"KAZHDAN'S PROPERTY (T)" ERRATUM

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1. Page 85, correction of Remark 2.3.7.ii

It is not true that a separable locally compact group which has Kazhdan's Property (T) necessarily has Serre's Property (FA) as a discrete group. Indeed, a (discrete) group with Property (FA) cannot be the union of an increasing sequence of proper subgroups (see [Serre–77], Remarque 1 in Chap.I, §6, 6.1). However, there are examples of compact groups (which have therefore Property (T) as topological groups) which can be written as union of a strictly increasing sequence of subgroups. For a concrete example, let $G = \prod_{\mathbf{N}} \mathbf{Z}/2\mathbf{Z}$ be the direct product of countably many copies of the two-element group. Since G can be viewed as a vector space over $\mathbf{Z}/2\mathbf{Z}$, there exists a surjective homomorphism $\varphi : G \to H$, where $H = \bigoplus_{\mathbf{N}} \mathbf{Z}/2\mathbf{Z}$ is the direct sum of countably many copies of $\mathbf{Z}/2\mathbf{Z}$. We can write $H = \bigcup_{n\in\mathbf{N}} H_n$ for an increasing sequence of proper subgroups H_n. So, G is the union of the increasing sequence of proper subgroups $\varphi^{-1}(H_n)$.

The statement in Remark 2.3.7.ii has to be corrected as follows (and this is indeed what Alperin shows in [Alper–82]): let G be a separable locally compact group with Property (T). Then G, viewed as a discrete group, has Property (FA'), that is, whenever G acts on a tree without inversion of edges then every element in G has a fixed vertex.

2. Page 91, correction in the definition of ξ_0

As defined, the function ξ_0 is not in $\mathcal{H}$, because the origin 0 is not fixed by α. A way out is to introduce representatives $(s_i)_{i\in I}$ for the left cosets of H modulo G, so that $G = \bigsqcup_{i\in I} s_i H$, and to define ξ by

$$\xi_0(s_i h) = \alpha\left(h^{-1}\right)\eta_0 \quad \text{for all } i \in I \text{ and } h \in H.$$

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3. Page 92, correction in the proof of Proposition 2.5.9

One should have "Let H be a maximal torsion–free abelian subgroup of G" instead of "Let H be a maximal free abelian subgroup of G".

Indeed, the existence of a maximal torsion–free subgroup is a straightforward consequence of the Zorn Lemma, whereas a discrete abelian group G need not have any maximal free abelian subgroup.

4. Page 94, on the hyperbolic distance

Proof that $d([x],[y])$ is well defined (lines 8-11): in order for the last inequality (line 9) to hold, one has to assume from the beginning that $x_{n+1}y_{n+1} \geq 0$; this can be achieved by replacing x by $-x$, if necessary.

5. Page 210, on the group G^X of Proposition 4.3.10

The topology of uniform convergence does not make sense here, unless X is assumed to be compact.

More precisely, a topological group has *two* canonical uniform structures, the left-invariant one and the right-invariant one (Bourbaki, Topologie générale, chapitre III, § 3). Very often, they are distinct, and they don't make G^X a topological group.

However, if X is compact, which is the case of interest in this part of our book, the compact-open topology on G^X has all desired properties (Bourbaki, Topologie générale, chapitre X, § 3), and this is the topology which has to be considered here.

6. Pages 185 and 215, on elementary matrices over a field

Remark 4.1.2.i and Exercise 4.4.2: read $\nu_n(\mathbf{K}) \leq n^2$ instead of $\nu_n(\mathbf{K}) \leq n(n-1)$.

7. Page 403, on the proof of Proposition F.2.2

It is indeed clear (by Definitions F.1.1 and F.2.1) that, if $(\pi_i)_{i\in I}$ converges to π, then π is weakly contained in $\bigoplus_{j\in J} \pi_j$ for every subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$. Since the proof of the converse is less clear, we give more details.

Assume, by contradiction, that π is weakly contained in $\bigoplus_{j\in J} \pi_j$ for every subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$ and that $(\pi_i)_{i\in I}$ does not converge to π.

On the one hand, there exists a neighbourhood $W = W(\pi, \varphi_1, ..., \varphi_n, \epsilon)$ of π (as in Definition F.2.1) and a subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$ such that no

π_j belongs to W (take $J = \{i \in I : \pi_i \notin W\}$). Upon passing to a subnet of $(\pi_j)_{j\in J}$, we can even assume that $n = 1$, that is, $W = W(\pi, \varphi, \epsilon)$ for a function φ of positive type associated to π.

