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Irreducible affine isometric actions on Hilbert spaces

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Abstract. Let G be locally compact group. We undertake a systematic study of irreducible affine isometric actions of G on Hilbert spaces. It turns out that, while that are a few parallels of this study to the by now classical theory of irreducible unitary representations, these two theories differ in several aspects (for instance, the direct sum of two irreducible affine actions can still be irreducible). One of the main tools we use is an affine version of Schur's lemma characterizing the irreducibility of an affine isometric action of G. This enables us to describe for instance the irreducible affine isometric actions of nilpotent groups. As another application, a short proof is provided for the following result of Neretin: the restriction to a cocompact lattice of an irreducible affine action of G remains irreducible. We give a necessary and sufficient condition for a fixed unitary representation π to be the linear part of an irreducible affine action. In particular, when π is a multiple of the regular representation of a discrete group Γ, we show how this question is related to the L^2-Betti number $\beta^1_{(2)}(\Gamma)$. After giving a necessary and sufficient condition for a direct sum of irreducible affine actions to be irreducible, we show the following super-rigidity result: if G is product of two or more locally compact groups and Γ an irreducible co-compact lattice in G, then any irreducible affine action α of Γ extends to an affine action of G, provided the linear part of α does not weakly con

1. Introduction

The theory of unitary representations of locally compact groups is by now a central and classical part of representation theory. Very quickly, the theory centers on the study of unitary irreducible representations which, for suitable classes of groups (e.g. compact Lie groups, nilpotent Lie groups, semi-simple Lie groups, to name just a few), has reached a very satisfactory state.

The theory of affine isometric actions on Hilbert spaces is, comparatively, a much more recent subject, that developed through connections with property (T), the Haagerup property, or operator algebras (see e.g. [3]). To the best of our knowledge, *irreducible* affine isometric actions were first considered by Neretin [23], who also provides many examples. So let α be an affine isometric action of the group G on the complex or real Hilbert space $\mathcal{H}$, i.e. a group

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homomorphism $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ from G to the group of affine isometries of $\mathcal{H}$.

Definition 1.1. The action α is *irreducible* if $\mathcal{H}$ has no non-empty, closed and proper $\alpha(G)$-invariant affine subspace.

In the sequel, our Hilbert spaces will often, but not always (as in Section 4.24 and Proposition 5.4), be indifferently complex or real; following common terminology, a representation of G by linear isometries on a such a Hilbert space $\mathcal{H}$ is called a unitary representation if $\mathcal{H}$ is complex and an orthogonal representation if $\mathcal{H}$ is real.

The following two classes of examples of irreducible affine isometric actions should be kept in mind.

Example 1.2. Let $b : G \to \mathcal{H}$ be a homomorphism to the additive group of $\mathcal{H}$. It gives rise to an affine action of G by translations on $\mathcal{H}$, which is irreducible if and only if the linear span of $b(G)$ is dense in $\mathcal{H}$.

Example 1.3. Let π be an irreducible unitary or orthogonal representation of G on $\mathcal{H}$, such that $H^1(G, \pi) \neq 0$. Choose a 1-cocycle $b \in Z^1(G, \pi)$ which is not a 1-coboundary. Then the affine action α of G on $\mathcal{H}$, defined

$$\alpha(g)v := \pi(g)v + b(g) \quad \text{for} \quad g \in G,\, v \in \mathcal{H},$$

is irreducible. Indeed, assume by contradiction that $\mathcal{K}$ is a non-empty, closed, proper, $\alpha(G)$-invariant affine subspace. Then its linear part $\mathcal{K}_0$, is a proper and closed $\pi(G)$-invariant linear subspace; by irreducibility of π, it follows that $\mathcal{K}_0 = \{0\}$. So α has a fixed point, contradicting the fact that b is not a coboundary.

In this paper, we undertake a systematic study of irreducible affine isometric actions of the locally compact group G on Hilbert spaces. The theory of irreducible affine isometric actions has some parallels with the theory of irreducible unitary representations, but to a limited extent. To illustrate this, we contrast the classical case and the affine case in two columns, where the left column is about a unitary representation π, the right column is about an affine isometric action α on a complex or real Hilbert space $\mathcal{H}$ with linear part π and translation part b.

(1) Characterization

π is irreducible if and only $\pi(G)\xi$ is total for every non-zero vector ξ if and only if every positive-definite function $g \mapsto \langle \pi(g)\xi \mid \xi \rangle$ lies on an extremal ray in the cone of positive-definite functions on G.	α is irreducible if and only if, for every vector v, the cocycle $g \mapsto b(g) + \pi(g)v - v$ has total image; if and only if $b(G)$ is total and, for every decomposition $\lVert b(g) \rVert^2 = \psi_0(g) + \psi_1(g)$, with ψ_0, ψ_1 functions conditionally of negative type with $\psi_0 \neq 0$, the function ψ_0 is unbounded (see Proposition 2.3).

(2) Existence (G locally compact)	
Irreducible unitary representations of G separate points (Gelfand-Raikov).	For G compactly generated, G admits an irreducible affine action if and only if G doesn't have property (T), as follows from Theorem 0.2 in [31]. Even then, irreducible affine actions do not separate points in general (see Corollary 4.21 below).
(3) Commutants	
$\pi(G)'$ is the commutant of $\pi(G)$ in $B(\mathcal{H})$ (it is a von Neumann algebra).	$\alpha(G)'$ is the commutant of $\alpha(G)$ in the monoid of continuous affine maps on $\mathcal{H}$. The affine map $Av := Tv + t$ is in $\alpha(G)'$ if and only if $T \in \pi(G)'$ and $(T-1)b(g) = \pi(g)t - t$ for all $g \in G$ (see Lemma 3.3).
(4) Schur's lemma	
π is irreducible if and only if $\pi(G)' = \mathbf{C}.1$.	α is irreducible if and only if $\alpha(G)'$ consists of translations (in this case, exactly the set of translations along $\mathcal{H}^{\pi(G)}$; see Proposition 3.6).
(5) Abelian groups	
Every irreducible unitary representation is one-dimensional.	Every irreducible action is given by some homomorphism $b : G \to \mathcal{H}$ with $b(G)$ having dense linear span (see Proposition 4.11).
(6) Nilpotent groups	
Usually, the irreducible unitary representations of G are infinite dimensional (think of Kirillov's orbit method).	Same as for abelian groups, see Corollary 4.21.

Apart from allowing us to determine the irreducible affine actions of abelian or nilpotent groups, our affine Schur lemma has several other applications:

- We give in Theorem 4.2 a short proof of Neretin's result [23] that, upon restricting to a co-compact lattice in a locally compact group, an irreducible affine action remains irreducible[1].
- We are able to study the question: "when is a given unitary representation π the linear part of an irreducible affine action?" In particular, taking for π a multiple of the regular representation of a non-amenable, ICC discrete group Γ, we get a new definition of the first L^2-Betti number $\beta^1_{(2)}(\Gamma)$; namely $\beta^1_{(2)}(\Gamma)$ is the supremum of all non-negative t's such that the unique

[1] It is well-known that, in general, restricting a unitary irreducible representation to a co-compact lattice, does not yield an irreducible representation.

module over the von Neumann algebra $L(\Gamma)$ of Γ with $L(\Gamma)$-dimension t is the linear part of some irreducible affine action (see Corollary 4.28).

- The definition of L^2-Betti numbers $\beta^n_{(2)}$ has been extended from discrete to locally compact unimodular groups, in two papers by Petersen [26] and Kyed-Petersen-Vaes [16]. We prove in Theorem 7.2 that, if G is a locally compact group containing a co-compact lattice, then

$$\beta^1_{(2)}(G) \geq \sum_{\sigma\in\hat{G}_d} d_\sigma \dim_{\mathbf{C}} H^1(G,\sigma) \tag{1}$$

  where $\hat{G}_d$ is the discrete series of G (i.e. the set of square-integrable unitary irreducible representations of G, up to unitary equivalence), and $d_\sigma > 0$ is the formal dimension of σ. The proof depends crucially on irreducible affine actions, even if the inequality involves no such actions .

Here is a short summary of the paper. We give in Section 2 a number of characterizations of irreducible affine actions. Commutants are introduced in Section 3, where the affine Schur lemma is also proved. Section 4 contains several applications of the affine Schur lemma: to the restriction of affine actions to lattices, to the behavior of an irreducible affine action on the center of a group, to abelian and nilpotent groups, and to the regular representation of a discrete group. Observing that (unlike what happens for unitary representations!), the direct sum of two irreducible affine actions can still be irreducible, we give in Section 5 a necessary and sufficient condition for this to happen. In Section 6, we combine this with a super-rigidity result of Shalom [31] and show that, if Γ is an irreducible co-compact lattice in a product of two or more locally compact groups, any irreducible affine action of Γ extends to an affine action of the ambient group, provided the linear part of α does not weakly contain the trivial representation. Section 7 is devoted to the proof of inequality (1) mentioned above. Finally, in Section 8 we compare our notion of irreducibility for affine actions with other possible definitions, already introduced in [8].

2. Characterizations of irreducible affine actions

2.1. **Notations.** Let G be a topological group with identity element e; a continuous function $\psi : G \to \mathbf{R}$ is *conditionally of negative type* (CNT) if $\psi(e) = 0$, $\psi(g^{-1}) = \psi(g)$ for every $g \in G$, and

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j\psi(g_i^{-1}g_j) \leq 0,$$

for every $n \geq 1$, $g_1, \ldots, g_n \in G$, and $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ with $\sum_{i=1}^n \lambda_i = 0$. Equivalently, by the GNS construction, there exists a real (or complex) Hilbert space $\mathcal{H}_\psi$ and a (continuous) affine isometric action α_ψ of G on $\mathcal{H}_\psi$ such that $\psi(g) = \|\alpha_\psi(g)(0)\|^2$ for every $g \in G$ (see [3], Theorem C.2.3 and Proposition 2.10.2; observe that although the result there is stated for a real Hilbert space $\mathcal{H}_\psi$, the proof produces an isometric action on the complexification of $\mathcal{H}_\psi$).

