Random products of automorphisms of Heisenberg nilmanifolds and Weil's representation

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Abstract

For $n \geq 1$, let H be the $(2n+1)$-dimensional real Heisenberg group, and let Λ be a lattice in H. Let Γ be a group of automorphisms of the corresponding nilmanifold $\Lambda\backslash H$ and U the associated unitary representation of Γ on $L^2(\Lambda\backslash H)$. Denote by T the maximal torus factor associated to $\Lambda\backslash H$. Using Weil's representation (also known as the metaplectic representation), we show that a dense set of matrix coefficients of the restriction of U to the orthogonal complement of $L^2(T)$ in $L^2(\Lambda\backslash H)$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for every $\varepsilon > 0$.

We give the following application to random walks on $\Lambda\backslash H$ defined by a probability measure μ on $\mathrm{Aut}(\Lambda\backslash H)$. Denoting by Γ the subgroup of $\mathrm{Aut}(\Lambda\backslash H)$ generated by the support of μ and by U^0 and V^0 the restrictions of U respectively to the subspaces of $L^2(\Lambda\backslash H)$ and $L^2(T)$ with zero mean, we prove the following inequality:

$$\|U^0(\mu)\| \leq \max\left\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)}\right\},$$

where λ_Γ is the left regular representation of Γ on $\ell^2(\Gamma)$. In particular, the action of Γ on $\Lambda\backslash H$ has a spectral gap if and only if the corresponding action of Γ on T has a spectral gap.

1 Introduction

Let (X, m) be a probability space and G a locally compact group of measure preserving transformations of X. Given a probability measure μ on G, consider a sequence of independent μ-distributed random variables X_n^ω with values in G and the corresponding random products $S_n^\omega = X_n^\omega \dots X_1^\omega$ for $n \in \mathbf{N}$. This defines a random walk on X with initial distribution m and trajectories $S_n^\omega(x)$ for $x \in X$ and $n \in \mathbf{N}$. A question of interest is whether this random walk has a spectral gap. To define this notion, let U be the unitary representation of G on $L^2(X, m)$ defined by $U_g(\xi) = \xi(g^{-1}(x))$ for $g \in G$, $\xi \in L^2(X, m)$, and $x \in X$. Let $L_0^2(X, m)$ be the $U(G)$-invariant subspace of functions ξ in $L^2(X, m)$ with zero mean, that is, with $\int_X \xi(x) dm(x) = 0$. Denote by U^0 the restriction of U to $L_0^2(X, m)$. Let $U^0(\mu)$ be the convolution operator defined on $L_0^2(X, m)$ by

$$U^0(\mu)\xi = \int_G U_g^0(\xi) d\mu(g) \qquad \text{for all} \quad \xi \in L_0^2(X, m).$$

Observe that $\|U^0(\mu)\| \leq 1$. We say that μ has a spectral gap in (X, m) if $\|U^0(\mu)\| < 1$. This spectral gap property has several interesting applications; the most immediate one is the exponentially fast convergence of the sequence of functions $x \mapsto \mathbf{E}(\xi(S_n^\omega(x))$ to $\int_X \xi dm$ in the L^2-norm for every $\xi \in L^2(X, m)$. Other applications include the existence of a rate of convergence for random ergodic theorems, a central limit theorem, and the uniqueness of invariant means on $L^\infty(X, m)$; see [FuSh99], [Guiv05], [Lubo94], [Sarn90].

The spectral gap property can be formulated in terms of weak containment of group representations (see [BeHV08, G.4.2]). Assume that the subgroup generated by the support of μ is dense in G. Assume moreover that μ is aperiodic (that is, the support of μ is not contained in the coset of a proper closed subgroup of G). Then μ has a spectral gap in (X, m) if and only if there is no G-almost invariant vectors in $L_0^2(X, m)$. If this is the case, we say for short that the G-action on X has a spectral gap.

We emphasize that the existence of a spectral gap property is a phenomenon which can occur only in the context of non-amenable groups: when G is a discrete amenable group and m is non-atomic, then G has *never* a spectral gap on X (see [JuRo79] or [Schmi80]).

When X is the n-dimensional torus $\mathbf{R}^n/\mathbf{Z}^n$, equipped with the normalized Lebesgue measure m, sufficient conditions were given in [FuSh99] for the existence of a spectral gap for the action of a subgroup of $GL_n(\mathbf{Z})$ by automorphisms on $\mathbf{R}^n/\mathbf{Z}^n$ (see also Example 4 below). In this paper, we will consider the case where X is a Heisenberg nilmanifold and G a group of automorphisms of X.

