$. Moreover, U intertwines the restriction of $\pi|_L$ to $\mathcal{H}_s$ and $\mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L})$: for $g, x \in L$ and $f \in \mathcal{H}'_s$, we have

$$\begin{aligned}\left(U\pi(g)U^{-1}f\right)(x) &= \left(\pi(g)U^{-1}f\right)(sx) \\ &= (U^{-1}f(sxg) \\ &= f(xg) \\ &= \left(\mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L})(g)f\right)(x)\end{aligned}$$

□

We will need the following elementary lemma about induced representations containing a finite dimensional representation. Recall that a representation π of a group G contains another representation σ of G if σ is equivalent to a subrepresentation of π. Recall also that, if π is finite dimensional representation of a group G, then $\pi\otimes\bar{\pi}$ contains the trivial representation 1_G, where $\bar{\pi}$ is the conjugate representation of π and $\pi\otimes\rho$ denotes the (inner) tensor product of the representations π and ρ (see [BHV–08, Proposition A. 1.12]).

Proposition 10. *Let G be a countable group, H a subgroup of G, and σ a representation of H. Assume that the induced representation $\mathrm{Ind}_H^G\sigma$ contains a finite dimensional representation of G. Then H has finite index in G.*

Proof. By assumption, $\pi := \mathrm{Ind}_H^G\sigma$ contains a finite dimensional representation σ. Hence, $\pi\otimes\bar{\pi}$ contains 1_G. On the other hand,

$$\pi\otimes\bar{\pi} = (\mathrm{Ind}_H^G\sigma)\otimes\bar{\pi}$$

is equivalent to $\mathrm{Ind}_H^G(\rho)$, where $\rho=\sigma\otimes(\bar{\pi}|_H)$; see [BHV–08, Proposition E. 2.5]. So, there exists a non-zero map $f: G\to\mathcal{K}$ in the Hilbert space of $\mathrm{Ind}_H^G(\rho)$ which is G-invariant, that is, such that $f(xg)=f(x)$ for all $g,x\in G$. This implies that the L^2-function $x\mapsto\|f(x)\|^2$ is constant on $H\backslash G$. This is only possible if $H\backslash G$ is finite.

□

2.3. **Amenability.** Let $\mathbf{G}$ be a topological group and $UCB(\mathbf{G})$ the Banach space of the left uniformly continuous bounded functions on $\mathbf{G}$, equipped with the uniform norm. Recall that $\mathbf{G}$ is amenable if there exists a $\mathbf{G}$-invariant mean on $UCB(\mathbf{G})$ (see Appendix G in [BHV–08]).

The following proposition characterizes the integral domains R for which $\mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$ is amenable; the proof is an easy extension of the proof given in Proposition 9 in [HoR–89] for the case where R is a field.

Proposition 11. *Let R be a countable unital commutative ring which is an integral domain. Let $\mathbf{K}$ be the field of fractions of R and $G=\mathrm{GL}_n(R)$ or $G=\mathrm{SL}_n(R)$ for an integer $n\geq 2$. The following properties are equivalent:*

(i) *G is not amenable.*
(ii) *$\mathbf{K}$ is not an algebraic extension of a finite field.*
(iii) *R contains $\mathbf{Z}$ if the characteristic of $\mathbf{K}$ is 0 or the polynomial ring $\mathbf{F}_p[T]$ if the characteristic of $\mathbf{K}$ is $p>0$.*

Proof. Assume that $\mathbf{K}$ is an algebraic extension of a finite field $\mathbf{F}_q$. Then $\mathbf{K}=\bigcup_m\mathbf{K}_m$ for an increasing family of finite extensions $\mathbf{K}_m$ of $\mathbf{F}_q$; hence, $\mathrm{GL}_n(\mathbf{K})=\bigcup_m\mathrm{GL}_n(\mathbf{K}_m)$ is the inductive limit of the finite and hence amenable groups $\mathrm{GL}_n(\mathbf{K}_m)$; it follows that $\mathrm{GL}_n(\mathbf{K})$ is amenable and therefore $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ are amenable. This shows that (i) implies (ii).