Let C be the cone of CNT functions on G. It is known (see [32], or Théorème 1 in [17] [2]) that a non-zero $\psi \in C$ lies on an extremal ray, if and only if the linear part π_ψ of the isometric action α_ψ on the associated real Hilbert is an irreducible orthogonal representation of G. Define two sub-cones C_b and C_u: the cone C_b is the set of bounded functions in C, and the cone C_u is the set of unbounded functions in C, together with 0. Clearly $C = C_b \cup C_u$, and $C_b \cap C_u = \{0\}$, and C_b is a face in C. For G locally compact σ-compact group, $C_u = \{0\}$ if and only if G has Kazhdan's property (T): this is a re-phrasing of the Delorme-Guichardet theorem (see [3], Theorem 2.12.4).

Let $(\pi, \mathcal{H})$ be a unitary or orthogonal representation of G on a complex or real Hilbert space $\mathcal{H}$; we denote by $Z^1(G,\pi)$ (resp. $B^1(G,\pi)$) the space of 1-cocycles (resp. 1-coboundaries) associated with π. The 1-cohomology $H^1(G,\pi)$ is the quotient $Z^1(G,\pi)/B^1(G,\pi)$.

Let $b \in Z^1(G,\pi)$ be a 1-cocycle. We denote by $\alpha_{\pi,b}$ the associated affine isometric action of G on $\mathcal{H}$ associated to b, defined by $\alpha_{\pi,b}(g)v = \pi(g)v + b(g)$ for $g \in G$ and $v \in \mathcal{H}$. When π and b are clear, we will write α for $\alpha_{\pi,b}$.

For $v \in \mathcal{H}$, we shall denote by ∂_v the 1-coboundary $\partial_v(\cdot) := \pi(\cdot)v - v$; this is the 1-cocycle associated with the affine isometric action $t_v^{-1} \circ \pi \circ t_v$, where t_v is the translation of vector v in $\mathcal{H}$, so this affine action has a fixed point and it is reducible.

Let π_0 be a sub-representation of π, on a closed subspace $V_0 \subset \mathcal{H}$. Let us denote by $b_0(g)$ the orthogonal projection of $b(g)$ on V_0. It is immediate to check that $g \mapsto b_0(g)$ is a cocycle with respect to π_0, so that $\alpha_0(g)v = \pi_0(g)v + b_0(g)$ defines an affine isometric action of G on V_0: we call it the *projected action* on V_0.

Recall that a subset of $\mathcal{H}$ is *total* if it generates a dense linear subspace of $\mathcal{H}$. Throughout this paper, all affine subspaces will be assumed to be non-empty.

2.2. **Characterizations of irreducibility.** Let $(\pi, \mathcal{H})$ be a unitary or orthogonal representation of G on a complex or real Hilbert space $\mathcal{H}$ and $b \in Z^1(G,\pi)$ a 1-cocycle. Recall that b is bounded if and only if b is a coboundary (see Proposition 2.2.9 in [3]). In this case, the corresponding affine action α is reducible. Thus, we assume *from now on that b is not a 1-coboundary.*

Proposition 2.3. *Keep notations as in subsection 2.1. The following properties are equivalent:*

- *(A1) The affine isometric action α is irreducible.*
- *(A2) For every $v \in \mathcal{H}$, the 1-cocycle $b + \partial_v$ has total image in $\mathcal{H}$.*
- *(A3) For every direct sum decomposition $\pi = \pi_0 \oplus \pi_1$ with $\pi_0 \neq 0$, in the corresponding decomposition $b = b_0 \oplus b_1$, the 1-cocycle b_0 is unbounded.*

[2] Note that the assumption $b \neq 0$ is missing in the statement of this result in [17]; also, it should have been said in the proof that the linear subspace spanned by $b(G)$ is $\pi(G)$-invariant (as follows easily from the 1-cocycle relation), hence by irreducibility it is dense in $\mathcal{H}$.

(A4) *$b(G)$ is total and, for every decomposition $\psi = \psi_0 + \psi_1$, with ψ_0, ψ_1 functions conditionally of negative type with $\psi_0 \neq 0$, the function ψ_0 is unbounded.*
(A5) *$b(G)$ is total and ψ belongs to a common face of C and C_u.*
(A6) *For every non-zero sub-representation π_0 of π, the projected action α_0 is irreducible.*

Proof. We follow the schemes $(A1) \Rightarrow (A6) \Rightarrow (A3) \Rightarrow (A2) \Rightarrow (A1)$ and $(A1) \Rightarrow (A4) \Leftrightarrow (A5) \Rightarrow (A3)$

$(A1) \Rightarrow (A6)$: Assume that there is a closed, $\pi(G)$-invariant subspace $V_0 \subset \mathcal{H}$ such that the projected action α_0 is reducible. So there exists a proper closed, $\alpha_0(G)$-invariant affine subspace $W \subset V_0$. Let $V_0^\perp$ denote the orthogonal complement of V_0. Then $W \oplus V_0^\perp$ is a proper closed, $\alpha(G)$-invariant affine subspace of $\mathcal{H}$, so that α is reducible.

$(A6) \Rightarrow (A3)$ is clear, as boundedness of b_0 implies reducibility of α_0.

$(A3) \Rightarrow (A2)$: Assume that, for some $v \in \mathcal{H}$, the set $(b + \partial_v)(G)$ is not total. Let W_1 be the closed linear subspace it generates. It follows from the 1-cocycle relation for $b+\partial_v$ that W_1 is $\pi(G)$-invariant. Let W_0 be the orthogonal complement of W_1, and let

$$\pi = \pi_0 \oplus \pi_1, \quad b = b_0 \oplus b_1, \text{ and } v = v_0 \oplus v_1$$

be the corresponding decompositions of π, b, and v. As $v + W_1$ is $\alpha(G)$-invariant, it follows that the affine action α_0 obtained by projecting to W_0 has v_0 as a fixed point, i.e. b_0 is bounded.

$(A2) \Rightarrow (A1)$: Assume by contraposition that α has a non-empty, closed invariant affine subspace $W \neq \mathcal{H}$; let $W_0 = W - W$ be the corresponding linear subspace, so that $W_0 \neq \mathcal{H}$. Then for $v \in W$ we have $\alpha(g)v - v \in W_0$ for every $g \in G$, i.e. $b(g) + \pi(g)v - v \in W_0$, showing that $(b + \partial_v)(G)$ is not total.

$(A1) \Rightarrow (A4)$: We proceed by contraposition. If $b(G)$ is not total, then α is reducible. So, we may assume that $b(G)$ is total and that there exists a decomposition $\psi = \psi_0 + \psi_1$ where ψ_0 is non-zero and bounded. By the GNS construction, there exist a unitary or orthogonal representation π_0 of G on a Hilbert space $\mathcal{H}_0$ and a 1-cocycle $b_0 \in Z^1(G, \pi_0)$ with total image such that $\psi_0(\cdot) = \|b(\cdot)\|^2$ (see [3], Theorem C.2.3 and Proposition 2.10.2). The associated affine isometric action $\alpha_0 = \alpha_{\pi_0, b_0}$ has a fixed point w, as ψ_0 is bounded. Now, by the proof of Theorem 1 in [17] (see in particular pp. 245-246), the map $\sum_i a_i b(g_i) \mapsto \sum_i a_i b_0(g_i)$, from the span of $b(G)$ to the span of $b_0(G)$, extends linearly and continuously to a bounded linear map $T_0 : \mathcal{H} \to \mathcal{H}_0$, which is onto and intertwines α and α_0. Hence $T_0^{-1}(w)$ is a proper, closed, affine subspace of $\mathcal{H}$ which is $\alpha(G)$-invariant, so α is reducible.

$(A4) \Rightarrow (A5)$: Set

$$F = \{\psi_0 \in C \mid \text{ there exists } \psi_1 \in C \text{ such that } \psi_0 + \psi_1 \in \mathbf{R}^+\psi\}.$$

This is clearly the smallest face of C containing ψ. The assumption implies that $F \subset C_u$, so F is a common face of C and C_u.

$(A5) \Rightarrow (A4)$ is obvious.

$(A4) \Rightarrow (A3)$: Set $\psi_i(\cdot) = \|b_i(\cdot)\|^2$ $(i = 0, 1)$ and notice that the assumption that $b(G)$ is total implies that $b_0 \neq 0$. $\square$

Example 2.4. If α is irreducible then by $(A1) \Rightarrow (A2)$ the set $b(G)$ is total in $\mathcal{H}$. The converse is *false*: the reason is that condition $(A2)$ is translation-invariant, while $b(G)$ being total is not. Concretely, let $G = \mathbf{Z}$ act isometrically on the Euclidean space $\mathbf{R}^2$ by

$$\alpha(n)(x, y) = (x + n, (-1)^n y + 1 - (-1)^n) \quad \text{for all} \quad n \in \mathbf{Z},\ (x, y) \in \mathbf{R}^2.$$

Geometrically, this is the action by powers of the glide symmetry with axis the horizontal line $y = 1$, and translation by $+1$ to the right. Then *all* orbits are total, in particular $\alpha(G)(0) = b(G)$, but α is reducible as the axis is invariant.

3. Use of commutants

3.1. **The commutant of an affine action.** Let α be an affine isometric action of a group G on a complex or real Hilbert space $\mathcal{H}$, with linear part π. We recall that the *commutant* of π is the subalgebra

$$\pi(G)' = \{T \in B(\mathcal{H}) \mid T\pi(g) = \pi(g)T \quad \text{for all} \quad g \in G\}$$

of $B(\mathcal{H})$. If b is a cocycle for π and $T \in \pi(G)'$, we observe that Tb is still a cocycle for π, so that $\pi(G)'$ acts on the space $Z^1(G, \pi)$ of 1-cocycles, and this action descends to the first cohomology space $H^1(G, \pi)$.

Definition 3.2. The *commutant* of α is the set of (continuous) affine transformations A on $\mathcal{H}$ such that $A \circ \alpha(g) = \alpha(g) \circ A$ for every $g \in G$.

