THE BOHR COMPACTIFICATION OF AN ARITHMETIC GROUP

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ABSTRACT. Given a group Γ, its Bohr compactification $\mathrm{Bohr}(\Gamma)$ and its profinite completion $\mathrm{Prof}(\Gamma)$ are compact groups naturally associated to Γ; moreover, $\mathrm{Prof}(\Gamma)$ can be identified with the quotient of $\mathrm{Bohr}(\Gamma)$ by its connected component $\mathrm{Bohr}(\Gamma)_0$. We study the structure of $\mathrm{Bohr}(\Gamma)$ for an arithmetic subgroup Γ of an algebraic group $\mathbf{G}$ over $\mathbf{Q}$. When $\mathbf{G}$ is unipotent, we show that $\mathrm{Bohr}(\Gamma)$ can be identified with the direct product $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0 \times \mathrm{Prof}(\Gamma)$, where $\Gamma^{\mathrm{Ab}} = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ. In the general case, using a Levi decomposition $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ (where $\mathbf{U}$ is unipotent and $\mathbf{H}$ is reductive), we show that $\mathrm{Bohr}(\Gamma)$ can be described as the semi-direct product of a certain quotient of $\mathrm{Bohr}(\Gamma \cap \mathbf{U})$ with $\mathrm{Bohr}(\Gamma \cap \mathbf{H})$. When $\mathbf{G}$ is simple and has higher $\mathbf{R}$-rank, $\mathrm{Bohr}(\Gamma)$ is isomorphic, up to a finite group, to the product $K \times \mathrm{Prof}(\Gamma)$, where K is the maximal compact factor of $\mathbf{G}(\mathbf{R})$.

1. Introduction

Given a topological group G, the **Bohr compactification** of G is a pair $(\mathrm{Bohr}(G), \beta)$ consisting of a compact (Hausdorff) group $\mathrm{Bohr}(G)$ and a continuous homomorphism $\beta : G \to \mathrm{Bohr}(G)$ with dense image, satisfying the following universal property: for every compact group K and every continuous homomorphism $\alpha : G \to K$, there exists a continuous homomorphism $\alpha' : \mathrm{Bohr}(G) \to K$ such that the diagram

$$\begin{tikzcd} & \mathrm{Bohr}(G) \arrow[d, dashed, "\alpha'"] \\ G \arrow[ur, "\beta"] \arrow[r, "\alpha"] & K \end{tikzcd}$$

commutes. The pair $(\mathrm{Bohr}(G), \beta)$ is unique in the following sense: if (K', β') is a pair consisting of a compact group K' and a continuous homomorphism $\beta' : G \to K'$ with dense image satisfying the same universal property (such a pair will be called a Bohr compactification of

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G), then there exists an isomorphism $\alpha : \mathrm{Bohr}(G) \to K'$ of topological groups such that $\beta' = \alpha \circ \beta$.

The compact group $\mathrm{Bohr}(G)$ was first introduced by A. Weil ([Wei40, Chap.VII]) as a tool for the study of almost periodic functions on G, a subject initiated by H. Bohr ([Boh25a], [Boh25b]) in the case $G = \mathbf{R}$ and generalized to other groups by J. von Neumann ([vN34]) among others. For more on this subject, see [Dix77, §16] or [BH, 4.C]).

The group $\mathrm{Bohr}(\Gamma)$ has been determined for only very few non abelian *discrete* groups Γ (for some general results, see [HK01] and [Hol64]; for the well-known case of abelian groups, see [AK43] and Section 11).

In contrast, there is a second much more studied completion of Γ, namely the **profinite completion** of Γ, which is a pair $(\mathrm{Prof}(\Gamma), \alpha)$ consisting of a profinite group (that is, a projective limit of finite groups) $\mathrm{Prof}(\Gamma)$ satisfying a similar universal property with respect to such groups, together with a homomorphism with $\alpha : \Gamma \to \mathrm{Prof}(\Gamma)$ with dense image. The group $\mathrm{Prof}(\Gamma)$ can be realized as the projective limit $\varprojlim \Gamma/H$, where H runs over the family of the normal subgroups of finite index of Γ. For all this, see [RZ00].

The universal property of $\mathrm{Bohr}(\Gamma)$ gives rise to a continuous epimorphism $\alpha' :\ \mathrm{Bohr}(\Gamma) \to \mathrm{Prof}(\Gamma)$. It is easy to see (see Proposition 7 below) that the kernel of α' is $\mathrm{Bohr}(\Gamma)_0$, the connected component of $\mathrm{Bohr}(\Gamma)$; so, we have a short exact sequence

$$1 \longrightarrow \mathrm{Bohr}(\Gamma)_0 \longrightarrow \mathrm{Bohr}(\Gamma) \longrightarrow \mathrm{Prof}(\Gamma) \longrightarrow 1.$$

In this paper, we will deal with the case where Γ is an arithmetic subgroup in a linear algebraic group. The setting is as follows. Let $\mathbf{G}$ be a connected linear algebraic group over $\mathbf{Q}$ with a fixed faithful representation $\rho : \mathbf{G} \to GL_m$. We consider the subgroup $\mathbf{G}(\mathbf{Z})$ of the group $\mathbf{G}(\mathbf{Q})$ of $\mathbf{Q}$-points of $\mathbf{G}$, that is,

$$\mathbf{G}(\mathbf{Z}) = \rho^{-1}\left(\rho(\mathbf{G}) \cap GL_m(\mathbf{Z})\right).$$

A subgroup Γ of $\mathbf{G}(\mathbf{Q})$ is called an **arithmetic subgroup** if Γ is commensurable to $\mathbf{G}(\mathbf{Z})$, that is, $\Gamma \cap \mathbf{G}(\mathbf{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbf{Z})$. Observe that Γ is a discrete subgroup of the real Lie group $\mathbf{G}(\mathbf{R})$.

We first deal with the case where $\mathbf{G}$ is unipotent. More generally, we describe the Bohr compactification of any finitely generated nilpotent group. Observe that an arithmetic subgroup in a unipotent algebraic $\mathbf{Q}$-group is finitely generated (see Corollary 2 of Theorem 2.10 in [Rag72]).

For two topological groups H and L, we write $H \cong L$ if H and L are topologically isomorphic. We observe that, when Δ is a finitely generated abelian group, $\mathrm{Bohr}(\Delta)$ splits as a direct sum $\mathrm{Bohr}(\Delta) = \mathrm{Bohr}(\Delta)_0 \oplus \mathrm{Prof}(\Delta)$; see Proposition 11.

Theorem 1. *Let Γ be a finitely generated nilpotent group. We have a direct product decomposition*

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0 \times \mathrm{Prof}(\Gamma),$$

where $\Gamma^{\mathrm{Ab}} = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ. This isomorphism is induced by the natural maps $\Gamma \to \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})$ and $\Gamma \to \mathrm{Prof}(\Gamma)$, together with the projection $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}}) \to \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$.

A crucial tool in the proof of Theorem 1 is the fact that elements in the commutator subgroup $[\Gamma, \Gamma]$ of a nilpotent group Γ are distorted (see Proposition 15).

We now turn to the case of a general algebraic group $\mathbf{G}$ over $\mathbf{Q}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{G}$. Then $\mathbf{U}$ is defined over $\mathbf{Q}$ and there exists a connected reductive $\mathbf{Q}$-subgroup $\mathbf{H}$ such that we have a **Levi decomposition** as semi-direct product $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ (see [Mos56]).

The group $\Lambda = \mathbf{H}(\mathbf{Z})$ acts by automorphisms on $\Delta = \mathbf{U}(\mathbf{Z})$ and hence on $\mathrm{Bohr}(\Delta)$, by the universal property of $\mathrm{Bohr}(\Delta)$. In general, this action does not extend to an action of $\mathrm{Bohr}(\Lambda)$ on $\mathrm{Bohr}(\Delta)$. However, as we will see below (proof of Theorem 2), $\mathrm{Bohr}(\Lambda)$ acts naturally by automorphisms on an appropriate quotient of $\mathrm{Bohr}(\Delta)$.

Observe that (see [BHC62, Corollary 4.6]) every arithmetic subgroup of $\mathbf{G}(\mathbf{Q})$ is commensurable to $\Delta(\mathbf{Z}) \rtimes \mathbf{H}(\mathbf{Z})$. Recall that two topological groups G_1 and G_2 are (abstractly) commensurable if there exist finite index subgroups H_1 and H_2 of G_1 and G_2 such that H_1 is topologically isomorphic to H_1. If this is the case, then $\mathrm{Bohr}(G_1)$ and $\mathrm{Bohr}(G_2)$ are commensurable; in fact, each one of the groups $\mathrm{Bohr}(G_1)$ or $\mathrm{Bohr}(G_2)$ can be described in terms of the other (see Propositions 8 and 9). For this reason, we will often deal with only one chosen representative of the commensurability class of an arithmetic group.

Theorem 2. *Let $\mathbf{G}$ be a connected linear algebraic group over $\mathbf{Q}$, with Levi decomposition $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$. Set $\Lambda := \mathbf{H}(\mathbf{Z}), \Delta := \mathbf{U}(\mathbf{Z})$, and $\Gamma := \Delta \rtimes \Lambda$. Let $\widehat{\Delta^{\mathrm{Ab}}}_{\Lambda-\mathrm{fin}}$ be the subgroup of the dual group $\widehat{\Delta^{\mathrm{Ab}}}$ of Δ^{Ab} consisting of the characters with finite Λ-orbit. We have a semi-direct decomposition*

$$\mathrm{Bohr}(\Gamma) \cong (Q \times \mathrm{Prof}(\Delta)) \rtimes \mathrm{Bohr}(\Lambda),$$

where Q is the connected component of $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$ and N is the annihilator of $\widehat{\Delta^{\mathrm{Ab}}}_{\Lambda-\mathrm{fin}}$ in $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})$. This isomorphism is induced by the natural homomophisms $\Delta \to \mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$ and $\Lambda \to \mathrm{Bohr}(\Lambda)$.