For $n \geq 1$, let $H = H_{2n+1}(\mathbf{R})$ be the $(2n+1)$-dimensional real Heisenberg group. This is a two step nilpotent Lie group with one-dimensional centre Z (see Section 2 below). Let Λ be a lattice in H: Λ is a discrete subgroup of H such that there exists a (unique) probability measure m on the Borel sets of the corresponding nilmanifold $\Lambda\backslash H$ which is invariant under right translation by elements from H. (Observe that Λ is cocompact in H.) Denote by $\mathrm{Aut}(H)$ the group of continuous automorphisms of H and by $\mathrm{Aut}(\Lambda\backslash H)$ the subgroup of all $g \in \mathrm{Aut}(H)$ such that $g(\Lambda) = \Lambda$; every automorphism $g \in \mathrm{Aut}(\Lambda\backslash H)$ induces a homeomorphism of $\Lambda\backslash H$.

Let Γ be a subgroup of $\mathrm{Aut}(\Lambda\backslash H)$. The action of Γ on $\Lambda\backslash H$ preserves the H-invariant probability measure m on $\Lambda\backslash H$. Let U be the associated unitary representation of Γ on $L^2(\Lambda\backslash H, m)$. Let $T = \Lambda Z\backslash H$ be the maximal torus factor of $\Lambda\backslash H$. Observe that $T \cong \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ and $\mathrm{Aut}(T) \cong GL_{2n}(\mathbf{Z})$. Since $\mathrm{Aut}(\Lambda\backslash H)$ preserves $Z\Lambda$, we have a homomorphism $p : \mathrm{Aut}(\Lambda\backslash H) \to \mathrm{Aut}(T)$ and an induced action of Γ on T. This defines a unitary representation of Γ on $L^2(T)$, where T is equipped with normalized Lebesgue measure. We can (and will) identify $L^2(T)$, as Γ-space, with a closed $U(\Gamma)$-invariant subspace of $L^2(\Lambda\backslash H)$. Denote by $\mathcal{H}$ the orthogonal complement of $L^2(T)$ in $L^2(\Lambda\backslash H)$, so that we have an orthogonal decomposition

$$L^2(\Lambda\backslash H) = L^2(T) \oplus \mathcal{H}$$

into $U(\Gamma)$-invariant subspaces. Here is our main result.

Theorem 1 *The matrix coefficients of the restriction of U to $\mathcal{H}$ are strongly $L^{4n+2+\varepsilon}$: there are dense subspaces D_1 and D_2 of $\mathcal{H}$ such that, for any $v \in D_1$ and $w \in D_2$, the matrix coefficient $\gamma \mapsto \langle U_\gamma v, w\rangle$ belongs to $\ell^{4n+2+\varepsilon}(\Gamma)$, for every $\varepsilon > 0$.*

Concerning the proof of the previous theorem, we first show that the representation U is linked with Weil's representation, which is also known as Segal-Shale-Weil, metaplectic, or oscillator representation (see [Shal62], [Weil64]). The crucial tool is then a result from [HoMo79] about the decay of the matrix coefficients of Weil's representation.

Here is an immediate consequence of Theorem 1. Recall that, if X is a locally compact space, $C_0(X)$ denotes the space of complex-valued continuous functions on X which tend to zero at infinity.

Corollary 2 *The restriction of the unitary representation U to $\mathcal{H}$ is mixing: the matrix coefficients $\gamma \mapsto \langle U_\gamma v, w\rangle$ belong to $c_0(\Gamma)$ for all $v, w \in \mathcal{H}$.*

The previous corollary immediately implies that the ergodicity or mixing of the Γ-action on $\Lambda\backslash H$ is equivalent to the ergodicity or mixing of the Γ-action on T (see Corollary 6 below).

We apply Theorem 1 to the existence of a spectral gap for the random walk on $\Lambda\backslash H$ associated to a probability measure μ on $\mathrm{Aut}(\Lambda\backslash H)$.

Theorem 3 *Let μ be a probability measure on $\mathrm{Aut}(\Lambda\backslash H)$. Denote by Γ be the subgroup of $\mathrm{Aut}(\Lambda\backslash H)$ generated by the support of μ. Let U^0 and V^0 be the associated unitary representations of Γ on $L_0^2(\Lambda\backslash H)$ and $L_0^2(T)$ respectively. Then*

$$\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)}\},$$

where λ_Γ is the left regular representation of Γ on $\ell^2(\Gamma)$. In particular, the action of Γ on $\Lambda\backslash H$ has a spectral gap if and only if the corresponding action of Γ on T has a spectral gap.