Write an affine transformation A on $\mathcal{H}$ as $Av = Tv + t$ for $v \in \mathcal{H}$, where $T \in B(\mathcal{H})$ is the linear part. It is easy to see that A is in the commutant of α if and only if $T \in \pi(G)'$ and $(T - 1)b(g) = \partial_t(g)$ for every $g \in G$. From this the following lemma is immediate:

Lemma 3.3. *For $T \in \pi(G)'$, the following properties are equivalent:*

i) There exists $t \in \mathcal{H}$ such that the affine transformation $Av := Tv + t$ is in the commutant of α.

ii) There exists $t \in \mathcal{H}$ such that $(T - 1)b(g) = \partial_t(g)$ for every $g \in G$.

iii) $(T - 1)[b] = 0$, where $[b]$ denotes the class of b in $H^1(G, \pi)$.

Remark 3.4. We observe that, if $Av = Tv + t$ is in the commutant of an affine action $\alpha = \alpha_{\pi,b}$ without fixed point, then 1 is a spectral value of T, as the operator $T - 1$ maps the unbounded set $b(G)$ to the bounded set $\partial_t(G)$.

3.5. **A Schur-type lemma.** Let α be an affine isometric action of a group G on a complex or real Hilbert space $\mathcal{H}$, with linear part π and associated 1-cocycle b. We denote by $\mathcal{H}^{\pi(G)}$ the space of $\pi(G)$-fixed vectors in $\mathcal{H}$.

Proposition 3.6. *The following properties are equivalent.*

i) The affine isometric action α is irreducible.

ii) The commutant of α is the set of translations along $\mathcal{H}^{\pi(G)}$.
iii) The commutant of α consists of translations.

Proof. $(i) \Rightarrow (ii)$ Let $Av = Tv + t$ be an affine transformation of $\mathcal{H}$, in the commutant of α. Then $T \in \pi(G)'$ and

$$(T-1)b(g) = \pi(g)t - t \quad \text{for every} \quad g \in G. \tag{2}$$

So it is enough to show that $T = 1$. For this, consider the positive operator

$$S = T^*T - T - T^* + 2 = (T-1)^*(T-1) + 1;$$

if we show $S = 1$, then $T = 1$. As S is self-adjoint, it is enough to show that the spectrum of S is $\{1\}$. Indeed, this fact, which is well-known when $\mathcal{H}$ is a complex Hilbert space, is an easy consequence of the functional calculus for the self-adjoint operator S. However, the functional calculus is still valid for a self-adjoint operator S on a real Hilbert space $\mathcal{H}$ and a real Borel function f on $\mathbf{R}$; to see this, one can extend S to a self-adjoint operator $S_{\mathbf{C}}$ on the complexification $\mathcal{H}_{\mathbf{C}}$ and one checks that $f(S_{\mathbf{C}})$ maps $\mathcal{H}$ to $\mathcal{H}$.

Assume by contradiction that there some spectral value $s \neq 1$ of S. Let $[a,b]$ be a closed interval of $\mathbf{R}$ containing s in its interior, and not containing 1. Let $E = \mathbf{1}_{[a,b]}(S)$ be the spectral projector of S associated with $[a,b]$. Then $E \neq 0$ and $E \in \pi(G)'$. Denote by ρ the sub-representation of π on $\mathrm{Im}(E)$. Apply $(T-1)^*$ to Equation 2:

$$(S-1)b(g) = (\pi(g)-1)(T^*-1)t.$$

Then apply E and restrict to $\mathrm{Im}(E)$:

$$(S-1)Eb(g) = (\rho(g)-1)E(T^*-1)t.$$

But $S-1$ is invertible as a bounded operator on $\mathrm{Im}(E)$ (since $1 \notin [a,b]$); denoting by R its inverse, we obtain

$$Eb(g) = (\rho(g)-1)RE(T^*-1)t.$$

The projection Eb of b on $\mathrm{Im}(E)$ is therefore bounded, contradicting condition (A3) in Proposition 2.3.

$(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (i)$ Assume that α is reducible, and let W be a non-trivial closed, invariant, affine subspace of $\mathcal{H}$. Let $E : \mathcal{H} \to W$ be the projection onto W; so Ev is the point of W closest to v, for every $v \in \mathcal{H}$. Since every $\alpha(g)$ is an isometry, it follows that the affine transformation E is in the commutant of α. □

We already observed that the first cohomology $H^1(G,\pi)$ is a module over the algebra $M := \pi(G)'$; recall that a vector ξ in a module over M is *separating* if $S\xi = 0$ implies $S = 0$ for every $S \in M$.

Corollary 3.7. *Let π be a unitary or orthogonal representation of G. There exists an irreducible affine action α with linear part π if and only if $H^1(G,\pi)$ admits a separating vector for $\pi(G)'$.*

Proof. According to Proposition 3.6, the existence of α is equivalent to the existence of a 1-cocycle b such that, for every $T \in \pi(G)'$ and $t \in \mathcal{H}$ such that $(T-1)b(g) = \partial_t(g)$ for every $g \in G$, we have $T = 1$; in turn, by Lemma 3.3, this is equivalent to the existence of a class $[b] \in H^1(G, \pi)$ such that $(T-1)[b] = 0$ for $T \in \pi(G)'$, implies $T = 1$; this exactly means that $[b]$ is a separating vector for $\pi(G)'$. □

4. Applications

4.1. **Restriction to lattices.** We give a short proof of a result of Neretin (Theorem 3.6 in [23][3]) asserting that the restriction of an irreducible affine action to a co-compact lattice, remains irreducible. Since we do not use induction of affine actions, we are able to remove the assumption of discreteness of the subgroup in [23]. In order to treat non-co-compact lattices, we introduce a definition: for H a lattice in G and $b \in Z^1(G, \pi)$, we say that the cocycle b is *integrable on G/H* if there exists a measurable fundamental domain Ω for the right action of H on G, such that $\int_\Omega \|b(g)\|\, dg < +\infty$, where dg denotes Haar measure on G.

Theorem 4.2. *Let H be a closed subgroup of the locally compact group G, such that G/H carries a G-invariant probability measure μ. Let $\alpha(g)v = \pi(g)v + b(g)$ be an affine isometric action of G on a complex or real Hilbert space. Assume either that H is co-compact or that H is discrete and the cocycle b is integrable on G/H. If α is irreducible, then the restriction $\alpha|_H$ is irreducible.*

Proof. Let $\mathcal{K}$ be a closed affine subspace of $\mathcal{H}$, which is invariant under $\alpha|_H$, and let E be the projection onto $\mathcal{K}$. We want to show that E is the identity of $\mathcal{H}$, or equivalently that its linear part E_0 is the identity. Write $Ev = E_0 v + t$ for $v \in \mathcal{H}$.

Let $\mathrm{Aff}(\mathcal{H})$ be the set of continuous affine maps from $\mathcal{H}$ to $\mathcal{H}$. Consider the map

$$G \to \mathrm{Aff}(\mathcal{H}),\ g \mapsto \alpha(g)E\alpha(g)^{-1}.$$

This map factors through G/H, and we wish to integrate it on G/H. For this, we compute (using $b(g^{-1}) = -\pi(g)^{-1}b(g)$):

$$\alpha(g)E\alpha(g)^{-1}v = \pi(g)E_0\pi(g)^{-1}v + \pi(g)t + [1 - \pi(g)E_0\pi(g)^{-1}]b(g).$$

The first two terms are bounded, and the third one is integrable on G/H under either of our assumptions. So we may define

$$A = \int_{G/H} \alpha(x)E\alpha(x)^{-1}\, d\mu(x) \tag{3}$$

[3] We seize this opportunity to correct an error in [23]: the proof of Theorem 3.6 rests on Proposition 2.5 of the same paper, which claims that, if an affine isometric action α has a closed, affine invariant subspace L such that $\alpha|_L$ is irreducible, then every closed, affine invariant subspace of α contains L: this is false, as shown by an action of $\mathbf{Z}$ by translations on the plane. It can be checked however that Neretin's proof holds for irreducible affine actions whose linear part has no non-zero fixed vector.

as an element of $\mathrm{Aff}(\mathcal{H})$. By G-invariance of μ, we see that A belongs to the commutant of α. By Proposition 3.6, the affine transformation A is a translation. Taking linear parts in Equation (3), we get $1 = \int_{G/H} \pi(x)E_0\pi(x)^{-1}\, d\mu(x)$, expressing the identity 1 on $\mathcal{H}$ as an average of operators of norm ≤ 1. Since 1 is an extreme point in the unit ball of $B(\mathcal{H})$ (see e.g. Proposition 1.4.7 in [24]), we deduce $E_0 = 1$. □

Remark 4.3. Let us take a closer look at the condition of integrability of the cocycle in the case of a non-uniform lattice Γ in G. Assume that the ambient group G is compactly generated, and denote by $|g|_S$ the word length of $g \in G$ with respect to some compact generating set $S \subset G$. If $b \in Z^1(G, \pi)$, it is an easy consequence of the triangle inequality that there exists $C > 0$ such that $\|b(g)\| \leq C|g|_S$; so, for a lattice Γ in G, a sufficient condition for every cocycle to be integrable on G/Γ is the existence of a measurable fundamental domain Ω for the right action of Γ on G such that:

$$\int_{\Omega} |g|_S \, dg < \infty \tag{4}$$

This is of course clear for uniform lattices. Margulis proves it for S-arithmetic groups in [21, Prop. VIII.1.2]. Using the Garland-Raghunathan description of cusps [12], it can be checked that this condition is also satisfied by all lattices in rank 1 simple Lie groups. It also holds for twin buildings lattices, see [6, Lemma 4.2].