Now, φ can be approximated, uniformly on compact subsets of G, by sums of functions of positive type associated to *irreducible* unitary representations which are weakly contained in π (see Proposition F.2.7, which is independent of Proposition F.2.2). So, there exist irreducible unitary representations $\sigma_1, \ldots, \sigma_n$ which are weakly contained in π and functions of positive type $\psi_1, \ldots, \psi_n$ associated to $\sigma_1, \ldots, \sigma_n$ such that no π_j belongs to $W(\sigma, \psi, \epsilon/2)$, where $\sigma = \bigoplus_{i=1}^n \sigma_i$ and $\psi = \sum_{i=1}^n \psi_i$.

On the other hand, π and hence every σ_i is weakly contained in $\bigoplus_{j\in J} \pi_j$. Since σ_i is irreducible, it follows that every ψ_i can be approximated, uniformly on compact subsets of G, by functions of positive type associated to the π_j's (and not just sums of such functions; see Proposition F.1.4). Upon passing to a subnet of $(\pi_j)_{j\in J}$, we can approximate $\psi_1, ..., \psi_n$, simultaneously and uniformly on compact subsets of G, by functions of positive type associated to the π_j's. It follows that $\pi_j \in W(\sigma, \psi, \epsilon/2)$ for appropriate j and this is a contradiction.

8. Page 405, correction in the proof of Proposition F.2.7

The proof (lines 10-19) that $\mathrm{ext}(\mathcal{C})$ is contained in $\mathrm{ext}(\mathcal{P}_{\leq 1}(G))$ is incorrect. This has to be corrected as follows:

Let $\varphi \in \mathrm{ext}(\mathcal{C})$; decompose φ as $\varphi = t\varphi_1 + (1-t)\varphi_2$ for $0 < t < 1$ and $\varphi_1, \varphi_2 \in \mathcal{P}_{\leq 1}(G)$. There exists a (cyclic) representation ρ with cyclic vector η such that $\varphi(x) = \langle \rho(x)\eta, \eta\rangle$ for all $x \in G$. As in the proof of Lemma F.1.3, φ_1 can be approximated by positive definite functions associated to ρ of the form

$$\Big\langle \rho(x)\Big(\sum_{i=1}^n \lambda_i \rho(x_i)\eta\Big), \sum_{i=1}^n \lambda_i \rho(x_i)\eta \Big\rangle,$$

that is, by sums of the form $\sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \varphi(x_j^{-1} x x_i)$. Such a sum ψ lies in $\mathcal{C}$, since φ belongs to $\mathcal{C}$ and so every ψ can be approximated by convex combinations of corresponding functions of positive type associated to π. This shows that φ_1 is in $\mathcal{C}$. Similarly, φ_2 is in $\mathcal{C}$. Since φ is an extreme point in $\mathcal{C}$, it follows that $\varphi = \varphi_1 = \varphi_2$, showing that $\mathrm{ext}(\mathcal{C})$ is contained in $\mathrm{ext}(\mathcal{P}_{\leq 1}(G))$.

9. Page 428, on the proof of Theorem G.3.1

Page 428, Line -6: In order to make the proof that (ii) implies (iii) more transparent, one should mention that the following formula holds:

$$\langle f * \varphi, g \rangle = \langle \varphi, \overline{f^*} * g \rangle \text{ for all } f, g \in L^1(G), \varphi \in L^\infty(G).$$

Here $\langle \varphi, f \rangle = \int_G \varphi(x) f(x) dx$ denotes the duality between $L^\infty(G)$ and $L^1(G)$; recall that $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ and observe that $\overline{f^*} \in L^1(G)_{1,+}$ if $f \in L^1(G)_{1,+}$.

10. Some typos

Page 47, Line 11: read "with $g_\alpha(e_i) = \lambda_\alpha e_i$".

Page 204, Line 4 of Lemma 4.3.4: Sign := is missing in $\gamma \in F := \{E_{1,2}(\pm 1), E_{2,1}(\pm 1), E_{1,2}(\pm t), E_{2,1}(\pm t)\}$.

Page 313, Line 8: read "each $X \in \mathcal{K}$ is either a constant or a centred Gaussian random variable" instead of "each $X \in \mathcal{K}$ is a centred Gaussian random variable".

Page 333, Line 3, Definition B.2.3: read "$h_1, h_2 \in H$" instead of "$h_1, h_2 \in G$".

Page 393 (Exercise E.4.1): read $\pi_{it}^{\pm}$ instead of $\pi_t^{\pm}$.

Page 399, Line 3 of Proposition F.1.4: read "a unit vector in $\mathcal{H}$" instead of "a unit vector $\mathcal{H}$".

Page 407, Line 1: read "a linear functional" instead of "a positive linear functional".

Page 410 (Example F.3.6): read π_{it}^{-} instead of π_t^{-}.

Page 455, for [Fell–62] the pages should read 237–268.

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