Example 4 Let Γ be a subgroup of $\mathrm{Aut}(\Lambda\backslash H)$ such that its image $p(\Gamma) \subset GL_{2n}(\mathbf{Z})$ under the homomorphism $p : \mathrm{Aut}(\Lambda\backslash H) \to \mathrm{Aut}(T) \cong GL_{2n}(\mathbf{Z})$ acts irreducibly on $\mathbf{R}^{2n}$ and does not have an abelian subgroup of finite index (this is for instance the case if $p(\Gamma)$ is Zariski dense in $GL_{2n}(\mathbf{R})$). Then, as shown in [FuSh99, Theorem 6.5], the action of Γ on T has a spectral gap.

In the case $n = 1$, we have the following more precise result.

Corollary 5 *Let $= H_3(\mathbf{R})$ be the 3–dimensional Heisenberg group and Λ a lattice in H. Let μ be a probability measure on $\mathrm{Aut}(\Lambda\backslash H)$. Then*

$$\|U^0(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/4},$$

where Γ is the subgroup generated by the support of μ. In particular, if μ is aperiodic, $\|U^0(\mu)\| < 1$ if and only if Γ is non-amenable.

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2 Proofs

We first recall the definition of the Heisenberg group $H = H_{2n+1}(\mathbf{R})$; we then describe the automorphism group of H as well as its irreducible unitary representations.

Let $n \geq 1$ be an integer. Consider the symplectic form β on $\mathbf{R}^{2n}$ given by

$$\beta((x,y),(x',y')) = (x,y)^t J(x',y') \qquad \text{for all} \quad (x,y),(x',y') \in \mathbf{R}^{2n},$$

where J is the $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is the $n \times n$–identity matrix. The $(2n+1)$-dimensional Heisenberg group (over $\mathbf{R}$) is the group $H = H_{2n+1}(\mathbf{R})$ with underlying set $\mathbf{R}^{2n} \times \mathbf{R}$ and product

$$((x,y),s)((x',y'),t) = \left((x+x', y+y'), s+t+\frac{1}{2}\beta((x,y),(x',y'))\right),$$

for $(x,y),(x',y') \in \mathbf{R}^{2n}$, $s,t \in \mathbf{R}$. This is a two-step nilpotent Lie group. Its centre Z coincides with its commutator subgroup and is given by

$$Z = \{((0,0),s) \ : \ s \in \mathbf{R}\}.$$

The symplectic group $Sp_{2n}(\mathbf{R})$, which is the subgroup of $GL_{2n}(\mathbf{R})$ of all matrices g with ${}^t gJg = J$, acts by automorphisms on $H = H_{2n+1}(\mathbf{R})$:

$$g((x,y),t) = (g(x,y),t) \qquad \text{for all} \quad g \in Sp_{2n}(\mathbf{R}),\ (x,y) \in \mathbf{R}^{2n},\ t \in \mathbf{R}.$$

As is well-known (see [Foll89, 1.22]), the automorphism group $\mathrm{Aut}(H)$ of H is generated by:

- the inner automorphisms,
- the automorphisms defined by matrices from $Sp_{2n}(\mathbf{R})$ as above,
- the dilations $((x,y),t) \mapsto ((rx,ry),r^2t)$ for $r > 0$, and
- the inversion $i : ((x,y),t) \mapsto ((y,x),-t)$.

The connected component $\mathrm{Aut}(H)_0$ of the identity in $\mathrm{Aut}(H)$ is a subgroup of index two and can be viewed as the group of $(2n+1)\times(2n+1)$-matrices of the form

$$\begin{pmatrix} rA & 0 \\ a^t & r^{2n} \end{pmatrix}$$

with $A \in Sp_{2n}(\mathbf{R})$, $r > 0$, and a a column vector in $\mathbf{R}^{2n}$ (the action on H corresponding to the usual action on $\mathbf{R}^{2n+1}$). The subgroup of automorphisms of H fixing pointwise the centre can be identified with the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ a^t & 1 \end{pmatrix}$$

and is hence isomorphic to the semi-direct product $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$, for the standard action of $Sp_{2n}(\mathbf{R})$ on $\mathbf{R}^{2n}$.