Remark 4.4. Integrability of cocycles does not hold in general, as counterexamples can be found into the automorphism group $\mathrm{Aut}(T_k)$ of the k-regular tree, with $k \geq 3$. First, it is a result of Nebbia [22] that $\mathrm{Aut}(T_k)$ has a unique unitary irreducible representation σ_0 with non-zero first cohomology; moreover $H^1(\mathrm{Aut}(T_k), \sigma_0)$ is 1-dimensional. Let $b \in Z^1(\mathrm{Aut}(T_k), \sigma_0)$ defining a non-zero class in H^1. By Example 1.3, the affine isometric action $\alpha_{\sigma_0,b}$ is irreducible. By compactness of vertex-stabilizers in $\mathrm{Aut}(T_k)$, we may assume that b vanishes on the stabilizer of some vertex x_0. By Theorem 1.1 in [14], there exist constants $A, B > 0$ such that, for $g \in \mathrm{Aut}(T_k)$,

$$\|b(g)\|^2 = Ad(gx_0, x_0) - B + B(k-1)^{-d(gx_0,x_0)}.$$

The following example of a non-uniform lattice for which b fails to be integrable was shown to us by T.Gelander. Consider the graph of groups based on the infinite ray with vertices $x_0, x_1, x_2, \ldots$. Denote by Γ_n the vertex group at x_n, and H_n the edge group at the edge $[x_n, x_{n+1}]$ for $n \geq 0$. Assume that indices satisfy

$$[\Gamma_n : H_{n-1}] + [\Gamma_n : H_n] = k,$$

so that the fundamental group Γ of the graph of groups (in the sense of Bass-Serre [30]) acts on T_k. Assume now that the Γ_n's are finite groups, whose orders satisfy

$$\sum_{n=0}^{\infty} \frac{1}{|\Gamma_n|} < +\infty \quad \text{but} \quad \sum_{n=0}^{\infty} \frac{n^{1/2}}{|\Gamma_n|} = +\infty.$$

The former condition ensures that Γ sits in $\mathrm{Aut}(T_k)$ as a non-uniform lattice (see [30], Section 1.5 in Chapter II), while the latter condition implies the non-existence of a fundamental domain $\Omega \subset \mathrm{Aut}(T_k)$ on which b is integrable (this follows from the explicit form for $\|b(g)\|^2$ given above). The construction of the Γ_n's requires some care, due to the constraints on the indices of H_n and H_{n+1}. For example, assuming that k is even, one can define a sequence $(a_i)_{i\geq 0}$ of positive integers in a recursive way, by requiring

$$a_0 = 0 \quad \text{and} \quad a_i - a_{i-1} = \lfloor \frac{(k-1)^i}{i^{3/2}} \rfloor,$$

choose H_n with $|H_n| = (k-1)^i$ for $a_{i-1} \leq n < a_i$, and then choose $|\Gamma_0| = k(k-1)$, and $|\Gamma_n| = \frac{k}{2}|H_{n-1}| = \frac{k}{2}|H_n|$ for $a_{i-1} < n < a_i$, and $\Gamma_{a_i} = H_{a_i}$ for $i > 0$. All this can be realized with finite cyclic groups.

We do not know whether the restriction of $\alpha_{\sigma_0,b}$ to the lattice Γ is irreducible or not.

Remark 4.5. Let Γ be a co-compact lattice in the locally compact group G. Given an action α of Γ by affine isometries on a complex or real Hilbert space $\mathcal{H}$, it is possible to define an *induced* affine action $\mathrm{Ind}_\Gamma^G\alpha$ of G, as discussed in [31, Section II]. Let us briefly review the construction. Let π be the linear part of α and $b \in Z^1(\Gamma, \pi)$ the corresponding 1-cocycle. Let Ω be a compact fundamental domain for the right action of Γ on G and $c : G \times \Omega \to \Gamma$ the associated cocycle defined by $c(g, x) = \gamma$ if and only if $gx\gamma \in \Omega$. The induced unitary or orthogonal representation $\mathrm{Ind}_\Gamma^G\pi$ of G can be realized on $L^2(\Omega, \mathcal{H})$ by means of the formula

$$(\mathrm{Ind}_\Gamma^G\pi)(g)f(x) = \pi(c(g^{-1}, x))f(g^{-1}x) \quad f \in L^2(\Omega, \mathcal{H}),\ g \in G,\ x \in \Omega.$$

The map $\widetilde{b} : G \to L^2(\Omega, \mathcal{H})$, defined by

$$\widetilde{b}(g)(x) = b(c(g^{-1}, x)) \quad g \in G,\ x \in \Omega,$$

belongs to $Z^1(G, \mathrm{Ind}_\Gamma^G\pi)$; observe that, since Ω is compact, $\widetilde{b}$ takes indeed its values in $L^2(\Omega, \mathcal{H})$. The induced affine action $\mathrm{Ind}_\Gamma^G\alpha$ of G is the action with linear part $\mathrm{Ind}_\Gamma^G\pi$ and translation part given by $\widetilde{b}$.

One may ask whether $\mathrm{Ind}_\Gamma^G\alpha$ is irreducible when α is irreducible. This is not the case, even when Γ has finite index in G, as the following simple example shows. Let $G = C_2 \times \mathbf{Z}$ be the direct product of the cyclic group of order two and the group of integers and let $\Gamma = \mathbf{Z}$. Let α be the affine isometric action of Γ on $\mathbf{R}$ defined by

$$\alpha(n)y = y + n, \quad n \in \mathbf{Z},\ y \in \mathbf{R}.$$

So, the linear part of α is the identity and the injection $\mathbf{Z} \to \mathbf{R}$ is the corresponding cocycle. The induced affine action $\mathrm{Ind}_\Gamma^G\alpha$ of G is easily seen to be defined on $\mathbf{R}^2$ by

$$(\mathrm{Ind}_\Gamma^G\alpha)(a, n)(x) = (x, y + n) \quad n \in \mathbf{Z},\ (x, y) \in \mathbf{R}^2.$$

Clearly, $\mathrm{Ind}_\Gamma^G\alpha$ is not irreducible.

4.6. **Center and FC-center.** We denote by $Z(G)$ the center of the topological group G.

Proposition 4.7. *In an irreducible affine action α of G on a complex or real Hilbert space $\mathcal{H}$, the center $Z(G)$ acts by translations in the direction of $\mathcal{H}^{\pi(G)}$.*

Proof. This follows immediately from Proposition 3.6. □

Corollary 4.8. *Assume that* $\mathrm{Hom}(G, \mathbf{R}) = 0$. *Then every irreducible affine action α of G on a complex or real Hilbert space $\mathcal{H}$ factors through $G/Z(G)$.*

Proof. Let b be the cocycle defining α, and let b_0 be its projection on $\mathcal{H}^{\pi(G)}$, so that b_0 is a continuous homomorphism from G to the additive group of $\mathcal{H}^{\pi(G)}$, hence $b_0 \simeq 0$ by our assumption. This forces $\mathcal{H}^{\pi(G)} = 0$ (otherwise we would contradict condition (A3) in Proposition 2.3). By Proposition 4.7, the center $Z(G)$ acts by the identity. □

As a consequence, we get a very short proof of a result of J.-P. Serre (see Theorem 1.7.11 in [3]).

Corollary 4.9. *Let G be a compactly generated, locally compact group. Assume that the separated abelianization $G/\overline{[G,G]}$ is compact. Let Z be a closed central subgroup of G. If G/Z has property (T), then so does G.*

Proof. Our assumption implies that $\mathrm{Hom}(G, \mathbf{R}) = 0$. Assume by contraposition that G does not have property (T). Since G is compactly generated, the group G admits an irreducible affine action α, by Shalom's theorem ([31, Theorem 0.2]). By Corollary 4.8, this action α is actually an irreducible affine action of G/Z, which therefore does not have property (T). □

The *FC-center* of G, denoted $\mathrm{FC}(G)$, is the set of elements in G with finite conjugacy class. Observe that the conjugacy class of an element γ is finite if and only its centralizer C_γ in G has finite index in G. The FC-center is a subgroup of G which is of course characteristic.

Observe that the FC-center of any group Γ is amenable. Indeed, every finitely generated subgroup of $\mathrm{FC}(\Gamma)$ has a center of finite index and is hence amenable; it follows that $\mathrm{FC}(\Gamma)$ is a union of amenable groups and is therefore amenable.

Proposition 4.10. *Let α be an irreducible affine action of the topological group G on a complex or real Hilbert space $\mathcal{H}$. The linear part of α is trivial on the FC-center* $\mathrm{FC}(G)$ *of G; more precisely, every $\gamma \in \mathrm{FC}(G)$ acts as a translation in the direction of $\mathcal{H}^{\pi(C_\gamma)}$.*

Proof. Let $\gamma \in \mathrm{FC}(G)$. Since C_γ is a closed subgroup with finite index, by Theorem 4.2, the restriction of α to C_γ is irreducible. Hence, by Proposition 3.6, $\alpha(\gamma)$ is a translation by a vector in $\mathcal{H}^{\pi(C_\gamma)}$. □

A group G is called an *FC-group* if $G = \mathrm{FC}(G)$. The following result is an immediate consequence of Proposition 4.10.

Proposition 4.11. *Let G be an FC-group. Every irreducible affine action of G on $\mathcal{H}$ is given by a homomorphism $b : G \to \mathcal{H}$ such that* $\mathrm{span}(b(G))$ *is dense.*

We now show that a result similar to Corollary 4.9 holds for discrete groups satisfying the following property introduced in [18].

Definition 4.12. A discrete group Γ has property (FAb) if, for every subgroup H of finite index of Γ, we have $\mathrm{Hom}(H, \mathbf{R}) = 0$.

It is shown in [18, Proposition 1.30] that Γ has property (FAb) if and only if $H^1(\Gamma, \pi) = 0$ for every complex representation π of Γ with finite image.

Corollary 4.13. *Let Γ be a group with property (FAb). Then every irreducible affine action α of Γ factors through $\Gamma/\mathrm{FC}(\Gamma)$.*

Proof. The proof is similar to the proof of Corollary 4.8. □

We obtain from the previous result the following extension of Serre's result from Corollary 4.9, with a similar proof.

Corollary 4.14. *Let Γ be countable discrete group with property (FAb). If $\Gamma/\mathrm{FC}(\Gamma)$ has property (T), then so does Γ.*

4.15. **Abelian groups.** In this section, A will denote a topological abelian group, written additively. Since A is an FC-group, we have from Proposition 4.11, that every irreducible affine action of A on a Hilbert space $\mathcal{H}$ is given by a continuous homomorphism $b : A \to \mathcal{H}$ such that $\mathrm{span}(b(A))$ is dense.