Assume that (ii) holds. If the characteristic of $\mathbf{K}$ is 0, then $\mathbf{K}$ contains $\mathbf{Q}$ and hence R contains $\mathbf{Z}$. So, we can assume that the characteristic of $\mathbf{K}$ is $p > 0$. We claim that R contains an element which is not algebraic over the prime field $\mathbf{F}_p$. Indeed, otherwise, every element in R is algebraic over $\mathbf{F}_p$. As the set of elements in $\mathbf{K}$ which are algebraic over $\mathbf{F}_p$ is a field, it would follow that the field fraction field $\mathbf{K}$ is algebraic over $\mathbf{F}_p$. This contradiction shows that (ii) implies (iii).

Assume that (iii) holds. Then $\mathrm{SL}_n(R)$ contains a copy of $\mathrm{SL}_2(\mathbf{Z})$ or a copy of $\mathrm{SL}_2(\mathbf{F}_p[T])$. It is well-known that both $\mathrm{SL}_2(\mathbf{Z})$ and $\mathrm{SL}_2(\mathbf{F}_p[T])$ contain a subgroup which is isomorphic to the free group on two generators. Therefore, G is not amenable and so (iii) implies (i). $\square$

Let $\mathbf{G}$ be a *locally compact* group, with Haar measure m. Recall that the amenability of $\mathbf{G}$ is characterized by the Hulanicki-Reiter theorem (see [BHV–08, Theorem G.3.2]): $\mathbf{G}$ is amenable if and only if the regular representation $(\lambda_{\mathbf{G}}, L^2(\mathbf{G}, m))$ weakly contains the trivial representation $1_{\mathbf{G}}$, where m is Haar measure on $\mathbf{G}$; when this is the case, $\lambda_{\mathbf{G}}$ weakly contains every representation of $\mathbf{G}$

The following result shows the amenability of $\mathbf{G}$ can be detected by the restriction of $\lambda_{\mathbf{G}}$ to a dense subgroup; for a more general result, see [Guiv–80, Proposition 1] or [Bek–16, Theorem 5.5].

Proposition 12. *Let $\mathbf{G}$ be a locally compact group and G a countable dense subgroup of $\mathbf{G}$. Assume that the restriction to G of the regular representation $\lambda_{\mathbf{G}}$ of $\mathbf{G}$ weakly contains the trivial representation 1_G. Then $\mathbf{G}$ is amenable.*

Proof. By assumption, there exists a sequence $(f_n)_n$ in $L^2(\mathbf{G}, m)$ with $\|f_n\| = 1$ such that

$$\lim_n \|\lambda_{\mathbf{G}}(g)f_n - f_n\| = 0 \qquad \text{for all} \quad g \in G.$$

Then, since $||f_n(g^{-1}x)| - |f_n(x)|| \leq |f_n(g^{-1}x) - f_n(x)|$ for $g, x \in \mathbf{G}$, we have

$$(*) \qquad \lim_n \|\lambda_{\mathbf{G}}(g)|f_n| - |f_n|\| = 0 \qquad \text{for all} \quad g \in G.$$

Set $\varphi_n := \sqrt{|f_n|}$. Then $\varphi_n \geq 0$ and $\int_{\mathbf{G}} \varphi_n dm = 1$. Every φ_n defines a mean $M_n : f \mapsto \int_{\mathbf{G}} f\varphi_n dm$ on $UCB(\mathbf{G})$. Let M be a limit of $(M_n)_n$ for the weak-*-topology on the dual space of $UBC(\mathbf{G})$. It follows from $(*)$ that M is invariant under G. Since, for every $f \in UCB(\mathbf{G})$, the map

$$\mathbf{G} \to UCB(\mathbf{G}), \quad g \mapsto_g f$$

is continuous (where ${}_g f$ denotes left translation by $g \in \mathbf{G}$), it follows that M is invariant under $\mathbf{G}$. Hence, $\mathbf{G}$ is amenable.