Theorems 1 and 2 reduce the determination of $\mathrm{Bohr}(\Gamma)$ for an arithmetic group Γ in $\mathbf{G}$ to the case where $\mathbf{G}$ is reductive. We have a further reduction to the case where $\mathbf{G}$ is simply connected and almost simple. Indeed, recall that a group L is the **almost direct product** of subgroups $L_1, \dots, L_n$ if the product map $L_1 \times \cdots \times L_n \to L$ is a surjective homomorphism with finite kernel.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbf{Q}$. The commutator subgroup $\mathbf{L} := [\mathbf{G}, \mathbf{G}]$ of $\mathbf{G}$ is a connected semi-simple $\mathbf{Q}$-group and $\mathbf{G}$ is an almost direct product $\mathbf{G} = \mathbf{TL}$ for a central $\mathbf{Q}$-torus $\mathbf{T}$ (see (14.2) and (18.2) in [Bor91]) Moreover, $\mathbf{L}$ is an almost direct product $\mathbf{L} = \mathbf{L}_1 \cdots \mathbf{L}_n$ of connected almost $\mathbf{Q}$-simple $\mathbf{Q}$-subgroups $\mathbf{L}_i$, called the almost $\mathbf{Q}$-simple factors of $\mathbf{L}$ (see [Bor91, (22.10)]). For every $i \in \{1, \dots, n\}$, let $\widetilde{\mathbf{L}}_i$ be the simply connected covering group $\mathbf{L}_i$. Set $\widetilde{\mathbf{G}} = \mathbf{T} \times \widetilde{\mathbf{L}_1} \times \cdots \times \widetilde{\mathbf{L}_n}$. Let $\widetilde{\Gamma}$ be the arithmetic subgroup $\mathbf{T}(\mathbf{Z}) \times \widetilde{\mathbf{L}_1}(\mathbf{Z}) \times \cdots \times \widetilde{\mathbf{L}_n}(\mathbf{Z})$ in $\widetilde{\mathbf{G}}(\mathbf{Q})$. The image Γ of $\widetilde{\Gamma}$ under the isogeny $p : \widetilde{\mathbf{G}} \to \mathbf{G}$ is an arithmetic subgroup of $\mathbf{G}(\mathbf{Q})$ (see Corollaries 6.4 and 6.11 in [BHC62]). The map $p : \widetilde{\Gamma} \to \Gamma$ induces an isomorphism $\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\widetilde{\Gamma})/F$, where F is the finite normal subgroup $F = \widetilde{\beta}(\ker p)$ and $\widetilde{\beta} : \widetilde{\Gamma} \to \mathrm{Bohr}(\widetilde{\Gamma})$ is the natural map (see Proposition 10).

As an easy consequence of Margulis' superrigidity results, we give a description of the Bohr compactification of an arithmetic lattice in a simple algebraic $\mathbf{Q}$-group $\mathbf{G}$ under a higher rank assumption. Such a description does not seem possible for arbitrary $\mathbf{G}$. For instance, the free non abelian group F_2 on two generators is an arithmetic lattice in $SL_2(\mathbf{Q})$, but we know of no simple description of $\mathrm{Bohr}(F_2)$.

Theorem 3. *Let $\mathbf{G}$ be a connected, simply connected, and almost simple $\mathbf{Q}$-group. Assume that the real semisimple Lie group $\mathbf{G}(\mathbf{R})$ is not locally isomorphic to any group of the form $SO(m,1) \times K$ or $SU(m,1) \times K$ for a compact Lie group K. Let $\mathbf{G}_{\mathrm{nc}}$ be the product of the almost $\mathbf{R}$-simple factors $\mathbf{G}_i$ of $\mathbf{G}$ for which $\mathbf{G}_i(\mathbf{R})$ is non compact. Let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be an arithmetic subgroup. We have a direct product decomposition*

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\Gamma)_0 \times \mathrm{Prof}(\Gamma)$$

and an isomorphism

$$\mathrm{Bohr}(\Gamma)_0 \cong \mathbf{G}(\mathbf{R})/\mathbf{G}_{\mathrm{nc}}(\mathbf{R}),$$

induced by the natural maps $\Gamma \to \mathbf{G}(\mathbf{R})/\mathbf{G}(\mathbf{R})_{\mathrm{nc}}$ and $\Gamma \to \mathrm{Prof}(\Gamma)$.

A group Γ as in Theorem 3 is an irreducible lattice in the Lie group $G = \mathbf{G}(\mathbf{R})$, that is, the homogeneous space G/Γ carries a G-invariant probability measure; moreover, Γ is cocompact in G if and only if $\mathbf{G}$ is anisotropic over $\mathbf{Q}$ (for all this, see [BHC62, (7.8), (11.6)]). The following corollary is a direct consequence of Theorem 3 and of the fact that a non cocompact arithmetic lattice in a semisimple Lie group has nontrivial unipotent elements (see [Mor15, (5.5.14)]).

Corollary 4. *With the notation as in Theorem 3, assume that* $\mathbf{G}$ *is isotropic over* $\mathbf{Q}$. *For every arithmetic subgroup* Γ *of* $\mathbf{G}(\mathbf{Q})$, *the natural map* $\mathrm{Bohr}(\Gamma) \to \mathrm{Prof}(\Gamma)$ *is an isomorphism.*

As shown in Section 6, it may happen that $\mathrm{Bohr}(\mathbf{G}(\mathbf{Z})) \cong \mathrm{Prof}(\mathbf{G}(\mathbf{Z}))$, even when $\mathbf{G}(\mathbf{Z})$ is cocompact in $\mathbf{G}(\mathbf{R})$..

A general arithmetic lattice Γ has a third completion: the **congruence completion** $\mathrm{Cong}(\Gamma)$ of Γ is the projective limit $\varprojlim \Gamma/H$, where H runs over the family of the congruence subgroups of Γ; recall that a normal subgroup of Γ is a congruence subgroup if it contains the kernel of the map $\mathbf{G}(\mathbf{Z}) \to \mathbf{G}(\mathbf{Z}/N\mathbf{Z})$ of the reduction modulo N, for some integer $N \geq 1$. There is a natural surjective homomorphism $\pi : \mathrm{Prof}(\Gamma) \to \mathrm{Cong}(\Gamma)$. The so-called **congruence subgroup problem** asks whether π is injective and hence an isomorphism of topological groups; more generally, one can ask for a description of the kernel of π. This problem has been extensively studied for arithmetic subgroups (and, more generally, for S-arithmetic subgroups) in various algebraic groups; for instance, it is known that π is an isomorphism when $\Gamma = SL_n(\mathbf{Z})$ for $n \geq 3$ or $\Gamma = Sp_{2n}(\mathbf{Z})$ for $n \geq 2$ (see [BMS67]); moreover, the same conclusion is true when $\Gamma = \mathbf{T}(\mathbf{Z})$ for a torus $\mathbf{T}$ (see [Che51]) and when $\Gamma = \mathbf{U}(\mathbf{Z})$ for a unipotent group $\mathbf{U}$ (see Proposition 16 below). For more on the congruence subgroup problem, see for instance [Rag76] or [PR94, §9.5].

This paper is organized as follows. In Section 2, we establish some general facts about the Bohr compactifications of commensurable groups and the relationship between Bohr compactifications and unitary representations; we also give an explicit description of the Bohr compactification for a finitely generated abelian group. In Section 3, we give the proof of Theorem 1. Section 4 contains the proof of Theorem 2 and Section 5 the proof of Theorem 3. Section 6 is devoted to the explicit computation of the Bohr compactification for various examples of arithmetic groups.

2. Some preliminaries

2.1. **Bohr compactifications and unitary representations.** Given a topological group G, we will consider finite dimensional unitary representations of G, that is, continuous homomorphisms $G \to U(n)$. Two such representations are equivalent if they are conjugate by a unitary matrix. A representation π is irreducible if $\mathbf{C}^n$ and $\{0\}$ there are only $\pi(G)$-invariant subspaces of $\mathbf{C}^n$. We denote by $\mathrm{Rep}_{\mathrm{fd}}(G)$ the set of equivalence classes of finite dimensional unitary representations of G and by $\widehat{G}_{\mathrm{fd}}$ the subset of irreducible ones. Every $\pi \in \mathrm{Rep}_{\mathrm{fd}}(G)$ is a direct sum of representations from $\widehat{G}_{\mathrm{fd}}$

When K is a compact group, every irreducible unitary representation of K is finite dimensional and $\widehat{K}_{\mathrm{fd}} = \widehat{K}$ is the unitary dual space of K. By the Peter-Weyl theorem, $\widehat{K}$ separates the points of K.

Let $\beta : G \to H$ be a a continuous homomorphism of topological groups G and H with dense image; then β induces *injective* maps

$$\widehat{\beta} : \mathrm{Rep}_{\mathrm{fd}}(H) \to \mathrm{Rep}_{\mathrm{fd}}(G) \qquad \text{and} \qquad \widehat{\beta} : \widehat{H}_{\mathrm{fd}} \to \widehat{G}_{\mathrm{fd}},$$

given by $\widehat{\beta}(\pi) = \pi \circ \beta$ for $\pi \in \mathrm{Rep}_{\mathrm{fd}}(H)$. The following proposition, which may be considered as well-known, is a useful tool for identifying the Bohr compactification of a group.

Proposition 5. *Let G be a topological group, K a compact group, and $\beta : G \to K$ a continuous homomorphism with dense image. The following properties are equivalent:*

(i) *(K, β) is a Bohr compactification of G;*
(ii) *the induced map $\widehat{\beta} : \widehat{K} \to \widehat{G}_{\mathrm{fd}}$ is surjective;*
(iii) *the induced map $\widehat{\beta} : \mathrm{Rep}_{\mathrm{fd}}(K) \to \mathrm{Rep}_{\mathrm{fd}}(G)$ is surjective.*

Proof. Assume that (i) holds and let $\pi : G \to U(n)$ be an irreducible representation of G; by the universal property of the Bohr compactification, there exists a continuous homomorphism $\pi' : K \to U(n)$ such that $\pi = \widehat{\beta}(\pi')$ and (ii) follows.

Conversely, assume that (ii) holds. Let L be a compact group and $\alpha : G \to L$ a continuous homomorphism with dense image. Choose a family $\pi_i : L \to U(n_i)$ of representatives of $\widehat{L}$. By the Peter-Weyl theorem, we may identify L with its image in $\prod_i U(n_i)$ under the map $x \mapsto \oplus_i \pi_i(x)$ For every i, we have $\pi_i \circ \alpha \in \widehat{G}_{\mathrm{fd}}$ and hence $\pi_i \circ \alpha = \widehat{\beta}(\pi_i') = \pi_i' \circ \beta$ for some representation $\pi_i' : K \to U(n_i)$ of K. Define a continuous homomorphism

$$\alpha' : K \to \prod_i U(n_i) \qquad x \mapsto \oplus_i \pi_i'(x).$$

We have $\alpha' \circ \beta = \alpha$ and hence

$$\alpha'(K) = \alpha'\left(\overline{\beta(G)}\right) \subset \overline{\alpha(G)} = L.$$

So, (i) and (ii) are equivalent. It is obvious that (ii) is equivalent to (iii). □

The profinite completion $(\mathrm{Prof}(G), \alpha)$ of G may be similarly characterized in terms of certain unitary representations of G. Recall first that $(\mathrm{Prof}(G), \alpha)$ is a pair consisting of a profinite group $\mathrm{Prof}(G)$ and a continuous homomorphism $\alpha : G \to \mathrm{Prof}(G)$ with dense image, satisfying the following universal property: for every profinite group K and every continuous homomorphism $f : G \to K$, there exists a continuous homomorphism $f' : \mathrm{Bohr}(G) \to K$ such that the diagram

$$\begin{tikzcd} & \mathrm{Prof}(G) \arrow[d, dashed, "f'"] \\ G \arrow[ur, "\alpha"] \arrow[r, "f"] & K \end{tikzcd}$$

commutes. Recall that the class of profinite groups coincides with the class of totally disconnected compact groups (see [BH, Proposition 4.C.10]).