Definition 4.16. (see [10]) A continuous function $Q : A \to \mathbf{R}^+$ is a *non-negative quadratic form* if $Q(x+y) + Q(x-y) = 2(Q(x) + Q(y))$ for every $x, y \in A$.

Lemma 4.17. *A continuous, non-negative function Q on A is a quadratic form if and only if there exists a complex or real Hilbert space $\mathcal{K}$ and a continuous homomorphism $\beta : A \to \mathcal{K}$ such that $Q(x) = \|\beta(x)\|^2$ for every $x \in A$.*

Proof. It is immediate to check that, if $Q(x) = \|\beta(x)\|^2$, then Q is a quadratic form. Conversely, start from a quadratic form Q, and observe that $Q(x) = Q(-x)$ and $Q(nx) = n^2 Q(x)$ for $n \in \mathbf{N}$ (the latter equality being proved by induction over n). Set

$$V := A \otimes_{\mathbf{Z}} \mathbf{K} \quad \text{and} \quad \tilde{Q}(x \otimes \lambda) = |\lambda|^2 Q(x),$$

where $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$; then $\tilde{Q}$ is a well-defined non-negative quadratic form on the complex or real vector space V, so we may define $\mathcal{K}$ as the separation-completion of V and

$$\beta : A \to V,\ x \mapsto x \otimes 1$$

does the job. Since the topology of $\mathcal{K}$ is determined by Q which is continuous, the homomorphism β is continuous by construction. □

Proposition 4.18. *Let $\alpha = \alpha_{\pi,b}$ be an affine action of A on a complex or real Hilbert space $\mathcal{H}$, with $b(A)$ total in $\mathcal{H}$. Let $\psi(\cdot) = \|b(\cdot)\|^2$. The following properties are equivalent:*

i) α is irreducible;
ii) ψ is a quadratic form.

Proof. $(i) \Rightarrow (ii)$ follows immediately from Proposition 4.11 and lemma 4.17. For $(ii) \Rightarrow (i)$, write $\psi(x) = \|\beta(x)\|^2$, with $\beta : A \to \mathcal{K}$ a continuous homomorphism in a complex or real Hilbert space $\mathcal{K}$ (depending on whether $\mathcal{H}$ is complex or real), as in Lemma 4.17. Clearly we may assume that $\beta(A)$ is total in $\mathcal{K}$. The actions α and β (viewed as actions by translations) both have total cocycles and define the same function conditionally of negative type, so they are conjugate by an A-equivariant affine isometry (see Proposition 2.10.2 in [3]). $\square$

Remark 4.19. When A is locally compact abelian, it is possible to give a proof of the implication $(i) \Rightarrow (ii)$ in Proposition 4.18, not depending on Proposition 4.11 (so that, together with Lemma 4.17, we get a direct proof of Proposition 4.11 in the case of an abelian group). Indeed, by the Levy-Khintchine formula (see Theorem 8 in [10]), ψ can be written as:

$$\psi(x) = Q(x) + \int_{\hat{A}\setminus\{1_A\}} (1 - \mathrm{Re}\chi(x))\, d\mu(\chi)$$

where Q is a quadratic form, $\hat{A}$ is the Pontryagin dual of A, and μ is a non-negative measure on $\hat{A}\setminus\{1_A\}$ that gives finite measure to the complement of any neighborhood of the unit 1_A of $\hat{A}$. If ψ is not a quadratic form, then $\mu \neq 0$. In this case, choose a point χ in the support of μ and a neighborhood V of χ which is disjoint from some neighborhood of 1_A. Set then

$$\psi_0(x) = \int_V (1-\mathrm{Re}\chi(x))\, d\mu(\chi), \quad \psi_1(x) = Q(x)+\int_{\hat{A}\setminus(\{1_A\}\cup V)} (1-\mathrm{Re}\chi(x))\, d\mu(\chi).$$

Then $\psi = \psi_0 + \psi_1$, the functions ψ_0, ψ_1 are conditionally of negative type, ψ_0 is bounded, and $\psi_0 \neq 0$ (because $\mu(V) > 0$). By condition (A4) in Proposition 2.3, the action α is reducible.

4.20. **Nilpotent groups and FC-nilpotent groups.** The following result generalizes Corollary 5 in [15], stating that for a nilpotent locally compact group, any non-trivial unitary irreducible representation has zero 1-cohomology.

Corollary 4.21. *Let G be a nilpotent group. Any irreducible affine action α of G on a complex or real Hilbert space $\mathcal{H}$ is given by a continuous homomorphism $b : G \to \mathcal{H}$ such that* $\mathrm{span}(b(G))$ *is dense.*

Proof. We proceed by induction on the nilpotency rank r of G, the case $r = 1$ being Proposition 4.11. For the general case, let α be an irreducible affine action of G, it is enough to show that π is the trivial representation, i.e. $\mathcal{H}^{\pi(G)} = \mathcal{H}$. Assume it is not the case, and let α_0 be the projected action on the orthogonal complement of $\mathcal{H}^{\pi(G)}$. By condition (A6) in Proposition 2.3, the action α_0 is irreducible. Since its linear part π_0 has no non-zero fixed vector, by Proposition 4.7 the center $Z(G)$ acts trivially in α_0, i.e. α_0 factors through $G/Z(G)$. By induction hypothesis α_0 is an action by translations, meaning that π_0 is the trivial representation of $G/Z(G)$. This contradiction ends the proof. $\square$

Denote by Q the convex cone of functions on G of the form $x \mapsto \|b(x)\|^2$, where b is a continuous homomorphism from G to the additive group of a Hilbert space (for G abelian, this is the cone of quadratic forms).

Corollary 4.22. *Let G be a nilpotent group. Then Q is the unique maximal face shared by C and C_u.*

The ascending FC-central series $(G_i)_i$ of a group G is defined inductively as follows: $G_1 = \mathrm{FC}(G)$ and G_{i+1} is the inverse image of $\mathrm{FC}(G/G_i)$ under the canonical map $G \to G/G_i$ for every $i \geq 1$. If $G_n = G$ and $G_{n-1} \neq G$, then G is said to be *FC-nilpotent* of rank n. Examples of FC-nilpotent groups include nilpotent-by-finite groups and (arbitrary) direct sums of finite groups.

Corollary 4.21 cannot be extended to the class of FC-nilpotent groups. Indeed, let G be the semi-direct product $\mathbf{Z} \rtimes C_2$, where the cyclic group $C_2 = \{\pm 1\}$ of order 2 acts on $\mathbf{Z}$ in the non trivial way. The group G is FC-nilpotent of rank 2; the affine action α of G on $\mathbf{C}$, defined by $\alpha(-1, m)x = -x + m$ for $m \in \mathbf{Z}, x \in \mathbf{C}$, is clearly irreducible and not given by a homomorphism $G \to \mathbf{C}$. Observe that the linear part of α factors though the finite quotient C_2. The next proposition is the proper generalization of this fact.

Corollary 4.23. *Let G be an FC-nilpotent and α an irreducible affine action of G on a complex or real Hilbert space $\mathcal{H}$, with linear part π. Then π can be decomposed as a direct sum $\pi = \bigoplus_i \pi$, where each π_i is a unitary or orthogonal representation of G which factors through a finite quotient of G.*

Proof. We proceed by induction on the FC-nilpotency rank r of G. When $r = 1$, the group G is an FC-group and the claim follows from Proposition 4.11.

Let $r \geq 2$. Denote by $\mathcal{K}$ be the closed linear space of $\mathcal{H}$ generated by all subrepresentations of π which factor through a finite quotient. It is clear that the restriction of π to $\mathcal{K}$ can be decomposed as a direct sum $\bigoplus_i \pi$, where each π_i is a subrepresentation of π which factors through a finite quotient of G.

The claim will be proved if we can show that $\mathcal{K} = \mathcal{H}$. Assume, by contradiction, that this is not the case. Let α_0 be the projected action on the orthogonal complement $\mathcal{H}_0$ of $\mathcal{K}$. By condition (A6) in Proposition2.3, the action α_0 is irreducible. Denote by π_0 the subrepresentation of π defined by $\mathcal{H}_0$. Observe that π_0 does not factor through a finite quotient of G.

Let $\gamma \in \mathrm{FC}(G)$. By Proposition 4.10, $\alpha_0(\gamma)$ is a translation in the direction of $\mathcal{H}_0^{\pi(C_\gamma)}$. Let N_γ be a normal subgroup of finite index of G contained in C_γ. Then $\mathcal{H}_0^{\pi(N_\gamma)}$ is a $\pi(G)$-invariant subspace of $\mathcal{H}_0$ and the corresponding subrepresentation of π_0 factors through the finite quotient G/N_γ. It follows that $\mathcal{H}_0^{\pi(N_\gamma)} = \{0\}$ and hence $\mathcal{H}_0^{\pi(C_\gamma)} = \{0\}$. So, $\alpha_0(\gamma)$ is the identity. We have therefore proved that α_0 factors through $G/\mathrm{FC}(G)$.

Observe that $G/\mathrm{FC}(G)$ is FC-nilpotent of rank $r-1$. By induction hypothesis, π_0 is a direct sum of subrepresentations which factor though finite quotients; hence, $\mathcal{H}_0 = \{0\}$ and this is a contradiction. $\square$

4.24. **The left regular representation of a discrete group.** In this subsection, all Hilbert spaces are over $\mathbf{C}$.

For a discrete group Γ, we will be interested in the question of the existence of an irreducible affine isometric action with linear part the left regular representation λ_Γ. More generally, we will consider the same question for a closed Γ-invariant subspace $\mathcal{H}$ of a countably many copies of $\ell^2(\Gamma)$; thus, $\mathcal{H}$ is a closed subspace of $\oplus_{n\in\mathbf{N}}\ell^2(\Gamma)$ which is invariant under the representation $\oplus_{n\in\mathbf{N}}\lambda_\Gamma$. Observe that such a space $\mathcal{H}$ is a Hilbert module over the left group von Neumann algebra $L(\Gamma)$ and every Hilbert module over $L(\Gamma)$ is of this form (see below).