2.4. **Special linear groups over a subring of a field.** We will use the following elementary lemma about subgroups of $\mathrm{SL}_n(\mathbf{K})$ which stabilize a line in $\mathbf{K}^n$.

Lemma 13. *For an infinite field* $\mathbf{K}$ *and* $n \geq 2$, *let* L *be a subgroup of* $\mathrm{SL}_n(\mathbf{K})$ *which stabilizes a line* ℓ *in* $\mathbf{K}^n$. *Then* $L \cap \mathrm{SL}_n(R)$ *has infinite index in* $\mathrm{SL}_n(R)$ *for every infinite unital subring* R *of* $\mathbf{K}$.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{K}^n$ with $\ell = \mathbf{K}v_1$. Fix $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and, for $\lambda \in \mathbf{K}$, let $E_{ij}(\lambda)$ be the corresponding elementary matrix in $\mathrm{SL}_n(\mathbf{K})$, that is,

$$E_{ij}(\lambda) = I_n + \lambda \Delta_{ij},$$

where Δ_{ij} denotes the matrix with 1 at the position (i, j) and 0 otherwise.

For every $l = 1, \ldots, n$, let $\varphi_l : \mathbf{K} \to \mathbf{K}$ be defined by

$$E_{ij}(\lambda)(v_1) = \sum_{i=1}^{n} \varphi_l(\lambda) v_i \quad \text{for} \quad \lambda \in \mathbf{K}.$$

Every φ_l is a polynomial function (in fact, an affine function) on $\mathbf{K}$ and, for $l = 2, \ldots, n$, we have $\varphi_l(\lambda) = 0$ for every $\lambda \in \mathbf{K}$ such that $E_{ij}(\lambda) \in L$.

Assume, by contradiction, that $L \cap \mathrm{SL}_n(R)$ has finite index in $\mathrm{SL}_n(R)$ for an infinite subring R of $\mathbf{K}$. Then the subgroup

$$L_{i,j}(R) := L \cap \{E_{ij}(\lambda) \mid \lambda \in R\}$$

has finite index in the subgroup $\{E_{ij}(\lambda) \mid \lambda \in R\}$ of $SL_n(R)$. In particular, $L_{i,j}(R)$ is infinite. It follows that φ_l has infinitely many roots in $\mathbf{K}$ and hence that $\varphi_l = 0$, for every $l = 2, \ldots, n$. Therefore, every elementary matrix $E_{ij}(\lambda)$ fixes the line ℓ, for $i, j \in \{1, \ldots, n\}$ and $\lambda \in \mathbf{K}$. Since $\mathrm{SL}_n(\mathbf{K})$ is generated by elementary matrices, it follows that every matrix in $\mathrm{SL}_n(\mathbf{K})$ fixes the line ℓ; this of course is impossible. □

3. Proofs of Theorem 1 and Corollary 2

3.1. **Proof of Theorem 1.** Let X be a tree, with X^0 the set of vertices and X^1 the set of edges of X. Let $\mathbf{G}$ be a locally compact group acting on X.

Julg and Valette constructed in [JuV–84] (see also [Szw–91] and [Jul–15]) a remarkable family of representations $(\pi_t)_{t \in [0,1]}$ of $\mathbf{G}$, all defined on $\ell^2(X^0)$, with the following properties:

(i) π_0 is the natural representation of $\mathbf{G}$ on $\ell^2(X^0)$ and π_1 is equivalent to $1_{\mathbf{G}} \oplus \rho_1$, where ρ_1 is the natural representation of $\mathbf{G}$ on $\ell^2(X^1)$;

(ii) for every $t \in [0,1]$, there exists a bounded operator T_t on $\ell^2(X^0)$ with inverse T_t^{-1} defined on the subspace of functions of X^0 with finite support such that $\pi_t(g) := T_t^{-1}\pi_0(g)T_t$ extends to a unitary operator on $\ell^2(X^0)$ for every $g \in \mathbf{G}$; so, a unitary representation $g \mapsto \pi_t(g)$ of $\mathbf{G}$ is defined on $\ell^2(X^0)$;