Denote by $\mathrm{Rep}_{\mathrm{finite}}(G)$ the set of equivalence classes of finite dimensional unitary representations π of G for which $\pi(G)$ is finite; let $\widehat{G}_{\mathrm{finite}}$ be the subset of irreducible representations from $\mathrm{Rep}_{\mathrm{finite}}(G)$.

If $\alpha : G \to H$ is a continuous homomorphism of topological groups G and H with dense image, then β induces *injective* maps

$$\widehat{\alpha} : \mathrm{Rep}_{\mathrm{finite}}(H) \to \mathrm{Rep}_{\mathrm{finite}}(G) \qquad \text{and} \qquad \widehat{\alpha} : \widehat{H}_{\mathrm{finite}} \to \widehat{G}_{\mathrm{finite}}.$$

Observe that $\widehat{K} = \widehat{K}_{\mathrm{finite}}$ if K is a profinite group. (Conversely, it follows from Peter-Weyl theorem that, if K is a compact group with $\widehat{K} = \widehat{K}_{\mathrm{finite}}$, then K is profinite.) The proof of the following proposition is similar to the proof of Proposition 5 and will be omitted.

Proposition 6. *Let K be a totally disconnected compact group and $\alpha : G \to K$ a continuous homomorphism with dense image. The following properties are equivalent:*

(i) *(K, α) is a profinite completion of G;*
(ii) *the induced map $\widehat{\alpha} : \widehat{K} \to \widehat{G}_{\mathrm{finite}}$ is surjective;*
(ii) *the induced map $\widehat{\beta} : \mathrm{Rep}_{\mathrm{finite}}(K) \to \mathrm{Rep}_{\mathrm{finite}}(G)$ is surjective.*

The universal property of $\mathrm{Bohr}(G)$ implies that there is a continuous epimorphism $\alpha' : \mathrm{Bohr}(G) \to \mathrm{Prof}(G)$ such that the diagram

$$\begin{tikzcd} & \mathrm{Bohr}(G) \arrow[d, dashed, "\alpha'"] \\ G \arrow[ur, "\beta"] \arrow[r, "\alpha"] & \mathrm{Prof}(G) \end{tikzcd}$$

commutes. We record the following elementary but basic fact mentioned in the introduction.

Proposition 7. *The kernel of* $\alpha' : \mathrm{Bohr}(G) \to \mathrm{Prof}(G)$ *coincides with the connected component* $\mathrm{Bohr}(G)_0$ *of* $\mathrm{Bohr}(G)$.

Proof. Since $\mathrm{Bohr}(G)_0$ is connected and $\mathrm{Prof}(G)$ is totally disconnected, $\mathrm{Bohr}(G)_0$ is contained in $\mathrm{Ker}\alpha'$. So, α' factorizes to a continuous epimorphism $\alpha'' : K \to \mathrm{Prof}(G)$, where $K := \mathrm{Bohr}(G)/\,\mathrm{Bohr}(G)_0$ and we have a commutative diagram

$$\begin{tikzcd} & K \arrow[d, "\alpha''"] \\ G \arrow[ur, "p\circ\beta"] \arrow[r, "\alpha"] & \mathrm{Prof}(G). \end{tikzcd}$$

where $p : \mathrm{Bohr}(G) \to K$ is the canonical epimorphism. Since K is a totally disconnected compact group, there exists a continuous epimorphism $f : \mathrm{Prof}(G) \to K$ and we have a commutative diagram

$$\begin{tikzcd} & K \\ G \arrow[ur, "p\circ\beta"] \arrow[r, "\alpha"] & \mathrm{Prof}(G). \arrow[u, "f"] \end{tikzcd}$$

For every $g \in G$, we have

$$f(\alpha''(p \circ \beta(g))) = f(\alpha(g)) = p \circ \beta(g);$$

since $p \circ \beta(G)$ is dense in K, it follows that $f \circ \alpha''$ is the identity on K. This implies that α'' is injective and hence an isomorphism. $\square$

2.2. **Bohr compactifications of commensurable groups.** Let G be a topological group and H be a closed subgroup of finite index in G. We first determine $\mathrm{Bohr}(H)$ in terms of $\mathrm{Bohr}(G)$.

Proposition 8. *Let* $(\mathrm{Bohr}(G), \beta)$ *be the Bohr compactification of* G. *Set* $K := \overline{\beta(H)}$.

(i) K *is a subgroup of finite index of* $\mathrm{Bohr}(G)$.
(ii) $(K, \beta|_H)$ *is a Bohr compactification of* H.

(iii) *K and* $\mathrm{Bohr}(G)$ *have the same connected component of the identity.*

Proof. Item (i) is obvious and Item (iii) follows from Item (i). To show Item (ii), let π be a unitary representation of H on $\mathbf{C}^n$. Since H has finite index in H, the induced representation $\rho := \mathrm{Ind}_H^G \pi$, which is a unitary representation of G, is finite dimensional. Hence, there exists $\rho' \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{Bohr}(G))$ such that $\rho = \rho' \circ \beta$. Now, π is equivalent to a subrepresentation of the restriction of ρ to H (see [BH, 1.F]); so, we may identify π with the representation of H defined by a $\rho(H)$-invariant subspace W of the space of ρ. Then W is $\rho'(K)$-invariant and defines therefore a representation π' of K. We have $\pi = \pi' \circ (\beta|_H)$ and Proposition 5 shows that Item (ii) holds. □

Next, we want to determine $\mathrm{Bohr}(G)$ in terms of $\mathrm{Bohr}(H)$.

Given a compact group K and a finite set X, we define another compact group, we call the **induced group** of (K, X), as

$$\mathrm{Ind}(K, X) := K^X \rtimes \mathrm{Sym}(X),$$

where the group $\mathrm{Sym}(X)$ of bijections of X acts by permutations of indices on K^X :

$$\sigma((g_x)_{x\in X}) = (g_{\sigma^{-1}(x)})_{x\in X} \qquad \text{for all} \quad \sigma \in \mathrm{Sym}(X), (g_x)_{x\in X} \in K^X$$

Observe that, if $\pi : K \to U(n)$ is a representation of K on $V = \mathbf{C}^n$, then a unitary representation $\mathrm{Ind}(\pi)$ of $\mathrm{Ind}(K, X)$ on on V^X is defined by

$$\mathrm{Ind}(\pi)((g_x)_{x\in X}, \sigma)(v_x)_{x\in X} = (\pi(g_x)v_{\sigma^{-1}(x)})_{x\in X},$$

for $((g_x)_{x\in X}, \sigma) \in \mathrm{Ind}(K, X)$ and $(v_x)_{x\in X} \in V^X$.

Coming back to our setting, where H is a closed subgroup of finite index in G, we fix a transversal X for the right cosets of H; so, we have a disjoint union $G = \bigsqcup_{x\in X} Hx$. For every $g \in G$ and $x \in X$, let $x \cdot g$ and $c(x, g))$ be the unique elements in X and H such that $xg = c(x, g)(x \cdot g)$. Observe that

$$X \times G \to X, \qquad (x, g) \mapsto x \cdot g$$

is an action of G on X (on the right), which is equivalent to the natural action of G on $H\backslash G$ given by right multiplication. In particular, for every $g \in G$, the map $\sigma(g) : x \mapsto x \cdot g^{-1}$ belongs to $\mathrm{Sym}(X)$ and we have a homomorphism

$$G \mapsto \mathrm{Sym}(X), \ \ g \mapsto \sigma(g).$$

Proposition 9. *Let* $(\mathrm{Bohr}(H), \beta)$ *be the Bohr compactification of* H. *Let* $\mathrm{Ind}(\mathrm{Bohr}(H), X)$ *be the compact group defined as above. Consider the map* $\widetilde{\beta} : G \to \mathrm{Ind}(\mathrm{Bohr}(H), X)$ *defined by*

$$\widetilde{\beta}(g) = ((\beta(c(x,g)))_{x\in X}, \sigma(g)) \qquad \textit{for all} \quad g \in G.$$

The closure of $\widetilde{\beta}(G)$ *in* $\mathrm{Ind}(\mathrm{Bohr}(H), X)$, *together with the map* $\widetilde{\beta}$, *is a Bohr compactification of* G.

Proof. It is readily checked that $\widetilde{\beta} : G \to \mathrm{Ind}(\mathrm{Bohr}(H), X)$ is a continuous homomorphism. Let $\rho : G \to U(n)$ be a finite dimensional unitary representation of G. Set $\pi := \rho|_H \in \mathrm{Rep}_{\mathrm{fd}}(H)$. There exists $\pi' \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{Bohr}(H))$ such that $\pi = \pi' \circ \beta$. Let $\widetilde{\pi} := \mathrm{Ind}_H^G \pi$. As is well-known (see [BH, 1.F]), $\widetilde{\pi}$ can be realized on V^X for $V := \mathbf{C}^n$ by the formula

$$\widetilde{\pi}(g)(v_x)_{x\in X}) = (\pi(c(x,g))v_{x\cdot g})_{x\in X} = (\pi(c(x,g))v_{\sigma(g^{-1})x})_{x\in X},$$

for all $g \in G$ and $(v_x)_{x\in X} \in V^X$. With the unitary representation $\mathrm{Ind}(\pi')$ of $\mathrm{Ind}(\mathrm{Bohr}(H), X)$ defined as above, we have therefore

$$(*) \qquad \widetilde{\pi}(g) = \mathrm{Ind}(\pi')(\widetilde{\beta}(g)) \qquad \text{for all} \quad g \in G,$$

that is, $\widetilde{\pi} = \mathrm{Ind}(\pi') \circ \widetilde{\beta}$. Now,

$$\widetilde{\pi} = \mathrm{Ind}_H^G \pi = \mathrm{Ind}_H^G(\rho|_H)$$

is equivalent to the tensor product representation $\rho \otimes \lambda_{G/H}$, where $\lambda_{G/H}$ is the regular representation of G/H (see [BHV08, E.2.5]). Since $\lambda_{G/H}$ contains the trivial representation of G, it follows that ρ is equivalent to a subrepresentation of $\widetilde{\pi}$; so, we can identify ρ with the representation of G defined by a $\widetilde{\pi}(G)$-invariant subspace W of V^X. Denoting by L the closure of $\widetilde{\beta}(G)$, it follows from $(*)$ that W is invariant under $\mathrm{Ind}(\pi')(L)$ and so defines a representation ρ' of L. Then $\rho = \rho' \circ \widetilde{\beta}$ and the claim follows from Proposition 5. ☐

We will also need the following well-known (see [HK01, Lemma 2.2]) description of the Bohr compactification of a quotient of G in terms of the Bohr compactification of G.