The unitary dual $\widehat{H}$ of H (that is, the set of classes of irreducible unitary representations of H under unitary equivalence) consists of the equivalence classes of the following representations (see [Foll89, 1.50]):

- the unitary characters of the abelianized group H/Z;
- for every $t \in \mathbf{R} \setminus \{0\}$, the infinite dimensional representation π_t defined on $L^2(\mathbf{R}^n)$ by the formula

$$\pi_t((a,b),s)\xi(x) = \exp(2\pi its)\exp\left(2\pi\langle a, x - \frac{b}{2}\rangle\right)\xi(x-b)$$

  for $((a,b),s) \in H$, $\xi \in L^2(\mathbf{R}^n)$, and $x \in \mathbf{R}^n$.

For $t \neq 0$, the representation π_t is, up to unitary equivalence, the unique irreducible unitary representation of H whose restriction to the centre Z is a multiple of the unitary character $s \mapsto \exp(2\pi its)$.

The group $\mathrm{Aut}(H)$ acts on $\widehat{H}$ by

$$\pi^g(h) = \pi(g^{-1}(h)) \qquad \text{for all} \quad \pi \in \widehat{H},\ g \in \mathrm{Aut}(H),\ h \in H.$$

Let $g \in Sp_{2n}(\mathbf{R})$. For $t \in \mathbf{R} \setminus \{0\}$, the representation π_t^g is unitary equivalent to π_t, since both representations have the same restriction to Z. Therefore, there exists a unitary operator $\sigma(g)$ on $L^2(\mathbf{R}^n)$ such that

$$\sigma(g)\pi_t(g^{-1}(h))\sigma(g)^{-1} = \pi_t(h) \qquad \text{for all} \quad h \in H.$$

By Schur's lemma, $\sigma(g)$ is unique up to a scalar multiple of the identity operator. Hence, for $g_1, g_2 \in Sp_{2n}(\mathbf{R})$, there exists a complex number $c(g_1, g_2)$ of modulus one such that $\sigma(g_1)\sigma(g_2) = c(g_1, g_2)\sigma(g_1 g_2)$. This means that $g \mapsto \sigma(g)$ is a projective unitary representation of $Sp_{2n}(\mathbf{R})$. We extend σ to a projective unitary representation ω_t, called Weil's representation, of $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$ by setting

$$\omega_t(g, a) = \sigma(g)\pi_t(a) \qquad \text{for all} \quad (g, a) \in Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}.$$

Although we will not need this fact, it is worth mentioning that ω_t lifts to an ordinary representation of a two-fold cover of $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$ (see [Foll89, Chapter 4]).

How, let Λ be a lattice in H. (As an example, Λ can be the standard lattice $\{((x, y), s/2) \ : \ x, y \in \mathbf{Z}^n, s \in \mathbf{Z}\}$; a full classification of the lattices in H is given in [Ausl77, I. 2].)

The Lebesgue measure on $\mathbf{R}^{2n} \times \mathbf{R}$ is a Haar measure on H and induces an invariant measure m on the nilmanifold $\Lambda\backslash H$. (For the classification of Γ-invariant measures on $\Lambda\backslash H$ for "large"groups $\Gamma \subset \mathrm{Aut}(\Lambda\backslash H)$, see [Heu09].)

Proof of Theorem 1.

Let Γ be a subgroup of $\mathrm{Aut}(\Lambda\backslash H)$. Then Γ is a discrete subgroup of $\mathrm{Aut}(H)$, for the topology of uniform convergence on compact subsets of H. Moreover, the subgroup of Γ consisting of the automorphisms fixing pointwise the centre of H has finite index in Γ. Indeed, the mapping

$$\mathrm{Aut}(H)_0 \to \mathbf{R}^*, \qquad \begin{pmatrix} rA & 0 \\ a^t & r^{2n} \end{pmatrix} \mapsto r$$

is a homomorphism and the image of $\Gamma \cap \mathrm{Aut}(H)_0$ is a discrete subgroup of $\mathbf{R}^*$.

It is clear that, if Theorem 1 is true for a subgroup of finite index in Γ, then it is true for Γ. So, we can (and will) assume that Γ is a subgroup of $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$.

Since every $\gamma \in \Gamma$ preserves the measure m on $\Lambda\backslash H$, we have an associated unitary representation $U : \gamma \mapsto U_\gamma$ of Γ on $L^2(\Lambda\backslash H, m)$.