Let $\mathcal{M}$ be finite von Neumann algebra, that is, $\mathcal{M}$ is a von Neumann algebra equipped with a faithful normal finite trace $\tau : \mathcal{M} \to \mathbf{C}$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained from τ by the GNS construction. We identify $\mathcal{M}$ with the subalgebra of $B(L^2(\mathcal{M}))$ of operators given by left multiplication with elements from $\mathcal{M}$. The commutant of $\mathcal{M}$ in $B(L^2(\mathcal{M}))$ is $\mathcal{M}' = J\mathcal{M}J$, where $J : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is the conjugate linear isometry which extends the mapping $\mathcal{M} \to \mathcal{M}, x \mapsto x^*$. The trace on $\mathcal{M}'$, again denoted by τ, is defined by $JxJ \mapsto \tau(x)$ for $x \in \mathcal{M}$.

Let $\mathcal{H}$ be a Hilbert $\mathcal{M}$-module, that is, a separable complex Hilbert space with a unital normal homomorphism $\mathcal{M} \to B(\mathcal{H})$. Then $\mathcal{H}$ can be identified as $\mathcal{M}$-module to a submodule of $L^2(\mathcal{M}) \otimes \mathcal{K}$ for an infinite dimensional separable Hilbert space $\mathcal{K}$, where $\mathcal{M}$ acts on $L^2(\mathcal{M}) \otimes \mathcal{K}$ by

$$\xi \otimes \eta \mapsto T\xi \otimes \eta, \quad T \in \mathcal{M}, \xi \in L^2(\mathcal{M}), \eta \in \mathcal{K}.$$

Let $P : L^2(\mathcal{M}) \otimes \mathcal{K} \to \mathcal{H}$ be the orthogonal projection. Then P belongs to the commutant of $\mathcal{M}$ in $B(L^2(\mathcal{M}) \otimes \mathcal{K})$, which is $\mathcal{M}' \otimes B(\mathcal{K})$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $B(L^2(\mathcal{M}))$.

Let $\{e_n\}_n$ be a Hilbert space basis of $\mathcal{K}$. Let $(P_{ij})_{i,j}$ be the matrix of P with respect to the decomposition $L^2(\mathcal{M}) \otimes \mathcal{K} = \oplus_i (L^2(\mathcal{M}) \otimes \mathbf{C}e_i)$. Then each P_{ij} belongs to $\mathcal{M}'$. The von Neumann dimension of the $\mathcal{M}$-module $\mathcal{H}$, which takes values in $[0, +\infty[\cup\{+\infty\}$, is defined by

$$\dim_{\mathcal{M}}\mathcal{H} = \sum_i \tau(P_{ii}).$$

When $\mathcal{M}$ is a factor, $\mathcal{H}$ is characterized as $\mathcal{M}$-module by its von Neumann dimension, up to unitary equivalence (see e.g. Proposition 3.2.5 in [13]).

Let Γ be a discrete countable group and λ_Γ the left regular representation of Γ on the complex Hilbert space $\ell^2(\Gamma)$. Denote by $L(\Gamma)$ the left regular von Neumann algebra of Γ. Recall that $L(\Gamma)$ is the closure of the linear span of $\{\lambda_\Gamma(\gamma) \ : \ \gamma \in \Gamma\}$ in the weak (or strong) operator topology. The commutant $L(\Gamma)'$ of $L(\Gamma)$ in $B(\ell^2(\Gamma))$ is the right group von Neumann algebra $R(\Gamma)$, the von Neumann algebra generated by the right regular representation of Γ. The algebras $L(\Gamma)$ and $R(\Gamma)$ are finite von Neumann algebras: a faithful normal trace τ on $L(\Gamma)$ or $R(\Gamma)$ is given by

$$\tau(T) = \langle T\delta_e|\delta_e\rangle, \text{ for all } T \in L(\Gamma) \quad \text{or} \quad T \in R(\Gamma).$$

Assume now that Γ is non amenable and finitely generated. By [4], there exists a $R(\Gamma)$-equivariant isomorphism between the first cohomology $H^1(\Gamma, \lambda_\Gamma)$ and the first L^2-cohomology $H^1_{(2)}(\Gamma)$; it follows that $H^1(\Gamma; \lambda_\Gamma)$ has a Hilbert space structure. The first L^2-Betti number of Γ is

$$\beta^1_{(2)}(\Gamma) = \dim_{R(\Gamma)} H^1_{(2)}(\Gamma).$$

Recall that $L(\Gamma)$ or $R(\Gamma)$ is a factor (that is, their common center consists only of the scalar multiples of the identity) if and only if Γ is ICC, i.e. every non-trivial conjugacy class in Γ is infinite; otherwise said, $\mathrm{FC}(\Gamma)$ is trivial.

The following result was initially obtained in the special case of the $L(\Gamma)$-module $\ell^2(\Gamma)$ under the additional assumption that Γ is an ICC group; we thank S. Vaes for suggesting to jack it up to arbitrary $L(\Gamma)$-modules.

Theorem 4.25. *Let Γ be a non-amenable, finitely generated group, and let $\mathcal{H}$ be a non zero Hilbert $L(\Gamma)$-module with finite von Neumann dimension. Denote by $\lambda^{\mathcal{H}}$ the corresponding unitary representation of Γ in $\mathcal{H}$. The following properties are equivalent:*

i) there exists an irreducible affine isometric action of Γ with linear part $\lambda^{\mathcal{H}}$;
ii) $\mathrm{FC}(\Gamma)$ is finite, $\mathrm{FC}(\Gamma)$ acts trivially on $\mathcal{H}$, and

$$\beta^1_{(2)}(\Gamma/\mathrm{FC}(\Gamma)) \geq \dim_{L(\Gamma/\mathrm{FC}(\Gamma))}\mathcal{H}.$$

Proof. First step: we assume that Γ is an ICC group, so that $L(\Gamma)$ is a factor.

Since $\dim_{L(\Gamma)}\mathcal{H}$ is finite, we can find an integer k such that $\mathcal{H}$ is a submodule of $\ell^2(\Gamma) \otimes \mathbf{C}^k$.

Let $P : \ell^2(\Gamma) \otimes \mathbf{C}^k \to \mathcal{H}$ be the corresponding orthogonal projection with range $\mathcal{H}$. Set $\mathcal{M} = L(\Gamma) \otimes I_{\mathbf{C}^k} \cong L(\Gamma)$. The commutant of $\mathcal{M}$ in $B(\ell^2(\Gamma) \otimes \mathbf{C}^k)$ is

$$\mathcal{M}' = R(\Gamma) \otimes B(\mathbf{C}^k) = M_k(R(\Gamma)).$$

So, we can write

$$P = (P_{ij})_{1\leq i,j\leq k} \in R(\Gamma) \otimes B(\mathbf{C}^k) = M_k(R(\Gamma))$$

and

$$\dim_{L(\Gamma)}\mathcal{H} = \sum_{i=1}^{k} \tau(P_{ii})$$

The subalgebras $\mathcal{M}P$ and $P\mathcal{M}'P$ of $B(\mathcal{H})$ are finite factors and we have $P\mathcal{M}'P = (\mathcal{M}P)'$; thus, the commutant of $\lambda^{\mathcal{H}}(\Gamma)$ is $P\mathcal{M}'P$.

Next, since Γ is not amenable, the 1-cohomology group $H^1(\Gamma, \oplus_{i=1}^k \lambda_\Gamma)$ coincides with the *reduced cohomology group* $\overline{H}^1(\Gamma, \oplus_{i=1}^k \lambda_\Gamma)$, that is, the quotient of Z^1 by the closure of B^1, for the topology of pointwise convergence of Γ ([15, Corollaire1]); moreover, we have

$$\overline{H}^1(\Gamma, \oplus_{i=1}^k \lambda_\Gamma) = \oplus_{i=1}^k \overline{H}^1(\Gamma, \lambda_\Gamma) = H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k,$$

which is a module over $\mathcal{M}'$. It follows that the 1-cohomology of $\lambda^{\mathcal{H}}$ is given by the $P\mathcal{M}'P$-module $P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k)$.

By Corollary 3.7, there exists an irreducible affine isometric action of Γ with linear part $\lambda^{\mathcal{H}}$ if an only if $P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k)$ admits a separating vector for $P\mathcal{M}'P$. Now, $\dim_{P\mathcal{M}'P} P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k)$ is the coupling constant for $P\mathcal{M}'P$ acting on $P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k)$; see [13, Proposition 3.2.5]. Hence $P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k)$ admits a separating vector for $P\mathcal{M}'P$ if only if

$$\dim_{P\mathcal{M}'P} P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k) \geq 1$$

(see [9, Chap. III, §6, Proposition 3]).

On the other hand, by [9, Chap. III, §6, Proposition 2] or [13, Proposition 3.2.5], we have

$$\dim_{P\mathcal{M}'P} P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k) \delta_{\mathcal{M}'}(P) = \dim_{\mathcal{M}'}(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k),$$

where $\delta_{\mathcal{M}'}$ is the canonical normalized trace on $\mathcal{M}' = M_k(R(\Gamma))$. We have, for every $T = (T_{ij})_{1 \leq i,j \leq k} \in M_k(R(\Gamma))$,

$$\delta_{\mathcal{M}'}(T) = \frac{1}{k} \sum_{i=1}^k \tau(T_{ii})$$

and hence

$$\delta_{\mathcal{M}'}(P) = \frac{1}{k} \dim_{L(\Gamma)} \mathcal{H}.$$

Moreover

$$\dim_{M_k(R(\Gamma))}(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k) = \frac{\dim_{R(\Gamma)} H^1_{(2)}(\Gamma)}{k} = \frac{\beta^1_{(2)}(\Gamma)}{k}.$$

We have therefore

$$\dim_{P\mathcal{M}'P} P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k) \dim_{L(\Gamma)} \mathcal{H} = \beta^1_{(2)}(\Gamma).$$

As a consequence,

$$\dim_{P\mathcal{M}'P} P(H^1_{(2)}(\Gamma) \otimes \mathbf{C}^k) \geq 1$$

if and only if $\beta^1_{(2)}(\Gamma) \geq \dim_{L(\Gamma)} \mathcal{H}$.