(iii) $\pi_t(g) - \pi_0(g)$ is a finite-rank operator on $\ell^2(X^0)$, for every $t \in [0,1]$ and $g \in \mathbf{G}$;

(iv) we have

$$\langle \pi_t(g)T_t^{-1}\delta_x \mid T_t^{-1}\delta_y\rangle = t^{d(gx,y)},$$

for every $t \in (0,1)$, $g \in \mathbf{G}$ and $x, y \in X^0$, where d denotes the natural distance on X^0;

(v) the map

$$[0,1] \to \mathbf{R}^+,\ t \mapsto \|\pi_t(g) - \pi_0(g)\|$$

is continuous for every $g \in \mathbf{G}$.

(Our representation π_t is $g \mapsto U_\lambda \rho_\lambda(g) U_t^{-1}$ with $\lambda = -\log t$, for the representation ρ_λ and the operator U_λ appearing in §2 of [JuV–84].)

Let G be a countable group acting on X. Assume that

(a) either G is not amenable and the stabilizer in G of every vertex of X is finite or

(b) X is locally finite and the closure of the image of G in $\mathrm{Aut}(X)$ is not amenable.

Set $\mathbf{G} = G$ in case (a) and let $\mathbf{G}$ be the closure of G in $\mathrm{Aut}(X)$ in case (b). Let $(\pi_t)_{t\in[0,1]}$ be the family of representations of $\mathbf{G}$ as above.

• *First step.* For every $a \in C^*(G)$ and every $t \in [0,1]$, the operator $\pi_t(a) - \pi_0(a)$ is compact and the map

$$[0,1] \to \mathbf{R}^+,\ t \mapsto \|\pi_t(a) - \pi_0(a)\|$$

is continuous.

Indeed, this follows from Properties (iii) and (v) of the family $(\pi_t)_t$ and from the fact that $\mathbf{C}[G]$ is dense in $C^*(G)$.

• *Second step.* The restriction $\pi_0|_G$ of π_0 to G does not weakly contain the trivial representation 1_G.

Indeed, the representation π_0 of $\mathbf{G}$ is equivalent to the direct sum $\oplus_{s\in T}\lambda_{\mathbf{G}/\mathbf{G}_s}$, where S is a system of representatives for the $\mathbf{G}$-orbits in X^0 and $\mathbf{G}_s$ is the stabilizer in $\mathbf{G}$ of $s \in S$. Since $\mathbf{G}_s$ is compact (and even finite in case (a)) and hence amenable, $\lambda_{\mathbf{G}/\mathbf{G}_s} = \mathrm{Ind}_{\mathbf{G}_s}^{\mathbf{G}} 1_{\mathbf{G}_s}$

is weakly contained in the regular representation $\lambda_{\mathbf{G}}$ of $\mathbf{G}$ and so π_0 is weakly contained in $\lambda_{\mathbf{G}}$. Hence, π_0 does not weakly contain the trivial representation 1_G in case (a). In case (b), the claim follows from Proposition 12, since $\mathbf{G}$ is not amenable and G is dense in $\mathbf{G}$.

• *Third step.* There exists an element $a \in C^*(G)$ and $0 \leq t_0 < 1$ with the following properties: $\pi_{t_0}(a) = 0$, $\pi_t(a)$ is a non zero compact operator for every $t \in (t_0, 1]$, and the map

$$[t_0, 1] \to \mathbf{R}+, \ t \mapsto \|\pi_t(a)\|$$

is continuous.

Indeed, by the second step, there exists $a \in C^*(G)$ such that $\pi_0(a) = 0$ and $1_G(a) \neq 0$. Therefore, $\pi_1(a) \neq 0$ and $\pi_t(a) = \pi_t(a) - \pi_0(a)$ for every $t \in [0, 1]$ and so the claim follows from the first step.

• *Fourth step.* Let $a \in C^*(G)$ and $0 \leq t_0 < 1$ be as in the third step. There exists an irreducible infinite dimensional subrepresentation σ_t of π_t such that $\sigma_t(a)$ is a compact operator and such that $\|\sigma_t(a)\| = \|\pi_t(a)\|$ for every $t \in (t_0, 1)$.