Proposition 10. *Let* $(\mathrm{Bohr}(G), \beta)$ *be the Bohr compactification of the topological group* G *and let* N *be a closed normal subgroup of* G. *Let* K_N *be the closure of* $\beta(N)$ *in* $\mathrm{Bohr}(G)$

(i) K_N *is a normal subgroup of* $\mathrm{Bohr}(G)$ *and* β *induces a continuous homomorphism* $\overline{\alpha} : G/N \to \mathrm{Bohr}(G)/K_N$

(ii) $(\mathrm{Bohr}(G)/K_N, \overline{\alpha})$ *is a Bohr compactification of* G/N.

Proof. Let $(\mathrm{Bohr}(G/N), \overline{\beta})$ be the Bohr compactification of G/N. The canonical homomorphism $\alpha : G \to G/N$ induces a continuous homomorphism $\alpha' : \mathrm{Bohr}(G) \to \mathrm{Bohr}(G/N)$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathrm{Bohr}(G) \arrow[d, dashed, "\alpha'"] \\ G/N \arrow[r, "\overline{\beta}"] & \mathrm{Bohr}(G/N) \end{tikzcd}$$

commutes. It follows that $\beta(N)$ and hence K_N is contained in $\mathrm{Ker}\,\alpha'$. So, we have induced homomorphisms $\overline{\alpha} : G/N \to \mathrm{Bohr}(G)/K_N$ and $\overline{\alpha'} : \mathrm{Bohr}(G)/K_N \to \mathrm{Bohr}(G/N)$, giving rise to a commutative diagram

$$\begin{tikzcd} & \mathrm{Bohr}(G)/K_N \arrow[d, dashed, "\overline{\alpha'}"] \\ G/N \arrow[ur, dashed, "\overline{\alpha}"] \arrow[r, "\overline{\beta}"] & \mathrm{Bohr}(G/N). \end{tikzcd}$$

It follows that $(\mathrm{Bohr}(G)/K_N, \overline{\alpha})$ has the same universal property for G/N as $(\mathrm{Bohr}(G/N), \overline{\beta})$. Since $\overline{\alpha}$ has dense image, $(\mathrm{Bohr}(G)/K_N, \overline{\alpha})$ is therefore a Bohr compactification of G/N. □

2.3. **Bohr compactification of finitely generated abelian groups.** Let G be a locally compact abelian group. Its dual group $\widehat{G}$ consists of the continuous homomorphism from G to the circle group $\mathbf{S}^1$; equipped with the topology of uniform convergence on compact subsets, $\widehat{G}$ is again a locally compact abelian group. Let $\widehat{G}_{\mathrm{disc}}$ be the group $\widehat{G}$ equipped with the discrete topology. It is well-known (see e.g. [BH, Proposition 4.C.4]) that the Bohr compactification of G coincides with the dual group K of $\widehat{G}_{\mathrm{disc}}$, together with the embedding $i : G \to K$ given by $i(g)(\chi) = \chi(g)$ for all $g \in G$ and $\chi \in \widehat{G}$. Notice that this implies that, by Pontrjagin duality, the dual group of $\mathrm{Bohr}(G)$ coincides with $\widehat{G}_{\mathrm{disc}}$.

A more precise information on the structure of the Bohr compactification is available in the case of a (discrete) finitely generated abelian group. As is well-known, such a group Γ splits a direct sum $\Gamma = F \oplus A$ of a finite group F (which is its torsion subgroup) and a free abelian group A of finite rank $k \geq 0$, called the rank of Γ. Recall that $\mathbf{Z}_p$ denotes the ring of p-adic integers for a prime p and $\mathbf{A}$ the ring of adèles over $\mathbf{Q}$.

Proposition 11. *Let Γ be a finitely generated abelian group of rank k.*

(i) *We have a direct sum decomposition*

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\Gamma)_0 \oplus \mathrm{Prof}(\Gamma).$$

(ii) *We have*

$$\mathrm{Prof}(\Gamma) \cong F \oplus \prod_{p \text{ prime}} \mathbf{Z}_p^k,$$

where F is a finite group.

(iii) *We have*

$$\mathrm{Bohr}(\Gamma)_0 \cong \prod_{\omega \in \mathfrak{c}} \mathbf{A}^k/\mathbf{Q}^k,$$

a product of uncountably many copies of the adelic solenoid $\mathbf{A}^k/\mathbf{Q}^k$.

Proof. We have $\Gamma \cong F \oplus \mathbf{Z}^k$ for a finite group F and $\mathrm{Bohr}(\mathbf{Z}^k) = \mathrm{Bohr}(\mathbf{Z})^k$. So, it suffices to determine $\mathrm{Bohr}(\mathbf{Z})$. As mentioned above, $\mathrm{Bohr}(\mathbf{Z})$ can be identified with the dual group of the circle $\mathbf{S}^1$ viewed as discrete group. Choose a linear basis $\{1\} \cup \{x_\omega \mid \omega \in \mathfrak{c}\}$ of $\mathbf{R}$ over $\mathbf{Q}$. Then $\mathbf{S}^1 \cong \mathbf{R}/\mathbf{Z}$ is isomorphic to the abelian group

$$(\mathbf{Q}/\mathbf{Z}) \oplus \oplus_{\omega \in \mathfrak{c}} \mathbf{Q}.$$

Hence,

$$\mathrm{Bohr}(\mathbf{Z}) \cong \widehat{\mathbf{Q}/\mathbf{Z}} \oplus \prod_{\omega \in \mathfrak{c}} \widehat{\mathbf{Q}}.$$

Now,

$$\mathbf{Q}/\mathbf{Z} = \oplus_{p \text{ prime}} Z(p^\infty),$$

with $Z(p^\infty) = \varinjlim_k \mathbf{Z}/p^k\mathbf{Z}$ the p-primary component of $\mathbf{Q}/\mathbf{Z}$. Hence,

$$\widehat{Z(p^\infty)} \cong \varprojlim_k \mathbf{Z}/p^k\mathbf{Z} = \mathbf{Z}_p.$$

On the other hand, $\widehat{\mathbf{Q}}$ can be identified with the solenoid $\mathbf{A}/\mathbf{Q}$ (see e.g. [HR79, (25.4)]). It follows that

$$\mathrm{Bohr}(\Gamma) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \oplus \prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q}.$$

$\square$

2.4. **Restrictions of representations to normal subgroups.** Let Γ be a group and N a normal subgroup of Γ. Recall that Γ acts on $\widehat{N}_{\rm fd}$: for $\sigma \in \widehat{N}_{\rm fd}$ and $\gamma \in \Gamma$, the conjugate representation $\sigma^\gamma \in \widehat{N}_{\rm fd}$ is defined by

$$\sigma^\gamma(n) \,=\, \sigma(\gamma^{-1}n\gamma), \quad \text{ for all } \ n \in N.$$

The stabilizer Γ_σ of σ is the subgroup consisting of all $\gamma \in \Gamma$ for which σ^γ is equivalent σ; observe that Γ_σ contains N.

Given a unitary representation ρ of N on a finite dimensional vector space V and $\sigma \in \widehat{N}_{\rm fd}$, we denote by V^σ the σ-*isotypical component* of ρ, that is, the sum of all ρ-invariant subspaces W for which the restriction of ρ to W is equivalent to σ. Observe that V decomposes as direct sum $V = \oplus_{\sigma\in\Sigma_\rho} V^\sigma$, where Σ_ρ is the finite set of $\sigma \in \widehat{N}_{\rm fd}$ with $V^\sigma \neq \{0\}$.

Proposition 12. *Let π be an irreducible unitary representation of Γ on a finite dimensional vector space V. Let $V = \oplus_{\sigma\in\Sigma_{\pi|_N}} V^\sigma$ be the decomposition of the restriction $\pi|_N$ of π to N into isotypical components. Then $\Sigma_{\pi|_N}$ coincides with a Γ-orbit: there exists $\sigma \in \widehat{N}_{\rm fd}$ such that $\Sigma_{\pi|_N} = \{\sigma^\gamma \ : \ \gamma \in \Gamma\}$; in particular, Γ_σ has finite index in Γ.*

Proof. Let $\sigma \in \Sigma_{\pi|_N}$ and fix a transversal T for the left cosets of Γ_σ with $e \in T$. Then $V^{\sigma^t} = \pi(t)V^\sigma$ for all $t \in T$ Since π is irreducible and $\sum_{t\in T}\pi(t)V^\sigma$ is $\pi(\Gamma)$-invariant, it follows that $\Sigma_{\pi|_N}$ is a Γ-orbit. □

3. Proof of Theorem 1

3.1. **Distortion and Bohr compactification.** Let Γ be a finitely generated group with a finite set S of generators. For $\gamma \in \Gamma$, denote by $\ell_S(\gamma)$ the word length of γ with respect to $S \cup S^{-1}$ and set

$$t(\gamma) = \liminf_{n\to\infty} \frac{\ell_S(\gamma^n)}{n}.$$

The number $t(\gamma)$ is called the *translation number* of γ in [GS91]

Definition 13. An element $\gamma \in \Gamma$ is said to be **distorted** if $t(\gamma) = 0$.

In fact, since the sequence $n \mapsto \ell_S(\gamma^n)$ is subadditive, we have, by Fekete's lemma,

$$t(\gamma) = \lim_{n\to\infty} \frac{\ell_S(\gamma^n)}{n} = \inf\left\{\frac{\ell_S(\gamma^n)}{n} : n \in \mathbf{N}^*\right\}$$

The property of being distorted is independent of the choice of the set of generators. Distorted elements are called *algebraically parabolic* in [BGS85, (7.5), p.90], but we prefer to use the terminology from [FH06]. The relevance of distorsion to the Bohr compactification lies in the following proposition; for a related result with a similar proof, see [LMR00, (2.4)].

Proposition 14. *Let Γ be a finitely generated group and $\gamma \in \Gamma$ a distorted element. Then, for every finite dimensional unitary representation $\pi : \Gamma \to U(N)$ of Γ, the matrix $\pi(\gamma) \in U(N)$ has finite order.*

Proof. It suffices to show that all eigenvalues of the unitary matrix $\pi(\gamma)$ are roots of unity. Assume, by contradiction, that π has an eigenvalue $\lambda \in \mathbf{S}^1$ of infinite order.