Let $\rho_{\Lambda\backslash H}$ be the unitary representation of H on $L^2(\Lambda\backslash H, m)$ given by right translation:

$$\rho_{\Lambda\backslash H}(h)\xi(x) = \xi(xh) \qquad \text{for all} \quad h \in H,\ \xi \in L^2(\Lambda\backslash H, m),\ x \in \Lambda\backslash H.$$

The representations U and $\rho_{\Lambda\backslash H}$ are linked in the following way. For every $\gamma \in \Gamma$, we have:

$$(1) \qquad U_\gamma \rho_{\Lambda\backslash H}(h) U_{\gamma^{-1}} = \rho_{\Lambda\backslash H}(\gamma(h)) \qquad \text{for all} \quad h \in H.$$

We have a decomposition of $L^2(\Lambda\backslash H, m)$ into $\rho_{\Lambda\backslash H}$-invariant subspaces

$$L^2(\Lambda\backslash H, m) = \bigoplus_{m\in\mathbf{Z}} \mathcal{H}_m,$$

where

$$\mathcal{H}_m = \{\xi \in L^2(\Lambda\backslash H) : \ \rho_{\Lambda\backslash H}(0,0,s)\xi = e^{2\pi i m s}\xi \qquad \text{for all} \quad t \in \mathbf{R}\}.$$

The space $\mathcal{H}_0$ coincides with the space $L^2(T)$, where $T \cong \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ is the maximal torus factor associated to $\Lambda\backslash H$. Moreover, for every $m \in \mathbf{Z} \setminus \{0\}$, the subspace $\mathcal{H}_m$ is an isotypical component for $\rho_{\Lambda\backslash H}$ and is equivalent to a finite multiple of the irreducible representation π_m from above. (For a computation of the multiplicities, see [Toli78], [Moor65].)

Let $m \in \mathbf{Z} \setminus \{0\}$. Since Γ fixes pointwise Z, we see from (1) that

$$U_\gamma(\mathcal{H}_m) = \mathcal{H}_m \qquad \text{for all} \quad \gamma \in \Gamma.$$

Denote by $U^{(m)}$ the restriction of U to $\mathcal{H}_m$.

Since $\mathcal{H}_m$ is equivalent to a finite multiple of the irreducible representation π_m, we can assume that $\mathcal{H}_m$ is the tensor product

$$\mathcal{H}_m = \mathcal{K}_m \otimes \mathcal{L}_m$$

of the Hilbert space $\mathcal{K}_m$ of π_m with a finite dimensional Hilbert space $\mathcal{L}_m$, in such a way that

$$(2) \qquad \rho_{\Lambda\backslash H}(h)|_{\mathcal{H}_m} = \pi_m(h) \otimes I_{\mathcal{L}_m} \qquad \text{for all} \quad h \in H.$$

Let $\gamma \in \Gamma$. By (1) and (2) above, we have

$$(3) \qquad U^{(m)}_\gamma \left(\pi_m(h) \otimes I_{\mathcal{L}_m}\right) U^{(m)}_{\gamma^{-1}} = \pi_m(\gamma(h)) \otimes I_{\mathcal{L}_m} \qquad \text{for all} \quad x \in H.$$

Let ω_m be Weil's representation from above. Recall that ω_m is a projective representation of $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$ defined on $\mathcal{K}_m = L^2(\mathbf{R}^n)$ and that ω_m extends π_m. We have

$$(4) \qquad \omega_m(\gamma)\pi_m(h)\omega_m(\gamma)^{-1} = \pi_m(\gamma(h)) \qquad \text{for all} \quad h \in H.$$

It follows from (3) and (4) that, on $\mathcal{H}_m$, the operator $(\omega_m(\gamma)^{-1} \otimes I_{\mathcal{L}_m})\, U_\gamma^{(m)}$ commutes with $(\pi_m(h) \otimes I_{\mathcal{L}_m})$ for all $h \in H$. Hence, since π_m is irreducible, there exists a bounded operator $V_\gamma^{(m)}$ on $\mathcal{L}_m$ such that

$$\big(\omega_m(\gamma)^{-1} \otimes I_{\mathcal{L}_m}\big)\, U_\gamma^{(m)} = I_{\mathcal{K}_m} \otimes V_\gamma^{(m)},$$

that is,

$$U_\gamma^{(m)} = \omega_m(\gamma) \otimes V_\gamma^{(m)}. \tag{5}$$

Since $U^{(m)}$ is a unitary representation, it is clear that $\gamma \mapsto V_\gamma^{(m)}$ is a projective unitary representation of Γ.