Indeed, it follows from the third step and Lemma 7 that $\pi_t|_G$ contains an irreducible subrepresentation σ_t such that $\sigma_t(a)$ is a compact operator with $\|\sigma_t(a)\| = \|\pi_t(a)\|$. It remains to show that σ_t is infinite dimensional for every $t \in (t_0, 1)$.

Assume, by contradiction, σ_t is finite dimensional for some $t \in (t_0, 1)$. Since G is dense in $\mathbf{G}$, the closed subspace $\mathcal{K}_t$ of $\ell^2(X^0)$ defining σ_t is invariant under $\mathbf{G}$ and so σ_t is the restriction to G of a dimensional subrepresentation of π_t, again denoted by σ_t. On the one hand, $\mathbf{G}$ acts properly on X^0, since the stabilizers of vertices are compact (and even finite in case (a)). So, we have

$$\lim_{g \to +\infty : g \in \mathbf{G}} d(gx, x) = 0 \qquad \text{for all} \quad x \in X^0.$$

It follows from Property (iv) of the family $(\pi_t)_t$ that π_t (and hence σ_t) is a C_0-representation, that is,

$$\lim_{g \to +\infty : g \in \mathbf{G}} \langle \pi_t(g) v \mid w \rangle = 0$$

for every $v, w \in \ell^2(X^0)$. On the other hand, since σ_t is finite dimensional, $\sigma_t \otimes \overline{\sigma_t}$ contains $1_{\mathbf{G}}$. As $\mathbf{G}$ is not compact, this is a contradiction to the fact that σ_t is a C_0-representation.

• *Fifth step.* There exists uncountably many real numbers $t \in (t_0, 1)$ such that the subrepresentations σ_t of $\pi_t|_G$ as in the fourth step are pairwise non weakly equivalent.

Indeed, by the third step, the function $f : t \mapsto \|\pi_t(a)\|$ is continuous on $[t_0, 1]$, with $f(t_0) = 0$ and $f(1) > 0$. So, the range of f contains a whole interval. Let $t, s \in (t_0, 1)$ be such that $f(t) \neq f(s)$. Then

$$\|\sigma_t(a)\| = \|\pi_t(a)\| = f(t) \neq f(s) = \|\pi_s(a)\| = \|\sigma_s(a)\|,$$

and so σ_t and σ_s are not weakly equivalent.

3.2. **Proof of Corollary 2.** The following remarks show how Corollary 2 follows from Theorem 1.

(i) Let $\mathbf{K}$ be global field $\mathbf{K}$. Choose a non trivial discrete valuation $v : \mathbf{K}^* \to \mathbf{Z}$. The completion of $\mathbf{K}$ at v is a non archimedean local field $\mathbf{K}_v$. The tree X_v associated to v (see Chapter II in [Ser–80]) is a locally finite regular graph. The group $G = \mathrm{GL}_2(\mathbf{K})$ acts as a group of automorphisms of X_v, with vertex stabilizers conjugate to $\mathrm{GL}_2(\mathcal{O}_v \cap \mathbf{K})$, where $\mathcal{O}_v$ is the compact subring of the integers in $\mathbf{K}_v$. The closure of the image of G in $\mathrm{Aut}(X_v)$ coincides with $\mathrm{PGL}_2(\mathbf{K}_v)$ and is therefore non amenable. A similar remark applies to $G = \mathrm{SL}_2(\mathbf{K})$.

(ii) As is well-known, the group $G = \mathrm{SL}_2(\mathbf{Z})$ is an amalgamated product $\mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$. It follows that G acts on a tree with vertices of valence 2 or 3 with vertex stabilizers of order 4 or 6 (see Chapter I, Examples 4.2. in [Ser–80])

(iii) The free non abelian group F_2 acts freely on its Cayley graph X, which is a 4-regular tree. It follows that F_n acts freely on X for every $n \in \{2, \ldots, +\infty\}$. Observe that, in this case, the representations π_t and σ_t as in the proof of Theorem 1 are faithful for $t \neq 1$ (since there are even C_0-representations). □

4. Proofs of Theorem 3 and Corollary 4

4.1. **Proof of Theorem 3.** Let R be a countable unital commutative ring which is an integral domain and $\mathbf{K}$ its field of fractions. In case the characteristic of $\mathbf{K}$ is positive, assume that $\mathbf{K}$ is not an algebraic extension of its prime field.