Let S be a finite set of generators of Γ with $S = S^{-1}$. The group $\pi(\Gamma)$ is generated by the set $\{\pi(s) \mid s \in S\}$. Hence, $\pi(G)$ is contained in $GL_N(L)$, where L is the subfield of $\mathbf{C}$ generated by the matrix coefficients of the $\pi(s)$'s. It follows that λ is contained in a finitely generated extension ℓ of L. By a lemma of Tits ([Tit72, Lemma 4.1]), there exists a locally compact field k endowed with an absolute value $|\cdot|$ and a field embedding $\sigma : \ell \to k$ such that $|\sigma(\lambda)| \neq 1$. Upon replacing γ by γ^{-1}, we may assume that $|\sigma(\lambda)| > 1$.

Define a function ("norm") $\xi \mapsto \|\xi\|$ on k^N by

$$\|\xi\| = \max\{|\xi_1|, \dots, |\xi_N|\} \qquad \text{for all} \quad \xi = (\xi_1, \dots, \xi_N) \in k^N.$$

For a matrix $A \in GL_N(k)$, set $\|A\| = \sup_{\xi \neq 0} \|A\xi\|/\|\xi\|$. It is obvious that $\|A\xi\| \leq \|A\|\|\xi\|$ for all $\xi \in k^N$ and hence

$$(*) \qquad \|AB\| \leq \|A\|\|B\| \qquad \text{for all} \quad A, B \in GL_N(k).$$

In particular, we have $\|A^n\| \leq \|A\|^n$ for all $A \in GL_N(k)$ and $n \in \mathbf{N}$.

For a matrix $w \in GL_n(\ell)$, denote by $\sigma(w)$ the matrix in $GL_n(k)$ obtained by applying σ to the entries of w. Set $A_s = \sigma(\pi(s))$ for $s \in S$ and $A := \sigma(\pi(\gamma))$. With

$$C := \max\{\|A_s\| : s \in S\},$$

it is clear that Inequality $(*)$ implies that

$$(**) \qquad \|A^n\| = \|\sigma(\pi(\gamma^n))\| \leq C^{\ell_S(\gamma^n)} \qquad \text{for all} \quad n \in \mathbf{N}.$$

On the other hand, $\sigma(\lambda)$ is an eigenvalue of A; so, there exists $\xi \in k^N \setminus \{0\}$ such that $A\xi = \sigma(\lambda)\xi$ and hence $A^n\xi = \sigma(\lambda)^n\xi$ for all $n \in \mathbf{N}$. So, for every $n \in \mathbf{N}$, we have

$$\|A^n\xi\| = |\sigma(\lambda)|^n\|\xi\|$$

and this implies that

$$\|A^n\| \geq |\sigma(\lambda)|^n.$$

In view of $(**)$, we obtain therefore

$$\frac{\ell_S(\gamma^n)\log C}{n} \geq \log|\sigma(\lambda)| \qquad \text{for all} \quad n \in \mathbf{N}.$$

Since $|\sigma(\lambda)| > 1$, this contradicts the fact that $\liminf_{n\to\infty} \dfrac{\ell_S(\gamma^n)}{n} = 0$. $\square$

3.2. **Distorted elements in nilpotent groups.** Let Γ be a finitely generated nilpotent subgroup. For subsets A, B in Γ, we let $[A, B]$ denote the subgroup of Γ generated by all commutators $[a, b] = aba^{-1}b^{-1}$, for $a \in A$ and $b \in B$. Let

$$\Gamma^{(0)} \supset \Gamma^{(1)} \supset \cdots \supset \Gamma^{(d-1)} \supset \Gamma^{(d)} = \{e\}$$

be the lower central series of Γ, defined inductively by $\Gamma^{(0)} = \Gamma$ and $\Gamma^{(k+1)} = [\Gamma^{(k)}, \Gamma]$. The step of nilpotency of Γ is the smallest $d \geq 1$ such that $\Gamma^{(d-1)} \neq \{e\}$ and $\Gamma^{(d)} = \{e\}$.

Proposition 15. *Let Γ be a finitely generated nilpotent subgroup. Every $\gamma \in \Gamma^{(1)} = [\Gamma, \Gamma]$ is distorted.*

Proof. Let S be a finite set of generators of Γ with $S = S^{-1}$. Let $d \geq 1$ be the step of nilpotency of Γ. The case $d = 1$ being trivial, we will assume that $d \geq 2$. We will show by induction on $i \in \{1, \ldots, d-1\}$ that every $\gamma \in \Gamma^{(d-i)}$ is distorded.

- *First step.* Assume that $i = 1$. It is well-known that every element γ in $\Gamma^{(d-1)}$ is distorted (see for instance [BGS85, (7.6), p. 91]); in fact, more precise estimates are available: for every $\gamma \in \Gamma^{(d-1)}$, we have $\ell_S(\gamma^n) = O(n^{1/d})$ as $n \to \infty$ (see [Tit81, 2.3 Lemme] or [DK18, Lemma 14.15]).

- *Second step.* Assume that, for every finitely generated nilpotent subgroup Λ of step $d' \geq 2$, every element $\delta \in \Lambda^{(d'-i)}$ is distorded for $i \in \{1, \ldots, d'-2\}$. Let $\gamma \in \Gamma^{(d-i-1)}$ and fix $\varepsilon > 0$.

The quotient group $\overline{\Gamma} = \Gamma/\Gamma^{(d-1)}$ is nilpotent of step $d' = d-1$ and $p(\gamma) \in \overline{\Gamma}^{(d'-i)}$, where $p : \Gamma \to \overline{\Gamma}$ is the quotient map. By induction hypothesis, $p(\gamma)$ is distorted in $\overline{\Gamma}$ with respect to the generating set $\overline{S} := p(S)$. So, we have $\lim_{n\to\infty} \dfrac{\ell_{\overline{S}}(p(\gamma)^n)}{n} = 0$; hence, we can find an integer $N \geq 1$ such that

$$(*) \qquad \forall n \geq N, \exists \delta_n \in \Gamma^{(d-1)} \ : \ \frac{\ell_S(\gamma^n \delta_n)}{n} \leq \varepsilon.$$

By the first step, we have $\lim_{k\to\infty} \dfrac{\ell_S(\delta_N^k)}{k} = 0$, since $\delta_N \in \Gamma^{(d-1)}$; so, there exists $K \geq 1$ such that

$$(**) \qquad \forall k \geq K \ : \ \frac{\ell_S(\delta_N^k)}{k} \leq \varepsilon.$$

Let $k \geq K$. We have

$$(***)\qquad \frac{\ell_S(\gamma^{Nk})}{Nk} = \frac{\ell_S((\gamma^{Nk}\delta_N^k)(\delta_N^{-1})^k)}{Nk} \leq \frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} + \frac{\ell_S(\delta_N^k)}{Nk}.$$

Now, since $\Gamma^{(d-1)}$ is contained in the center of Γ, the elements δ_N and γ_N commute and hence, by $(*)$, we have

$$\frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} = \frac{\ell_S((\gamma^N\delta_N)^k)}{Nk} \leq k\frac{\ell_S(\gamma^N\delta_N)}{Nk} = \frac{\ell_S(\gamma^N\delta_N)}{N} \leq \varepsilon.$$

So, together with $(***)$ and $(**)$, we obtain

$$\forall k \geq K \ : \ \frac{\ell_S(\gamma^{Nk})}{Nk} \leq 2\varepsilon.$$

This shows that $t(\gamma) = 0$. $\square$

3.3. **Congruence subgroups in unipotent groups.** The following result, which shows that the congruence subgroup problem has a positive solution for unipotent groups, is well-known (see the sketch in [Rag76, p.108]); for the convenience of the reader, we reproduce its short proof.

Proposition 16. *Let* $\mathbf{U}$ *be a unipotent algebraic group over* $\mathbf{Q}$. *Let* Γ *be an arithmetic subgroup of* $\mathbf{U}(\mathbf{Q})$. *Then every finite index subgroup of* Γ *is a congruence subgroup.*

Proof. We can find a sequence

$$\mathbf{U} = \mathbf{U}_0 \supset \mathbf{U}_1 \supset \cdots \supset \mathbf{U}_{d-1} \supset \mathbf{U}_d = \{e\}$$

of normal $\mathbf{Q}$-subgroups of $\mathbf{U}$ such that the factor groups $\mathbf{U}_i/\mathbf{U}_{i+1}$ are $\mathbf{Q}$-isomorphic to $\mathbf{G}_a$, the additive group of dimension 1 (see [Bor63, (15.5)]).

We proceed by induction on $d \geq 1$. If $d = 1$, then Γ is commensurable to $\mathbf{Z}$ and the claim is obvious true. Assume that $d \geq 2$. Then $\mathbf{U}$ can be written as semi-direct product $\mathbf{U} = \mathbf{U}_1 \rtimes \mathbf{G}_a$. By [BHC62, Corollary 4.6], Γ is commensurable to $\mathbf{U}_1(\mathbf{Z}) \rtimes \mathbf{Z}$. Let H a subgroup of finite index in Γ. Then $H \cap \mathbf{U}_1(\mathbf{Z})$ has finite index in $\mathbf{U}_1(\mathbf{Z})$ and hence, by induction hypothesis, contains the kernel of the reduction $\mathbf{U}_1(\mathbf{Z}) \to \mathbf{U}_1(\mathbf{Z}/N_1\mathbf{Z})$ modulo some $N_1 \geq 1$. Moreover, $H \cap \mathbf{Z} = N_2\mathbf{Z}$ for some $N_2 \geq 1$. Hence, H contains the kernel of the reduction $\mathbf{U}(\mathbf{Z}) \to \mathbf{U}(\mathbf{Z}/N_1N_2\mathbf{Z})$ modulo N_1N_2. $\square$

3.4. **Proof of Theorem 1.** Let Γ be a finitely generated nilpotent group and $\alpha : \Gamma \to \mathrm{Prof}(\Gamma)$ the canonical homomorphism. Recall (see Proposition 11) that the Bohr compactification of $\Gamma^{\mathrm{Ab}} = \Gamma/[\Gamma,\Gamma]$ splits as a direct sum

$$\mathrm{Bohr}(\Gamma^{\mathrm{Ab}}) = \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0 \oplus B_1,$$

for a closed subgroup $B_1 \cong \mathrm{Prof}(\Gamma^{\mathrm{Ab}})$. Let $p : \mathrm{Bohr}(\Gamma^{\mathrm{Ab}}) \to \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$ be the corresponding projection. Denote by $\beta_0 : \Gamma \to \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})$ the map induced by the quotient homomorphism $\Gamma \to \Gamma^{\mathrm{Ab}}$. Set

$$K := \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0 \times \mathrm{Prof}(\Gamma),$$

and let $\beta : \Gamma \to K$ be the homomorphism $\gamma \mapsto (p \circ \beta_0(\gamma), \alpha(\gamma))$. We claim that (K, β) is a Bohr compactification for Γ.