Second step: we assume that $\mathrm{FC}(\Gamma)$ is non trivial. Observe that $\Gamma/\mathrm{FC}(\Gamma)$ is not amenable, since $\mathrm{FC}(\Gamma)$ is amenable and Γ is not amenable.

Assume first that there exists an irreducible affine isometric action α of Γ with linear part $\lambda^{\mathcal{H}}$. By Proposition 4.10, $\lambda^{\mathcal{H}}$ is trivial on $\mathrm{FC}(\Gamma)$. Since $\lambda^{\mathcal{H}}$ is a subrepresentation of a multiple of the regular representation λ_Γ, it follows that $\mathrm{FC}(\Gamma)$ is finite. As a consequence, $\ell^2(\Gamma/\mathrm{FC}(\Gamma))$ can be identified as $L(\Gamma)$-module (or as $R(\Gamma)$-module) with the closed subspace $\ell^2(\Gamma)^{\lambda_\Gamma(\mathrm{FC}(\Gamma))}$ of $\ell^2(\Gamma)$. So, the Hilbert module $\mathcal{H}$ over $L(\Gamma)$, on which $\mathrm{FC}(\Gamma)$ acts trivially, can be identified with a Hilbert module over $L(\Gamma/\mathrm{FC}(\Gamma))$.

Since $\mathrm{FC}(\Gamma)$ is finite, it is straightforward to check that $\Gamma/\mathrm{FC}(\Gamma)$ is ICC. By the first step, it follows that

$$\beta^1_{(2)}(\Gamma/\mathrm{FC}(\Gamma)) \geq \dim_{L(\Gamma/\mathrm{FC}(\Gamma))}\mathcal{H}.$$

Conversely, assume that $\mathrm{FC}(\Gamma)$ is finite, that $\mathrm{FC}(\Gamma)$ acts trivially on $\mathcal{H}$, and that

$$\beta^1_{(2)}(\Gamma/\mathrm{FC}(\Gamma))) \geq \dim_{L(\Gamma/\mathrm{FC}(\Gamma))}\mathcal{H}.$$

It follows by the first step that there exists an irreducible affine isometric action of $\Gamma/\mathrm{FC}(\Gamma)$ with linear part given by $\lambda^{\mathcal{H}}$. This concludes the proof. □

As a corollary, we obtain a necessary condition for the existence an irreducible affine isometric action of Γ with linear part $\lambda^{\mathcal{H}}$, in terms of $\beta^1_{(2)}(\Gamma)$ and $\dim_{L(\Gamma)}\mathcal{H}$.

Corollary 4.26. *Let Γ, $\mathcal{H}$ and $\lambda^{\mathcal{H}}$ be as in Theorem 4.25. If there exists an irreducible affine isometric action of Γ with linear part $\lambda^{\mathcal{H}}$, then*

$$\beta^1_{(2)}(\Gamma) \geq \dim_{L(\Gamma)}\mathcal{H}.$$

Proof. By Theorem 4.25, the cardinality N of $\mathrm{FC}(\Gamma)$ is finite. It is easily checked that $\dim_{L(\Gamma/\mathrm{FC}(\Gamma))}\mathcal{H} = N\dim_{L(\Gamma)}\mathcal{H}$; similarly, since $H^1_{(2)}(\Gamma/\mathrm{FC}(\Gamma))$ can be identified with the $R(\Gamma)$-submodule of $H^1_{(2)}(\Gamma)$ on which $\mathrm{FC}(\Gamma)$ acts trivially, we have

$$N\beta^1_{(2)}(\Gamma) \geq \beta^1_{(2)}(\Gamma/\mathrm{FC}(\Gamma))$$

and hence, using Theorem 4.25, we obtain

$$\beta^1_{(2)}(\Gamma) \geq \dim_{L(\Gamma)}\mathcal{H}.$$

□

The following corollaries are immediate consequences of Theorem 4.25.

Corollary 4.27. *Let Γ be a non-amenable, finitely generated group such that $\mathrm{FC}(\Gamma)$ is infinite. No non-zero $L(\Gamma)$-module $\mathcal{H}$ has an irreducible affine isometric action with linear part $\lambda^{\mathcal{H}}$.*

Corollary 4.28. *For Γ a non-amenable, finitely generated ICC group, we have*

$$\beta^1_{(2)}(\Gamma) = \sup\{t \geq 0 \mid t.\lambda_\Gamma \textit{ is the linear part of an irreducible affine action}\},$$

where $t.\lambda_\Gamma$ is the underlying Γ-representation of the unique $L(\Gamma)$-module of von Neumann dimension t.

Example 4.29.

(i) The group $PSL_2(\mathbf{Z})$ is ICC and satisfies $\beta^1_{(2)}(PSL_2(\mathbf{Z})) = \frac{1}{6}$ (see Section 4 in [7]), so there exists no irreducible affine action with linear part the left regular representation.

(ii) Let $\widetilde{G}$ be the universal cover of the Lie group $G = SL_2(\mathbf{R})$ and let Γ be the inverse image in $\widetilde{G}$ of $SL_2(\mathbf{Z})$ under the covering map $\widetilde{G} \to G$. Then, since $\mathrm{FC}(\Gamma)$ is infinite, no non-zero $L(\Gamma)$-module $\mathcal{H}$ has an irreducible affine isometric action with linear $\lambda^{\mathcal{H}}$.

(iii) Let G a unimodular locally compact group and $(\pi, \mathcal{H})$ be a square-integrable irreducible unitary representation of G (see the beginning of Section 7). Let Γ be a lattice in G and assume that Γ is an ICC group. Then $\mathcal{H}$ is a Hilbert module over the von Neumann algebra $L(\Gamma)$ with von Neumann dimension given by $\dim_{L(\Gamma)} \mathcal{H} = d.\mathrm{covol}(\Gamma)$, where d is the formal dimension of π (see e.g. Theorem 3.3.2 in [13]). If Γ is non amenable (that is, if G is non amenable) and finitely generated, it follows from Theorem 4.25 that $\pi|_\Gamma$ is the linear part of an irreducible affine action of Γ if and only if

$$d.\mathrm{covol}(\Gamma) \leq \beta^1_{(2)}(\Gamma).$$

For instance, let $G = PSL_2(\mathbf{R})$ and, for $k \geq 2$, let $(\pi_k, \mathcal{H}_k)$ be the discrete series representation of G as in §17 of [29]. For $g \geq 2$, let Γ_g be the fundamental group of a closed surface of genus g, viewed as a co-compact lattice in G. Then

$$\dim_{L(\Gamma_g)} \mathcal{H}_k = d_k.\mathrm{covol}(\Gamma_g) = (k-1)(g-1).$$

Since the first L^2-Betti number of Γ_g is $2g-2$, we see that $\pi|_\Gamma$ is the linear part of an irreducible affine action if and only if $(k-1)(g-1) \leq 2g-2$, that is, if and only if $k \leq 3$ (note that this does not depend on g). This implies that $H^1(\Gamma_g, \pi_2|_{\Gamma_g})$ and $H^1(\Gamma_g, \pi_3|_{\Gamma_g})$ are non trivial; by way of contrast, it is known that $H^1(G, \pi_2)$ is one-dimensional, while $H^1(G, \pi_3) = 0$.

For the free group $\mathbf{F}_n$ on n generators $(2 \leq n \leq +\infty)$, we have $\beta^1_{(2)}(\mathbf{F}_n) = n-1$ (see [7]) and it is possible to construct explicit irreducible affine isometric actions with linear part $\lambda_{\mathbf{F}_n}$. Indeed, let $(a_i)_{1 \leq i \leq n}$ be a free generating family of $\mathbf{F}_n$. Set $b(a_1) = \delta_1$ (the characteristic function of the identity of $\mathbf{F}_n$), and $b(a_i) = 0$ for $i \geq 2$. Since $\mathbf{F}_n$ is free, we may extend uniquely b to a 1-cocycle $b \in Z^1(\mathbf{F}_n, \lambda_{\mathbf{F}_n})$. It is easily seen that, for $k \geq 0$, we have $b(a_1^k) = \sum_{i=0}^{k-1} \delta_{a_1^i}$, so that b is unbounded.

Proposition 4.30. *For b as above, the affine isometric action of $\mathbf{F}_n$ on $\ell^2(\mathbf{F}_n)$ given by $\alpha(g)v = \lambda_{\mathbf{F}_n}(g)v + b(g)$, is irreducible.*

Proof. Let $Av = Tv+t$ be an affine transformation of $\ell^2(\mathbf{F}_n)$ in the commutant of α. Then $T \in R(\mathbf{F}_n)$ and $(T-1)b(g) = \lambda_{\mathbf{F}_n}(g)t - t$ for every $g \in \mathbf{F}_n$. For $g = a_2$, we get $\lambda_{\mathbf{F}_n}(a_2)t = t$, hence $t = 0$ since a_2 has infinite order. So $(T-1)b(g) = 0$ for every g. For $g = a_1$, this gives $(T-1)\delta_1 = 0$, hence $T = 1$ since δ_1 is separating for $R(\mathbf{F}_n)$. By Proposition 3.6, the action α is irreducible. □

The situation is completely different for the regular representation of amenable groups. Indeed we have the following result due to A.Thom, who kindly gave us permission to include it here.

Theorem 4.31. *Let Γ be a discrete, amenable group. Let α be an affine isometric action of Γ, with linear part λ_Γ. For every $\varepsilon > 0$, the action α*

admits a closed, affine invariant subspace $\mathcal{H}_\varepsilon$ such that the linear part $\mathcal{H}_\varepsilon^0$ satisfies $\dim_{L(\Gamma)} \mathcal{H}_\varepsilon^0 < \varepsilon$. In particular, there is no irreducible affine action of Γ with linear part λ_Γ.