Let $n \geq 3$ and $G = \mathrm{GL}_n(R)$. We consider the natural action of G on the projective space $\mathbf{P}(\mathbf{K}^n)$. Let $\ell_0 = \mathbf{K}e_1 \in \mathbf{P}(\mathbf{K}^n)$ be the line defined by the first unit vector e_1 in $\mathbf{K}^n$. The stabilizer of ℓ_0 in G is

$$H = \begin{pmatrix} R^\times & R^{n-1} \\ 0 & GL_{n-1}(R) \end{pmatrix}.$$

Let σ be a finite dimensional representation of H and $\pi := \mathrm{Ind}_H^G \sigma$. We claim that π is irreducible and that $\pi(C^*(G))$ contains a non zero compact operator. For the proof of this claim, we have to treat separately the cases $n = 3$ and $n \geq 4$.

4.1.1. ***Case*** $n = 3$. • *First step.* We claim that $gHg^{-1} \cap H$ is amenable, for every $g \in G \setminus H$.

Indeed, let $g \in G \setminus H$. Then ℓ_0 and $g\ell_0$ are distinct lines in $\mathbf{K}^n$ and are both stabilized by $gHg^{-1} \cap H$. Hence, $gHg^{-1} \cap H$ is isomorphic to a subgroup of the solvable group

$$\begin{pmatrix} \mathbf{K}^* & 0 & \mathbf{K} \\ 0 & \mathbf{K}^* & \mathbf{K} \\ 0 & 0 & \mathbf{K}^* \end{pmatrix}$$

and is therefore amenable.

• *Second step.* We claim that the representation π is irreducible.

Indeed, in view of Theorem 8, we have to show that $\mathrm{Comm}_G(H) = H$. Let $g \in G \setminus H$. On the one hand, $gHg^{-1} \cap H$ is amenable, by the first step. On the other hand, H is non amenable, by Proposition 11. This implies that $gHg^{-1} \cap H$ is not of finite index in H and so g is not in the commensurator of H in G.

• *Third step.* We claim that the C^*-algebra $\pi(C^*(G))$ contains a non-zero compact operator.

Indeed, let S be a system of representatives for the double cosets space $H\backslash G/H$ with $e \in S$. By Proposition 9, the restriction $\pi|_H$ of π to H is equivalent to the direct sum

$$\bigoplus_{s \in S} \mathrm{Ind}_{s^{-1}Hs \cap H}^H (\sigma^s|_{s^{-1}Hs \cap H}) = \sigma \oplus \bigoplus_{s \in S \setminus \{e\}} \mathrm{Ind}_{s^{-1}Hs \cap H}^H (\sigma^s|_{s^{-1}Hs \cap H})$$

Let $s \in S \setminus \{e\}$. By the first step, $s^{-1}Hs \cap H$ is amenable and hence $\sigma^s|_{s^{-1}Hs \cap H}$ is weakly contained in the regular representation $\lambda_{s^{-1}Hs \cap H}$ of $s^{-1}Hs \cap H$, by the Hulanicki-Reiter theorem. By continuity of induction (see [BHV–08, Theorem F.3.5]), it follows that $\mathrm{Ind}_{s^{-1}Hs \cap H}^H (\sigma^s|_{s^{-1}Hs \cap H})$ is weakly contained in the regular representation λ_H of H. Therefore,

$$\pi_0 := \bigoplus_{s \in S \setminus \{e\}} \mathrm{Ind}_{s^{-1}Hs \cap H}^H (\sigma^s|_{s^{-1}Hs \cap H})$$

is weakly contained in λ_H. It follows that π_0 does not weakly contain σ; indeed, assume by contradiction that σ is weakly contained in π_0. Then $\lambda_H \otimes \overline{\lambda_H}$, which is a multiple of λ_H, weakly contains $\sigma \otimes \overline{\sigma}$. However, since σ is finite dimensional, $\sigma \otimes \overline{\sigma}$ contains 1_H. Hence, 1_H is weakly contained in λ_H and this is a contradiction to the non amenability of H.