• *First step.* We claim that $\beta(\Gamma)$ is dense in K. Indeed, let L be the closure of $\beta(\Gamma)$ in K and L_0 its connected component. Since $\mathrm{Prof}(\Gamma)$ is totally disconnected, the projection of L_0 on $\mathrm{Prof}(\Gamma)$ is trivial; hence $L_0 = K_0 \times \{1\}$ for a connected closed subgroup K_0 of $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$. The projection of L on $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$ induces then a continuous homomorphism

$$f : L/L_0 \to \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0/K_0.$$

Observe that f has dense image, since $p\circ\beta_0(\Gamma)$ is dense in $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$; so, f is surjective by compactness of L/L_0. It follows, by compactness again, that $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0/K_0$ is topologically isomorphic to a quotient of L/L_0. As L/L_0 is totally disconnected, this implies (see [Bou71, Chap. 3, §4, Corollaire 3]) that $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0/K_0$ is also totally disconnected and hence that $K_0 = \mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$. So, $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0 \times \{1\}$ is contained in L. It follows that L is the product of $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$ with a subgroup of $\mathrm{Prof}(\Gamma)$. Since $\alpha(\Gamma)$ is dense in $\mathrm{Prof}(\Gamma)$, this subgroup coincides with $\mathrm{Prof}(\Gamma)$, that is, $L = K$ and the claim is proved.

• *Third step.* We claim that every irreducible unitary representation $\pi : \Gamma \to U(N)$ of Γ is of the form $\chi \otimes \rho$ for some $\chi \in \widehat{\Gamma^{\mathrm{Ab}}}$ and $\rho \in \widehat{\Gamma}_{\mathrm{finite}}$.

Indeed, Propositions 14 and 15, imply that $\pi([\Gamma,\Gamma])$ is a periodic subgroup of $U(N)$. Since Γ is finitely generated, $[\Gamma,\Gamma]$ is finitely generated (in fact, every subgroup of Γ is finitely generated; see [Rag72, 2.7 Theorem]). Hence, by Schur's theorem (see [Weh73, 4.9 Corollary]), $\pi([\Gamma,\Gamma])$ is finite. It follows that there exists a finite index normal subgroup H of $[\Gamma,\Gamma]$ so that $\pi|_H$ is the trivial representation of H.

Next, we claim that there exists a normal subgroup Δ of finite index in Γ such that $\Delta \cap [\Gamma,\Gamma] = H$. Indeed, since $\Gamma/[\Gamma,\Gamma]$ is abelian and finitely generated, we have $\Gamma/[\Gamma,\Gamma] \cong \mathbf{Z}^k \oplus F$ for some finite subgroup F and some integer $k \geq 0$. Let Γ_1 be the inverse image in Γ of the copy of $\mathbf{Z}^k$ in $\Gamma/[\Gamma,\Gamma]$. Then Γ_1 is a normal subgroup of finite index of Γ. Moreover, Γ_1 can be written as iterated semi-direct product

$$\Gamma_1 = (\dots(([\Gamma,\Gamma] \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z})).$$

Set

$$\Delta := (\dots((H \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z})).$$

Then Δ is a normal subgroup of finite index of Γ with $\Delta \cap [\Gamma,\Gamma] = H$.

Since $\pi|_H$ is trivial on H and since $[\Delta,\Delta] \subset H$, the restriction $\pi|_\Delta$ of π to Δ factorizes through Δ^{Ab}. So, by Proposition 12, there exists a finite Γ-orbit $\mathcal{O}$ in $\widehat{\Delta^{\mathrm{Ab}}}$ such that we have a direct sum decomposition $V = \bigoplus_{\chi\in\mathcal{O}} V^\chi$, where V^χ is the χ-isotypical component of $\pi|_\Delta$.

Fix $\chi \in \mathcal{O}$. Since χ is trivial on H and since $\Delta\cap[\Gamma,\Gamma] = H$, we can view χ as a unitary character of the subgroup $\Delta/(\Delta\cap[\Gamma,\Gamma])$ of Γ^{Ab}. Hence, χ extends to a character $\widetilde{\chi} \in \widehat{\Gamma^{\mathrm{Ab}}}$ (see, e.g. [HR79, (24.12)]). This implies that $\Gamma_\chi = \Gamma$; indeed,

$$\chi^\gamma(\delta) = \widetilde{\chi}(\gamma^{-1}\delta\gamma) = \widetilde{\chi}(\delta) = \chi(\delta)$$

for every $\gamma\in\Gamma$ and $\delta\in\Delta$. This shows that $\mathcal{O}$ is a singleton and so $V = V^\chi$. We write

$$\pi = \widetilde{\chi} \otimes (\overline{\widetilde{\chi}} \otimes \pi).$$

Then $\rho := \overline{\widetilde{\chi}} \otimes \pi$ is an irreducible unitary representation of Γ which is trivial on Δ; so, ρ has finite image and $\pi = \widetilde{\chi}\otimes\rho$.

• *Third step.* Let $\pi \in \widehat{\Gamma}_{\mathrm{fd}}$. We claim that there exists a representation $\pi' \in \widehat{K}$ such that $\pi = \pi'\circ\beta$. Once proved, Proposition 5 will imply that (K,β) is a Bohr compactification for Γ.

By the second step, we can write $\pi = \chi\otimes\rho$ for some $\chi \in \widehat{\Gamma^{\mathrm{Ab}}}$ and $\rho \in \widehat{\Gamma}_{\mathrm{finite}}$. On the one hand, we can write $\rho = \rho'\circ\alpha$ for some $\rho' \in \widehat{\mathrm{Prof}(\Gamma)}$, by the universal property of $\mathrm{Prof}(\Gamma)$. On the other hand, we can decompose χ as $\chi = \chi_0\chi_1$ with $\chi_0 \in \widehat{\Gamma^{\mathrm{Ab}}}$ of infinite order and $\chi_1 \in \widehat{\Gamma^{\mathrm{Ab}}}$ of finite order. We have $\chi_0 = \chi_0'\circ(p\circ\beta_0)$ and $\chi_1 = \chi_1'\circ\alpha$ for unitary characters χ_0' of $\mathrm{Bohr}(\Gamma^{\mathrm{Ab}})_0$ and χ_1' of $\mathrm{Prof}(\Gamma^{\mathrm{Ab}})$. For $\pi' = \chi_0\otimes(\chi_1'\otimes\rho')$, we have $\pi'\in\widehat{K}$ and $\pi = \pi'\circ\beta$.

4. Proof of Theorems 2

Let $\mathbf{G} = \mathbf{U}\rtimes\mathbf{H}$ be a Levi decomposition of $\mathbf{G}$ and set

$$\Lambda = \mathbf{H}(\mathbf{Z}),\quad \Delta = \mathbf{U}(\mathbf{Z}),\qquad \text{and}\qquad \Gamma = \Delta\rtimes\Lambda.$$

Denote by $\beta_\Delta : \Delta \to \mathrm{Bohr}(\Delta)$ and $\beta_\Lambda : \Lambda\to\mathrm{Bohr}(\Lambda)$ the natural homomorphisms. Observe that, by the universal property of $\mathrm{Bohr}(\Delta)$, every element $\lambda\in\Lambda$ defines a continuous automorphism $\theta_b(\lambda)$ of $\mathrm{Bohr}(\Delta)$

such that

$$\theta_b(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \qquad \text{for all} \quad \delta \in \Delta.$$

The corresponding homomorphism $\theta_b : \Lambda \to \mathrm{Aut}(\mathrm{Bohr}(\Delta))$ defines an action of Λ on $\mathrm{Bohr}(\Delta)$. By Theorem 1, we have

$$\mathrm{Bohr}(\Delta) = \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0 \times \mathrm{Prof}(\Delta).$$

The group Λ acts naturally on Δ^{Ab} and, by duality, on $\widehat{\Delta^{\mathrm{Ab}}}$. Let

$$H := \widehat{\Delta^{\mathrm{Ab}}}_{\Lambda-\mathrm{fin}} \subset \widehat{\Delta^{\mathrm{Ab}}}$$

be the subgroup of characters of Δ^{Ab} with finite Λ-orbits. Observe that H contains the torsion subgroup of $\widehat{\Delta^{\mathrm{Ab}}}$.

Let

$$\alpha : \Lambda \to \mathrm{Aut}(H)$$

be the homomorphism given by the action of Λ on H.

For a locally compact group G, the group $\mathrm{Aut}(G)$ of continuous automorphisms of G will be endowed with the compact-open topology for which it is also a (not necessarily locally compact) topological group (see [HR79, (26.3)]).

• *First step.* We claim that the closure of $\alpha(\Lambda)$ in $\mathrm{Aut}(H)$ is compact. Indeed, let us identify $\mathrm{Aut}(H)$ with a subset of the product space H^H. The topology of $\mathrm{Aut}(H)$ coincides with the topology induced by the product topology on H^H. Viewed this way, $\alpha(\Lambda)$ is a subspace of the product $\prod_{\chi\in H} \chi^\Lambda$ of the finite Λ-orbits χ^Λ. Since $\prod_{\chi\in H} \chi^\Lambda$ is compact and hence closed, the claim is proved.

Next, let N be the annihilator of H in $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})$. Then N is Λ-invariant and the induced action of Λ on $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$ is a quotient of the action given by θ_b.

Let C be the connected component of $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$. Then C coincides with the image of $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0$ in $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$ (see [Bou71, Chap. 3, §4, Corollaire 3]) and so

$$C \cong \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0/(N \cap \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0).$$

Since C is invariant under Λ, we obtain an action of Λ on C; let

$$\widehat{\alpha} : \Lambda \to \mathrm{Aut}(C)$$

be the corresponding homomorphism.