Let $\xi, \eta \in \mathcal{S}(\mathbf{R}^n)$ be Schwartz functions on $\mathbf{R}^n$. By [HoMo79, Proposition 6.4], for every $\varepsilon > 0$, the matrix coefficient

$$C_{\xi,\eta}^{\omega_m} : g \mapsto \langle \omega_m(g)\xi, \eta \rangle$$

of the metaplectic representation ω_m belongs to $L^{4n+2+\varepsilon}(Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n})$. Set $G = Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$; observe that Γ is a discrete and hence closed subgroup of G. Choosing a Borel subset $X \subset G$ which is a fundamental domain for the quotient space $\Gamma \backslash G$, we can write (compare with the proof of Proposition 6.4 in [Howe82])

$$\begin{aligned}
\int_G |C_{\xi,\eta}^{\omega_m}(g)|^{4n+2+\varepsilon} dg &= \int_G |\langle \omega_m(g)\xi, \eta \rangle)|^{4n+2+\varepsilon} dg \\
&= \int_X \left(\sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma g)\xi, \eta \rangle|^{4n+2+\varepsilon} \right) dg \\
&< \infty.
\end{aligned}$$

Therefore, by Fubini's theorem, for almost every $g \in X$, we have

$$\sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma)\omega_m(g)\xi, \eta \rangle|^{4n+2+\varepsilon} < \infty,$$

that is, $C_{\omega_m(g)\xi,\eta}^{\omega_m|_\Gamma} \in \ell^{4n+2+\varepsilon}(\Gamma)$.

Since $\mathcal{S}(\mathbf{R}^n)$ contains a countable set which is dense in $L^2(\mathbf{R}^n)$, il follows that there exist dense subspaces $D_1^{(m)}$ and $D_2^{(m)}$ of $L^2(\mathbf{R}^n)$ such that

$$C_{\xi,\eta}^{\omega_m|_\Gamma} \in \ell^{4n+2+\varepsilon}(\Gamma)$$

for all $\xi \in D_1^{(m)}$ and $\eta \in D_2^{(m)}$.

Since $U_\gamma^{(m)} = \omega_m(\gamma) \otimes V_\gamma^{(m)}$ and since matrix coefficients of projective unitary representations are bounded, the matrix coefficients $C^{U^{(m)}}_{\xi\otimes\xi',\eta\otimes\eta'}$ of $U^{(m)}$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for $\xi \in D_1^{(m)}, \eta \in D_2^{(m)}$ and $\xi' \in \mathcal{L}_m, \eta' \in \mathcal{L}_m$.

Let now D_1, D_2 be the linear subspaces of $\mathcal{H}$ generated respectively by

$$\{\xi \otimes \xi' \ : \ \xi \in D_1^{(m)},\ \xi' \in \mathcal{L}_m,\ m \in \mathbf{Z} \setminus \{0\}\}$$

and

$$\{\eta \otimes \eta' \ : \ \eta \in D_2^{(m)},\ \eta' \in \mathcal{L}_m,\ m \in \mathbf{Z} \setminus \{0\}\}.$$

Then D_1 and D_2 are dense in $\mathcal{H}$ and the matrix coefficients $C^U_{v,w}$ belong to $\ell^{4n+2+\varepsilon}(\Gamma)$ for $v \in D_1$ and $w \in D_2$. ■

Proof of Theorem 3

Let μ be a probability measure on $\mathrm{Aut}(\Lambda\backslash H)$. Denote by Γ the subgroup of $\mathrm{Aut}(\Lambda\backslash H)$ generated by the support of μ.

Let λ_Γ be the left regular representation of Γ on $\ell^2(\Gamma)$. Let U^0 and V^0 be the corresponding unitary representations of Γ on $L^2_0(\Lambda\backslash H)$ and $L^2_0(T)$, respectively. We claim that

$$\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/k}\} \tag{6}$$

for $k = 2n + 2$.

Denoting by $U^{\mathcal{H}}$ the restriction of U to $\mathcal{H}$, it suffices to show that

$$\|U^{\mathcal{H}}(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/k}. \tag{7}$$

By Theorem 1, the matrix coefficients $C^{U^{\mathcal{H}}}_{v,w}$ of $U^{\mathcal{H}}$ are in $\ell^{2k}(\Gamma)$ for v and w in dense subspaces D_1 and D_2 of $\mathcal{H}$. It follows that the k-fold tensor power $\left(U^{\mathcal{H}}\right)^{\otimes k}$ of $U^{\mathcal{H}}$ is unitarily equivalent to a subrepresentation of an infinite multiple $\infty\lambda_\Gamma$ of λ_Γ (see [HoTa92, Chapter V, 1.2.4]). Hence,

$$\| \left(U^{\mathcal{H}}\right)^{\otimes k}(\mu)\| \leq \|\infty\lambda_\Gamma(\mu)\| = \|\lambda_\Gamma(\mu)\|$$

and Inequality (7) will be proved if we show that

$$\|U^{\mathcal{H}}(\mu)\| \leq \| \left(U^{\mathcal{H}}\right)^{\otimes k}(\mu)\|^{1/k}. \tag{8}$$