It follows from Lemma 6 that $\pi(C^*(H))$ contains a non-zero compact operator. Since $C^*(H)$ can be viewed a subalgebra of $C^*(G)$, the claim is proved for $G = \mathrm{GL}_3(R)$.

4.1.2. ***Case*** $n \geq 4$. For every unital subring R' of R, set

$$L(R') := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SL}_{n-1}(R') \end{pmatrix},$$

which is a subgroup of H isomorphic to $SL_{n-1}(R')$.

• *First step.* Let $g_0 \in G \setminus H$ and R' an infinite unital subring of R. We claim that $g_0 H g_0^{-1} \cap L(R')$ has infinite index in $L(R')$.

Indeed, the group $L := g_0 H g_0^{-1} \cap L(R')$ stabilizes the two lines ℓ_0 and $g_0\ell_0$. Let V be the linear span of the $n-1$ unit vectors $e_2, \dots, e_n$. Denote by ℓ the projection on V of the line $g_0\ell_0$, parallel to ℓ_0. As $g\ell_0 \neq \ell_0$, we have $\ell \neq \{0\}$. Moreover, L stabilizes ℓ, since L stabilizes ℓ_0 and V. So, identifying $L(R')$ with the group $\mathrm{SL}_{n-1}(R')$, we see can view L as a subgroup of $\mathrm{SL}_{n-1}(\mathbf{K})$ which stabilizes a line in $\mathbf{K}^{n-1}$. Lemma 13 shows that L has infinite index in $\mathrm{SL}_{n-1}(R')$, as claimed.

• *Second step.* We claim that the representation π is irreducible. In view of Theorem 8, it suffices to show that $\mathrm{Comm}_G(H) = H$.

Let $g_0 \in G \setminus H$. By the first step, $g_0 H g_0^{-1} \cap L(R)$ has infinite index in $L(R)$; hence, $g_0 H g_0^{-1} \cap H$ has infinite index in H, since $L(R)$ is a subgroup of H.

• *Third step.* We claim that $\pi(C^*(G))$ contains a non-zero compact operator.

Indeed, since $\mathbf{K}$ is not an algebraic extension over its prime field, R contains a subring R' which is a copy $\mathbf{Z}$ or a copy of the polynomial ring $\mathbf{F}_p[T]$, by Proposition 11. The corresponding subgroup

$$L := L(R')$$

of G is isomorphic to $\mathrm{SL}_{n-1}(\mathbf{Z})$ or $\mathrm{SL}_{n-1}(\mathbf{F}_p[T])$. Observe that L is a lattice in the locally group $\mathbf{G} = SL_{n-1}(\mathbf{R})$ or $\mathbf{G} = SL_{n-1}(\mathbf{F}_p((T^{-1})))$, where $\mathbf{F}_p((T^{-1}))$ is the local field of Laurent series over $\mathbf{F}_p$. Since $n-1 \geq 3$, the group $\mathbf{G}$ and hence L has Kazhdan's Property (T); see [BHV–08, §. 1.4, 1.7].

Let S be a system of representatives for the double cosets space $H \backslash G / H$ with $e \in S$. By Proposition 9, the restriction $\pi|_L$ to L of π is equivalent to the direct sum $\sigma|_L \oplus \pi_0$, where

$$\pi_0 := \bigoplus_{s\in S\setminus\{e\}} \mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L}).$$

We claim that π_0 does not weakly contain $\sigma|_L$. Indeed, assume by contradiction that π_0 weakly contains $\sigma|_L$. Since σ is finite dimensional and L has Property (T), it follows that π_0 contains $\sigma|_L$ (see [BHV–08, Theorem 1.2.5]). Therefore, $\mathrm{Ind}_{s^{-1}Hs\cap L}^{L}(\sigma^s|_{s^{-1}Hs\cap L})$ contains a subrepresentation of $\sigma|_L$ for some $s \in S \setminus \{e\}$. Hence, $s^{-1}Hs \cap L$ has finite index in L, by Proposition 10. Since $L = L(R')$ for an infinite unital subring R' of R, this is a contradiction to the first step.