• *Second step.* We claim that the action $\widehat{\alpha}$ of Λ on C extends to an action of $\mathrm{Bohr}(\Lambda)$; more precisely, there exists a continuous homomorphism

$$\widehat{\alpha}' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(C)$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Bohr}(\Lambda) \arrow[d, dashed, "\widehat{\alpha}'"] \\ \Lambda \arrow[ur, "\beta_\Lambda"] \arrow[r, "\widehat{\alpha}"'] & \mathrm{Aut}(C) \end{tikzcd}$$

commutes. Indeed, by the first step, the closure K of $\alpha(\Lambda)$ in $\mathrm{Aut}(H)$ is a compact group. Hence, by the universal property of $\mathrm{Bohr}(\Lambda)$, there exists a continuous homomorphism

$$\alpha' : \mathrm{Bohr}(\Lambda) \to K \subset \mathrm{Aut}(H)$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Bohr}(\Lambda) \arrow[d, dashed, "\alpha'"] \\ \Lambda \arrow[ur, "\beta_\Lambda"] \arrow[r, "\alpha"'] & \mathrm{Aut}(H) \end{tikzcd}$$

commutes. Since $\widehat{H} = \mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$, we obtain by duality a continuous homomorphism $\widehat{\alpha}' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N)$. The connected component C of $\mathrm{Bohr}(\Delta^{\mathrm{Ab}})/N$ is invariant under $\mathrm{Bohr}(\Lambda)$. This proves the existence of the map $\widehat{\alpha}' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(C)$ with the claimed property.

Next, observe that, by the universal property of $\mathrm{Prof}(\Delta)$, every element $\lambda \in \Lambda$ defines a continuous automorphism $\theta_p(\lambda)$ of $\mathrm{Prof}(\Delta)$ such that

$$\theta_p(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \qquad \text{for all} \quad \delta \in \Delta.$$

The corresponding homomorphism $\theta_p : \Lambda \to \mathrm{Aut}(\mathrm{Prof}(\Delta))$ defines an action of Λ on $\mathrm{Prof}(\Delta)$.

• *Third step.* We claim that the action θ_p of Λ on $\mathrm{Prof}(\Delta)$ extends to an action of $\mathrm{Bohr}(\Lambda)$; more precisely, there exists a homomorphism $\theta' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(\mathrm{Prof}(\Delta))$ such that the diagram

$$\begin{tikzcd} & \mathrm{Prof}(\Lambda) \arrow[d, dashed, "\theta'"] \\ \Lambda \arrow[ur, "\beta_\Lambda"] \arrow[r, "\theta_p"'] & \mathrm{Aut}(\mathrm{Prof}(\Delta)) \end{tikzcd}$$

commutes. Indeed, since Δ is finitely generated and since its image in $\mathrm{Bohr}(\Delta)$ dense, the profinite group $\mathrm{Bohr}(\Delta)$ is finitely generated (that is, there exists a finite subset of $\mathrm{Bohr}(\Delta)$ which generates a dense subgroup). This implies that $\mathrm{Aut}(\mathrm{Bohr}(\Delta))$ is a profinite group (see [RZ00, Corollary 4.4.4]) and so there exists a homomorphism $\theta'_p : \mathrm{Prof}(\Lambda) \to \mathrm{Aut}(\mathrm{Prof}(\Delta))$ such that $\theta'_p \circ \alpha_\Lambda = \theta_p$. We then lift θ'_p to a homomorphism $\theta' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(\mathrm{Prof}(\Delta))$.

We set

$$Q := \mathrm{Bohr}(\Delta)/(N \cap \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0) = C \times \mathrm{Prof}(\Delta);$$

we have an action of Λ on Q given by the homomorphism

$$\widehat{\alpha} \oplus \theta_p : \Lambda \to \mathrm{Aut}(C) \times \mathrm{Aut}(\mathrm{Prof}(\Delta)) \subset \mathrm{Aut}(Q)$$

and, by the second and third step, an action of $\mathrm{Bohr}(\Lambda)$ on Q given by

$$\widehat{\alpha}' \oplus \theta' : \mathrm{Bohr}(\Lambda) \to \mathrm{Aut}(C) \times \mathrm{Aut}(\mathrm{Prof}(\Delta))$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Bohr}(\Lambda) \arrow[d, dashed, "\widehat{\alpha}' \oplus \theta'"] \\ \Lambda \arrow[ur, "\beta_\Lambda"] \arrow[r, "\widehat{\alpha} \oplus \theta_p"'] & \mathrm{Aut}(C) \times \mathrm{Aut}(\mathrm{Prof}(\Delta)) \end{tikzcd}$$

commutes.

Let

$$B := (C \times \mathrm{Prof}(\Delta)) \rtimes \mathrm{Bohr}(\Lambda)$$

be the semi-direct product defined by $\widehat{\alpha}' \oplus \theta'$. Let

$$p : \mathrm{Bohr}(\Delta) \to C = \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0/(N \cap \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0)$$

be the quotient epimorphism.

• *Fourth step.* We claim that B, together with the map $\beta : \Gamma \to B$, given by

$$\beta(\delta, \lambda) = (p(\beta_\Delta(\delta)), \beta_\Lambda(\lambda)) \qquad \text{for all} \quad (\delta, \lambda) \in \Gamma,$$

is a Bohr compactification for $\Gamma = \Delta \rtimes \Lambda$.

First, we have to check that β is a homomorphism with dense image. Since $p \circ \beta_\Delta$ and β_Λ are homomorphisms with dense image, it suffices to show that

$$\beta(\lambda\delta\lambda^{-1}, e) = ((\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))(p(\beta_\Delta(\delta)), e) \qquad \text{for all} \quad (\delta, \lambda) \in \Gamma.$$

This is indeed the case: since p is equivariant for the Λ-actions, we have

$$p(\beta_\Delta(\lambda\delta\lambda^{-1})) = p(\theta_b(\lambda)\beta_\Delta(\delta)) = (\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))p(\beta_\Delta(\delta)).$$

Next, let π be a unitary representation of Γ on a finite dimensional vector space V. By Proposition 5, we have to show that there exists a unitary representation $\widetilde{\pi}$ of B on V such that $\pi = \widetilde{\pi} \circ \beta$.

Consider a decomposition of $V = V_1 \oplus \cdots \oplus V_s$ into irreducible $\pi(\Delta)$-invariant subspaces V_i; denote by $\sigma_1, \ldots, \sigma_s$ the corresponding irreducible representations of Δ. By Theorem 1, every σ_i is of the form $\sigma_i = \chi_i \otimes \rho_i$ for some $\chi_i \in \widehat{\Delta^{\mathrm{Ab}}}$ and $\rho_i \in \widehat{\Delta}_{\mathrm{finite}}$.

We decompose every χ_i as a product $\chi_i = \chi_i' \chi_i''$ with $\chi_i' \in \widehat{\Delta^{\mathrm{Ab}}}$ of finite order and $\chi_i'' \in \widehat{\Delta^{\mathrm{Ab}}}$ of infinite order. Since χ_i' has finite image, upon replacing ρ_i by $\chi_i' \otimes \rho_i$, we may and will assume that every non trivial χ_i has infinite order.

Fix $i \in \{1, \ldots, s\}$. We can extend χ_i and ρ_i to unitary representations of $\mathrm{Bohr}(\Delta)$, that is, we can find representations $\widetilde{\chi_i}$ and $\widetilde{\rho_i}$ of $\mathrm{Bohr}(\Delta)$ on V_i such that $\chi_i = \widetilde{\chi_i} \circ \beta_\Delta$ and $\rho_i = \widetilde{\rho_i} \circ \beta_\Delta$. By Proposition 12, the stabilizer Γ_{σ_i} of σ_i has finite index in Γ. It follows that the Λ-orbit of σ_i is finite, and this implies that $\chi_i \in H$; hence, $\widetilde{\chi_i}$ factorizes through

$$C = \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0 / (N \cap \mathrm{Bohr}(\Delta^{\mathrm{Ab}})_0)$$

and we have $\chi_i = \widetilde{\chi_i} \circ (p \circ \beta_\Delta)$. Since ρ_i has finite image, $\widetilde{\rho_i}$ factorizes through $\mathrm{Prof}(\Delta)$. So, $\widetilde{\sigma_i} := \widetilde{\chi_i} \otimes \widetilde{\rho_i}$ is a unitary representation of $C \times \mathrm{Prof}(\Delta)$ on V_i. Set

$$\widetilde{\pi_\Delta} := \bigoplus_{i=1}^{s} \widetilde{\sigma_i}.$$

Then $\widetilde{\pi_\Delta}$ is a unitary representation of $C \times \mathrm{Prof}(\Delta)$ on V such that $\pi|_\Delta = \widetilde{\pi_\Delta} \circ (\beta|_\Delta)$.

On the other hand, since $\pi|_\Lambda$ is a finite dimensional representation of Λ, we can find a representation $\widetilde{\pi_\Lambda}$ of $\mathrm{Bohr}(\Lambda)$ on V such that $\pi|_\Lambda = \widetilde{\pi_\Lambda} \circ (\beta|_\Lambda)$. For $\lambda \in \Lambda$ and $\delta \in \Delta$, we have

$$\begin{aligned} \widetilde{\pi_\Delta}(\beta(\lambda)\beta(\delta)\beta(\lambda)^{-1}) &= \widetilde{\pi_\Delta}(\beta(\lambda\delta\lambda)^{-1}) \\ &= \pi(\lambda\delta\lambda)^{-1}) \\ &= \pi(\lambda)\pi(\delta)\pi(\lambda)^{-1} \\ &= \widetilde{\pi_\Lambda}(\beta(\lambda))\widetilde{\pi_\Delta}(\beta(\delta))\widetilde{\pi_\Lambda}(\beta(\lambda))^{-1}. \end{aligned}$$

Since β has dense image in B, it follows that

$$\widetilde{\pi_\Delta}(bab^{-1}) = \widetilde{\pi_\Lambda}(b)\widetilde{\pi_\Delta}(a)\widetilde{\pi_\Lambda}(b)^{-1} \qquad \text{for all} \quad (a,b) \in B$$

and therefore the formula

$$\widetilde{\pi}(a,b) = \widetilde{\pi_\Delta}(a)\widetilde{\pi_\Lambda}(b) \qquad \text{for all} \quad (a,b) \in B$$

defines a unitary representation of B on V such that $\pi = \widetilde{\pi} \circ \beta$.

5. Proof of Theorem 3

Recall that we are assuming that $\mathbf{G}$ is a connected, simply-connected and almost $\mathbf{Q}$-simple algebraic group. The group $\mathbf{G}$ can be obtained from an absolutely simple algebraic group $\mathbf{H}$ by the so-called restriction of scalars; more precisely (see [BT65, 6.21, (ii)]), there exists a number field K and an absolutely simple algebraic group $\mathbf{H}$ over K which is absolutely simple with the following property: $\mathbf{G}$ can be written as (more precisely, is $\mathbf{Q}$-isomorphic to) the $\mathbf{Q}$-group $\mathbf{H}^{\sigma_1} \times \cdots \times \mathbf{H}^{\sigma_s}$, where the σ_i's are the different (non conjugate) embeddings of K in $\mathbf{C}$. Assuming that $\sigma_1, \dots, \sigma_{r_1}$ are the embeddings such that $\sigma_i(K) \subset \mathbf{R}$, we can identify $\mathbf{G}(\mathbf{R})$ with

$$\mathbf{H}^{\sigma_1}(\mathbf{R}) \times \cdots \times \mathbf{H}^{\sigma_{r_1}}(\mathbf{R}) \times \mathbf{H}^{\sigma_{r_1+1}}(\mathbf{C}) \times \cdots \times \mathbf{H}^{\sigma_{r_s}}(\mathbf{C}).$$

Let $\mathbf{G}_c$ be the product of the $\mathbf{H}^{\sigma_i}$'s for which $\mathbf{H}^{\sigma_i}(\mathbf{R})$ is compact.