To show Inequality (8), we use the following argument from [Nevo98]. Denote by $\check{\mu}$ the probability measure on Γ defined by $\check{\mu}(\gamma) = \mu(\gamma^{-1})$. For every vector $v \in \mathcal{H}$, using Jensen's inequality, we have

$$\begin{aligned}
\|U(\mu)v\|^{2k} &= |\langle U(\check{\mu} * \mu)v, v\rangle|^k \\
&= \left|\sum_{\gamma\in\Gamma}\langle U(\gamma)v, v\rangle(\check{\mu} * \mu)(\gamma)\right|^k \\
&\le \sum_{\gamma\in\Gamma} |\langle U(\gamma)v, v\rangle|^k(\check{\mu} * \mu)(\gamma) \\
&= \sum_{\gamma\in\Gamma}\langle U(\gamma)^{\otimes k}v^{\otimes k}, v^{\otimes k}\rangle(\check{\mu} * \mu)(\gamma) \\
&= |\langle U^{\otimes k}(\check{\mu} * \mu)v^{\otimes k}, v^{\otimes k}\rangle| \\
&= \|U^{\otimes k}(\mu)v^{\otimes k}\|^2.
\end{aligned}$$

Hence, $\|U^{\mathcal{H}}(\mu)\| \le \| \left(U^{\mathcal{H}}\right)^{\otimes k}(\mu)\|^{1/k}$, as claimed.

Assume that μ is aperiodic and that the Γ action on T has a spectral gap. Then, as mentioned in the Introduction, Γ is not amenable. Hence, $\|\lambda_\Gamma(\mu)\| < 1$ (see [BeHV08, G.4.2]) and therefore $\|U^0(\mu)\| < 1$, that is, the Γ action on $\Lambda\backslash H$ has a spectral gap. ■

Proof of Corollary 5

Let $H = H_3(\mathbf{R})$ be the 3–dimensional Heisenberg group and Λ a lattice in H. The unitary representation V of $\mathrm{Aut}(\Lambda\backslash H)$ on $L^2(T)$ factors through $p : \mathrm{Aut}(\Lambda\backslash H) \to \mathrm{Aut}(T) \cong GL_2(\mathbf{Z})$ to the standard representation of $GL_2(\mathbf{Z})$ on $L^2(T) = L^2(\mathbf{R}^2/\mathbf{Z}^2)$. By Fourier duality, this last representation is unitarily equivalent to the representation of $GL_2(\mathbf{Z})$ on $\ell^2(\mathbf{Z}^2)$ obtained from the dual action of $GL_2(\mathbf{Z})$ on $\mathbf{Z}^2$. We have an orthogonal decomposition into $GL_2(\mathbf{Z})$-invariant subspaces

$$\ell^2(\mathbf{Z}^2) = \mathbf{C}\delta_0 \bigoplus \bigoplus_{t\in T} \ell^2(GL_2(\mathbf{Z})/\Gamma_t),$$

where T is a set of representatives for the $GL_2(\mathbf{Z})$-orbits in $\mathbf{Z}^2 \setminus \{0\}$ and Γ_t is the stabilizer of t in $GL_2(\mathbf{Z})$. Since every Γ_t is solvable (and hence amenable), it follows that V^0 is weakly contained in $\lambda_{GL_2(\mathbf{Z})} \circ p$ (see [BeHV08, Appendix F]). Hence, for every probability measure μ on $\mathrm{Aut}(\Lambda\backslash H)$, we have

$$\|V^0(\mu)\| \le \|(\lambda_{GL_2(\mathbf{Z})} \circ p)(\mu)\|.$$

Since

$$\ker p \subset \left\{ \left(\begin{array}{cc} I_2 & 0 \\ a^t & 1 \end{array} \right) \;:\; a \in \mathbf{R}^2 \right\} \cong \mathbf{R}^2,$$

$\ker p$ is amenable and it follows that

$$\|(\lambda_{GL_2(\mathbf{Z})} \circ p)(\mu)\| \leq \|\lambda_{\mathrm{Aut}(\Lambda\backslash H)}(\mu)\|.$$