As in the proof for the case $n = 3$, we conclude that $\pi(C^*(G))$ contains a non-zero compact operator.

This proves Theorem 3 for $G = \mathrm{GL}_n(R)$ when $n \geq 3$. The case $G = \mathrm{SL}_n(R)$ is proved in exactly the same way.

4.2. **Proof of Corollary 4.** For $n \geq 3$, let $G = GL_n(R)$ for a ring R as above. The irreducible traceable representations of G constructed in the proof of Theorem 3 are of the form $\pi = \mathrm{Ind}_H^G \sigma$ for a finite dimensional representation of the subgroup $H = \begin{pmatrix} R^\times & R^{n-1} \\ 0 & GL_{n-1}(R) \end{pmatrix}$. Observe that, $\pi = \mathrm{Ind}_H^G \sigma$ is trivial on the center Z of G, since H contains Z.

By Theorem 8, there are infinitely (respectively, uncountably) many non equivalent such representations π, provided there exists infinitely (respectively, uncountably) non equivalent finite dimensional irreducible representations of H. This will be the case if $GL_{n-1}(R) \ltimes R^{n-1}$, which is a quotient of H, has infinitely (respectively, uncountably) many non equivalent finite dimensional irreducible representations.

(i) Assume that $R = \mathbf{K}$ is an infinite field. It is easy to show that the finite dimensional irreducible representations of $GL_{n-1}(\mathbf{K}) \ltimes \mathbf{K}^{n-1}$ are all of the form

$$\begin{pmatrix} * & * \\ 0 & A \end{pmatrix} \mapsto \chi(\det A), \qquad A \in \mathrm{GL}_{n-1}(\mathbf{K})$$

for some χ in the unitary dual $\widehat{\mathbf{K}^*}$ of $\mathbf{K}^*$; as $\mathbf{K}^*$ is infinite, $\widehat{\mathbf{K}^*}$ is a compact infinite group and is therefore uncountable.

(ii) Assume that $R = \mathbf{Z}$.

• *Case* $n = 3$. The free group F_2 is a subgroup of finite index in $\mathrm{GL}_2(\mathbf{Z})$. There exists uncountably many unitary characters (that is one-dimensional unitary representations) of F_2. For every such unitary character χ, the representation $\mathrm{Ind}_{F_2}^{\mathrm{GL}_2(\mathbf{Z})}\chi$ is finite dimensional and so has a decomposition $\oplus_i \sigma_i^{(\chi)}$ as a direct sum of finite dimensional irreducible representations $\sigma_i^{(\chi)}$ of $\mathrm{GL}_2(\mathbf{Z})$. One can choose uncountably many pairwise non equivalent representations among the $\sigma_i^{(\chi)}$'s and we obtain in this way uncountably many non equivalent finite dimensional irreducible representations of $\mathrm{GL}_2(\mathbf{Z})$ and hence of $\mathrm{GL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$.

• *Case* $n \geq 4$. The group $\mathrm{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$ has Kazhdan's property (T) and so has at most countably many non equivalent finite dimensional representations (see [Wan–75, Theorem 2.1]). There are indeed *infinitely* many such representations: for every integer $N \geq 1$, the finite group

$$G_N = \mathrm{GL}_{n-1}(\mathbf{Z}/N\mathbf{Z})) \ltimes (\mathbf{Z}/N\mathbf{Z})^{n-1}$$

is a quotient of $\mathrm{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$; infinitely many representations among the irreducible representations of the G_N's are pairwise non equivalent when viewed as representations of $\mathrm{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$.

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