We assume now that the real semisimple Lie group $\mathbf{G}(\mathbf{R})$ is not locally isomorphic to a group of the form $SO(m,1)\times L$ or $SU(m,1)\times L$ for a compact Lie group L. Let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be an arithmetic subgroup.

Set $K := \mathbf{G}_c(\mathbf{R}) \times \mathrm{Prof}(\Gamma)$ and let $\beta : \Gamma \to K$ be defined by $\beta(\gamma) = (p(\gamma), \alpha(\gamma))$, where $p : \mathbf{G}(\mathbf{R}) \to \mathbf{G}_c(\mathbf{R})$ is the canonical projection and $\alpha : \Gamma \to \mathrm{Prof}(\Gamma)$ the map associated to $\mathrm{Prof}(\Gamma)$. We claim that (K, β) is a Bohr compactification of Γ,

First, we show that $\beta(\Gamma)$ has dense image. Observe that $\mathbf{G}_c(\mathbf{R})$ is connected (see [Bor91, (24.6.c)]). By the Strong Approximation Theorem (see [PR94, Theorem 7.12]), $p(\mathbf{G}(\mathbf{Z}))$ is dense in $\mathbf{G}_c(\mathbf{R})$. Since $\mathbf{G}_c(\mathbf{R})$ is connected and since Γ is commensurable to $\mathbf{G}(\mathbf{Z})$, it follows that $p(\Gamma)$ is dense in $\mathbf{G}_c(\mathbf{R})$. Now, $\alpha(\Gamma)$ is dense in $\mathrm{Prof}(\Gamma)$ and $\mathrm{Prof}(\Gamma)$ is totally disconnected. As in the first step of the proof of Theorem 1, we conclude that $\beta(\Gamma)$ is dense in K.

Let $\pi : \Gamma \to U(n)$ be a finite dimensional unitary representation of Γ. Then, by Margulis' superrigidity theorem (see [Mar91, Chap. VIII, Theorem B]), [Mor15, Corollary 16.4.1]), there exists a continuous homomorphism $\rho_1 : \mathbf{G}(\mathbf{R}) \to U(n)$ and a homomorphism $\rho_2 : \Gamma \to U(n)$ such that

(i) $\rho_2(\Gamma)$ is finite;
(ii) $\rho_1(g)\rho_2(\gamma) = \rho_2(\gamma)\rho_1(g)$ for all $g \in \mathbf{G}(\mathbf{R})$ and $\gamma \in \Gamma$;
(iii) $\pi(\gamma) = \rho_1(\gamma)\rho_2(\gamma)$ for all $\gamma \in \Gamma$.

By a classical result of Segal and von Neumann [SvN50], ρ_1 factorizes through $\mathbf{G}_c(\mathbf{R})$, that is, $\rho_1 = \rho_1' \circ p$ for a unitary representation ρ_1' of $\mathbf{G}_c(\mathbf{R})$. It follows from (i) that $\rho_2 = \rho_2' \circ \alpha$ for a unitary representation ρ_2' of $\mathrm{Prof}(\Gamma)$. Moreover, (ii) and (iii) show that $\pi = (\rho_1|_\Gamma) \otimes \rho_2$. Hence,

$\pi = (\rho_1' \otimes \rho_2') \circ \otimes \beta$. We conclude by Proposition 5 that (K, β) is a Bohr compactification of Γ.

6. A few examples

We compute the Bohr compactification for various examples of arithmetic groups.

(1) For an integer $n \geq 1$, the $(2n+1)$-dimensional Heisenberg group is the unipotent $\mathbf{Q}$-group $\mathbf{H}_{2n+1}$ of matrices of the form

$$m(x_1, \ldots, x_n, y_1, \ldots, y_n, z) := \begin{pmatrix} 1 & x_1 & \ldots & x_n & z \\ 0 & 1 & \ldots & 0 & y_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & y_n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

The arithmetic group $\Gamma = \mathbf{H}_{2n+1}(\mathbf{Z})$ is nilpotent of step 2; its commutator subgroup $[\Gamma, \Gamma]$ coincides with its center $\{m(0, 0, z) : z \in \mathbf{Z}\}$. So, $\Gamma^{\mathrm{Ab}} \cong \mathbf{Z}^{2n}$. We have, by Theorem 1,

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\mathbf{Z}^{2n})_0 \times \mathrm{Prof}(\Gamma)$$

and hence, by Proposition 11 and Proposition 16

$$\mathrm{Bohr}(\Gamma) \cong (\prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q}) \times \prod_{p \text{ prime}} \mathbf{H}_{2n+1}(\mathbf{Z}_p).$$

(2) Let $\mathbf{G} = SL_n$ for $n \geq 3$ or $\mathbf{G} = Sp_{2n}$ for $n \geq 2$. Then $SL_n(\mathbf{Z})$ and $Sp_{2n}(\mathbf{Z})$ are non cocompact arithmetic lattices in $SL_n(\mathbf{R})$ and $Sp_{2n}(\mathbf{R})$, respectively. Hence, we have, by Corollary 4, $\mathrm{Bohr}(SL_n(\mathbf{Z})) = \mathrm{Prof}(SL_n(\mathbf{Z}))$ and $\mathrm{Bohr}(Sp_{2n}(\mathbf{Z})) = \mathrm{Prof}(Sp_{2n}(\mathbf{Z}))$. Since $SL_n(\mathbf{Z})$ and $Sp_{2n}(\mathbf{Z})$ have the congruence subgroup property, it follows that

$$\mathrm{Bohr}(SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} SL_n(\mathbf{Z}_p) \cong SL_n(\mathrm{Prof}(\mathbf{Z}))$$

and similarly

$$\mathrm{Bohr}(Sp_{2n}(\mathbf{Z})) \cong \prod_{p \text{ prime}} Sp_{2n}(\mathbf{Z}_p) \cong Sp_{2n}(\mathrm{Prof}(\mathbf{Z})).$$

(3) The group $\Gamma = SL_2(\mathbf{Z}[\sqrt{2}])$ embeds as a non cocompact arithmetic lattice of $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$. So, by Corollary 4, we have

$$\mathrm{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \mathrm{Prof}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

Moreover, since Γ has the congruence subgroup property (see [Ser70, Corollaire 3]), it follows that

$$\mathrm{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \mathrm{Cong}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

(4) For $n \geq 4$, consider the quadratic form

$$q(x_1, \dots, x_n) = x_1^2 + \cdots + x_{n-1}^2 - \sqrt{2}x_n^2 - \sqrt{2}x_{n+1}^2$$

The group $\mathbf{G} = SO(q)$ of unimodular $(n+1) \times (n+1)$-matrices which preserve q is an almost simple algebraic group over the number field $\mathbf{Q}[\sqrt{2}]$. The subgroup $\Gamma = SO(q, \mathbf{Z}[\sqrt{2}])$ of $\mathbf{Z}[\sqrt{2}]$-rational points in $\mathbf{G}$ embeds as a cocompact lattice of the semisimple real Lie group $SO(n+1) \times SO(n-1, 2)$ via the map

$$SO(q, \mathbf{Q}[\sqrt{2}]) \to SO(n+1) \times SO(n-1, 2),\ \gamma \mapsto (\gamma^\sigma, \gamma),$$

where σ is the field automorphism of $\mathbf{Q}[\sqrt{2}]$ given by $\sigma(\sqrt{2}) = -\sqrt{2}$; so, $SO(n+1) \times SO(n-1, 2)$ is the group of real points of the $\mathbf{Q}$-group $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$ obtained by restriction of scalars from the $\mathbf{Q}[\sqrt{2}]$-group $\mathbf{G}$. Observe that $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$ is almost $\mathbf{Q}[\sqrt{2}]$-simple since $\mathbf{G}$ is almost $\mathbf{Q}$-simple. By Theorem 3, we have

$$\mathrm{Bohr}(SO(q, \mathbf{Z}[\sqrt{2}])) \cong SO(n+1) \times \mathrm{Prof}(SO(q, \mathbf{Z}[\sqrt{2}]).$$

(5) For $d \geq 2$, let D be a central division algebra over $\mathbf{Q}$ such that $D \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to the algebra $M_d(\mathbf{R})$ of real $d \times d$-matrices. There exists a subring $\mathcal{O}$ of D which is a $\mathbf{Z}$-lattice in D (a so-called order in D). There is an embedding $\varphi : D \to M_d(\mathbf{R})$ such that $\varphi(SL_1(D) \subset SL_d(\mathbf{Q})$ and such that $\Gamma := \varphi(SL_1(\mathcal{O})$ is an arithmetic cocompact lattice in $SL_d(\mathbf{R})$, where $SL_1(D)$ is the group of norm one elements in D; for all this, see [Mor15, §6.8.i]. For $d \geq 3$, we have

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Prof}(\Gamma).$$

So, this is an example of a *cocompact* lattice Γ in a simple real Lie group for which there exists no homomorphism $\Gamma \to U(n)$ with infinite image; the existence of such examples was mentioned in [Mor15, (16.4.3)]

(6) For $n \geq 2$, let Γ be the semi-direct product $\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})$, induced by the usual linear action of $SL_n(\mathbf{Z})$ on $\mathbf{R}^n$. The dual action of $SL_n(\mathbf{Z})$ on $\widehat{\mathbf{Z}^n} \cong \mathbf{R}^n/\mathbf{Z}^n$ is given by

$$SL_n(\mathbf{Z}) \times \mathbf{R}^n/\mathbf{Z}^n \to \mathbf{R}^n/\mathbf{Z}^n,\ (g, x + \mathbf{Z}^n) \mapsto {}^t g x + \mathbf{Z}^n.$$

It is well-known and easy to show that the subgroup of $SL_n(\mathbf{Z})$-periodic orbits in $\widehat{\mathbf{Z}^n}$ corresponds to $\mathbf{Q}^n/\mathbf{Z}^n$, that is, to the characters of finite image. It follows from Theorem 2 that

$$\mathrm{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \mathrm{Bohr}(SL_n(\mathbf{Z}))_0 \times \mathrm{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})).$$

For $n \geq 3$, we have therefore

$$\mathrm{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \mathrm{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \rtimes SL_n(\mathbf{Z}_p).$$

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