Therefore, we have

$$\|V^0(\mu)\| \leq \|\lambda_{\mathrm{Aut}(\Lambda\backslash H)}(\mu)\|.$$

Denote by Γ the subgroup generated by the support of μ. Since

$$\|\lambda_\Gamma(\mu)\| = \|\lambda_{\mathrm{Aut}(\Lambda\backslash H)}(\mu)\|,$$

it follows from Theorem 3 that

$$\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/4}\} = \|\lambda_\Gamma(\mu)\|^{1/4}.$$

Assume that μ is aperiodic. If Γ is non-amenable, then $\|\lambda_\Gamma(\mu)\| < 1$ and hence $\|U^0(\mu)\| < 1$. If Γ is amenable, then Γ has no spectral gap in $\Lambda\backslash H$. ■

3 Some further applications

Let G be a locally compact group acting by measure preserving transformations on a probability space (X, m). Let U denote the associated unitary representation of G on $L^2(X, m)$. The action of G on X is weakly mixing if $L^2_0(X, m)$ contains no non-zero finite dimensional $U(G)$-invariant subspace (equivalently: if the diagonal action of G on $X \times X$ is ergodic; see [BeMa00, Chapter I, 2.17]). The action is strongly mixing if, for all $\xi, \eta \in L^2_0(X, m)$, the matrix coefficient $g \mapsto \langle U_g \xi, \eta \rangle$ belongs to $C_0(G)$.

With the notation of Theorem 1, all matrix coefficients $C^U_{v,w}$ are in $c_0(\Gamma)$ for $v \in D_1$ and $w \in D_2$. By density of D_1 and D_2 in $\mathcal{H}$, the same is true for all $v, w \in \mathcal{H}$. It follows that $\mathcal{H}$ contains no non-zero finite dimensional $U(G)$-invariant subspace if Γ is infinite (see [BeMa00, Chapter I, 2.15.iii]). Therefore, we immediately obtain the following corollary.

Corollary 6 *Let Γ be a group of automorphisms of the compact Heisenberg nilmanilfold $\Lambda\backslash H$ and T the maximal T torus factor associated to $\Lambda\backslash H$. The following properties are equivalent.*

(i) The action of Γ *on* $\Lambda\backslash H$ *is ergodic (weakly mixing or strongly mixing, respectively).*

(ii) The action of Γ *on* T *is ergodic (weakly mixing or strongly mixing, respectively)*

Remark 7 In the case where Γ is generated by a single automorphism (or even an affine transformation) of an arbitrary compact nilmanifold, the previous corollary was obtained by W. Parry (see [Parr69], [Parr70]). The result concerning ergodicity has been generalized by J.-P. Conze ([Conz09]) to arbitrary groups of affine transformations of a general compact nilmanifold $\Lambda\backslash H$. Moreover, [Conz09] gives an example of an ergodic group Γ of automorphisms of the standard 7-dimensional Heisenberg nilmanifold $\Lambda\backslash H_7(\mathbf{R})$ such that no element $\gamma \in \Gamma$ acts ergodically on $\Lambda\backslash H_7(\mathbf{R})$.

Let G be a locally compact group acting by measure preserving transformations on a probability space (X, m) and U the associated unitary representation of G on $L^2(X, m)$. Assume that $\|(U^0 \otimes U^0)(\mu)\| < 1$. This condition, which is formely stronger than the spectral gap condition $\|U^0(\mu)\| < 1$, plays an important role in [FuSh99]. Indeed, it is shown in Theorem 1.4 there that, with the notation as in the Introduction, for every $\xi, \eta \in L^2(X, m)$, the correlation coefficient $\langle U(S_n^\omega)\xi, \eta\rangle$ converge almost surely to $\int_X \xi dm \int_X \eta dm$, with exponentially fast speed.

However, we can see that both conditions are equivalent in our situation.

Corollary 8 *With the notation as in Theorem 3, the following properties are equivalent.*

(i) $\|(U^0 \otimes U^0)(\mu)\| < 1$;

(ii) $\|U^0(\mu)\| < 1$;

(iii) $\|V^0(\mu)\| < 1$.

Indeed, a proof similar to the one of Theorem 3 shows that the following inequality holds:

$$\|(U^0 \otimes U^0)(\mu)\| \leq \max\{\|(V^0 \otimes V^0)(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/k}\}.$$

On the other hand, as was shown in [FuSh99, Theorem 6.4], the condition $\|V^0(\mu)\| < 1$ is equivalent to the condition $\|(V^0 \otimes V^0)(\mu)\| < 1$. This shows the equivalence of Conditions (i), (ii), and (iii).

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