Observe that, by a result of Guichardet [15, Corollaire1], we have $H^1(\Gamma, \lambda_\Gamma) \neq 0$ for every countable amenable group Γ, so there is indeed something to be proved.

Proof. Let $b \in Z^1(\Gamma, \lambda_\Gamma)$ be the 1-cocycle defining α. We will need the ring $\mathcal{U}(\Gamma)$ of operators affiliated to the von Neumann algebra $R(\Gamma) = \lambda_\Gamma(\Gamma)'$, as introduced e.g. in [19, Chap.8]. We recall that, as Γ-modules, we have the chain of inclusions $R(\Gamma) \subset \ell^2(\Gamma) \subset \mathcal{U}(\Gamma)$. Now we appeal to a special case of Theorem 2.2 in [28]: if a group Λ has vanishing first L^2-Betti number, then $H^1(\Lambda, \mathcal{U}(\Lambda)) = 0$. This applies to Γ, by the Cheeger-Gromov vanishing theorem for amenable groups (Theorem 0.2 in [7]). This means that, viewing our cocycle $b \in Z^1(\Gamma, \ell^2(\Gamma))$ as a cocycle in $Z^1(\Gamma, \mathcal{U}(\Gamma))$, we may trivialize it and find some $f \in \mathcal{U}(G)$ such that $b(g) = \lambda_\Gamma(g)f - f$ for every $g \in \Gamma$. We now proceed as is the proof of Corollary 2.4 in [28]: given $\varepsilon > 0$, we find a projector $Q \in R(\Gamma)$ such that $Qf \in \ell^2(\Gamma)$ and $\dim_{R(\Gamma)}(1 - Q)(\ell^2(\Gamma)) < \varepsilon$. It is then easy to check (as in the proof of our Proposition 2.3) that the closed affine subspace

$$\mathcal{H}_\varepsilon := -Qf + (1 - Q)(\ell^2(\Gamma))$$

is $\alpha(\Gamma)$-invariant. □

5. Direct sums of irreducible actions

For affine isometric actions α_1, α_2 of a group G, we may consider in an obvious way the direct sum $\alpha_1 \oplus \alpha_2$. Unlike the direct sum of unitary representations, which is always reducible, it may happen that the direct sum of two affine isometric actions is irreducible. For instance, if β_1, β_2 are linearly independent homomorphisms $G \to \mathbf{C}$, then $\beta_1 \oplus \beta_2$ defines an irreducible affine isometric action of G on $\mathbf{C}^2$. On the other hand, if α is any affine isometric action of G, then $\alpha \oplus \alpha$ is not irreducible (look at the diagonal). We shall give a sufficient and necessary condition for the direct sum of two irreducible actions to be irreducible.

In order to state the main result of this section (Theorem 5.2 below) we need to clarify the notion of equivalence between affine isometric actions.

Definition 5.1. Let α_1 and α_2 be two affine isometric actions of a group G on complex (or real) Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We say that α_1 and α_2 are equivalent if they are intertwined by an invertible continuous affine mapping, that is, if there exists an invertible continuous affine mapping $A : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying:

$$A\alpha_1(g) = \alpha_2(g)A \quad \text{for all } g \in G.$$

If we write $A(\cdot) = T(\cdot)+t$ and $\alpha_i(g)(\cdot) = \pi_i(g)(\cdot)+b_i(g)$, the above definition boils down to

$$T\pi_1(g) = \pi_2(g)T \quad \text{and} \quad Tb_1(g) = b_2(g) + \pi_2(g)t - t \quad \text{for all } g \in G.$$

Since the actions are by isometries, it may seem more natural to require the intertwining in the definition of equivalence to be given by an isometric operator, in which case we would say that the actions are isometrically equivalent. To motivate our definition, one should be reminded of the similar definition for unitary representations. It is well-known that, in this case, an equivalence can always be implemented via a unitary intertwiner. This is a consequence of the fact that every invertible intertwiner can be "straightened" by replacing it with its unitary part (see e.g. [3, Appendix A.1]). However, this fails for affine isometric actions: equivalent affine actions by isometries need not be isometrically equivalent [4].

Theorem 5.2. *Let α_1, α_2 be irreducible affine isometric actions of a group G on complex (or real) Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The following properties are equivalent:*

i) $\alpha_1 \oplus \alpha_2$ is reducible.
ii) α_1 and α_2 admit equivalent projected actions.

Before proving this theorem, we pinpoint two specific cases, important enough to be considered on their own.

Recall that two unitary or orthogonal representations $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ of G are said to be *disjoint* if $\mathrm{Hom}_G(\mathcal{H}_\pi, \mathcal{H}_\sigma) = \{0\}$, where $\mathrm{Hom}_G(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ is the space of all bounded linear operators $\mathcal{H}_\pi \to \mathcal{H}_\sigma$ intertwining π and σ.

Proposition 5.3. *Let $\alpha_1, \dots, \alpha_k$ be irreducible affine actions of G on complex (or real) Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_k$, with linear parts $\pi_1, \dots, \pi_k$. Assume that the π_i's are pairwise disjoint. Then the direct sum $\alpha := \alpha_1 \oplus \cdots \oplus \alpha_k$ is irreducible.*

Proof. Let $b = (b_1, \dots, b_k)$ be the 1-cocycle defining α. Let $Av = Tv + t$ be a continuous affine mapping in the commutant of α. Write T as a $k \times k$-matrix $(T_{ij})_{1 \le i,j \le k}$ where T_{ij} is a bounded operator $\mathcal{H}_j \to \mathcal{H}_i$; similarly, write $t = (t_1, \dots, t_k)$. Since T belongs to the commutant of $\pi_1 \oplus \cdots \oplus \pi_k$, we have $T_{ij} \in \mathrm{Hom}_G(\mathcal{H}_j, \mathcal{H}_i)$ and hence $T_{ij} = 0$ for $i \neq j$. The relation $(T-1)b(g) = \partial_t(g)$ then gives

$$(T_{ii} - 1)b_i(g) = \partial_{t_i}(g) \quad \text{for } 1 \le i \le k \quad \text{and } g \in G.$$

This means that the affine map

$$\mathcal{H}_i \to \mathcal{H}_i, \; v \mapsto T_{ii}v + t_i$$

is in the commutant of α_i. Since the latter is irreducible, we get $T_{ii} = 1$ by Proposition 3.6; hence $T = 1$ and α is irreducible.

[4] As an example, consider two actions of $\mathbf{Z}$ on $\mathbf{R}$, the first one by integer translations, the second one by even translations. These actions are equivalent in our sense, but clearly they are not isometrically equivalent.

□

For π a unitary or orthogonal representation of G and $k \in \mathbf{N}$, we denote by $k \cdot \pi$ the representation $\pi \oplus \cdots \oplus \pi$ (k times).

Proposition 5.4. *Let π be an irreducible unitary representation of G on a* complex *Hilbert space $\mathcal{H}$. Let $b_1, \dots, b_k$ be elements in $Z^1(G, \pi)$ whose classes $[b_1], \dots, [b_k]$ are linearly independent in $H^1(G, \pi)$. Then the affine isometric action $\alpha = \bigoplus_{i=1}^k \alpha_{\pi, b_i}$ is irreducible.*

Proof. Let $Av = Tv + t$ be a continuous affine mapping in the commutant of α. In view of Proposition 3.6, we have to show that A is a translation, that is, $T = 1$. We know that T is in the commutant of $k \cdot \pi$ and that $(T-1)b = \partial t$, where $b = \oplus_{i=1}^k b_i$.

Write T as a $k \times k$-matrix $(T_{ij})_{1 \le i,j \le k}$, where T_{ij} is a bounded operator $\mathcal{H} \to \mathcal{H}$. Then every T_{ij} intertwines π with itself and hence $T_{ij} = \lambda_{ij} 1$ for some $\lambda_{ij} \in \mathbf{C}$, by Schur's lemma (here, we use the fact that $\mathcal{H}$ is complex). On the other hand, since

$$H^1(G, k \cdot \pi) = \underbrace{H^1(G, \pi) \oplus \cdots \oplus H^1(G, \pi)}_{k \text{ times}},$$

we have

$$(T-1) \begin{pmatrix} [b_1] \\ \vdots \\ [b_k] \end{pmatrix} = 0;$$

since the $[b_i]$'s are linearly independent, we deduce that $T = 1$.

□

Remark 5.5. Proposition 5.4 does not hold for orthogonal representations. This was pointed out to us by Y. de Cornulier who provided the following counterexample.

Let π be an irreducible orthogonal representation of a group G on a real Hilbert space $\mathcal{H}$ such that $\pi(G)' \cong \mathbf{C}$ and such that $H^1(G, \pi)$ is non trivial. Fix an unbounded 1-cocycle $b \in Z^1(G, \pi)$. Let $J \in \pi(G)'$ with $J^2 = -I$. Then $Jb \in Z^1(G, \pi)$ and $[b]$ and $[Jb]$ are linearly independent in the real vector space $H^1(G, \pi)$, since otherwise J would have a real spectral value. However, the affine action $\alpha := \alpha_{\pi, b} \oplus \alpha_{\pi, Jb}$ is reducible. Indeed, the closed proper linear subspace $\{(v, Jv) \mid v \in \mathcal{H}\}$ is $\alpha(G)$-invariant.

As a concrete example, we may take as G the semi-direct product $\mathbf{Z} \ltimes \mathbf{Z^2}$ given by action $(n, \mathbf{m}) \mapsto i^n \mathbf{m}$ of $\mathbf{Z}$ on $\mathbf{Z^2}$,where $\mathbf{Z^2}$ is viewed as subgroup of $\mathbf{C}$. The orthogonal representation of G on $\mathbf{C}$, viewed as the real Hilbert space $\mathbf{R}^2$, defined by

$$\pi(n, \mathbf{m})z = i^n z \quad \text{for all} \quad (n, \mathbf{m}) \in G, z \in \mathbf{C}$$

is irreducible. Morever, $H^1(G, \pi)$ is non trivial. Indeed, the map $b : G \to \mathbf{C}$ given by $b(n, \mathbf{m}) = \mathbf{m}$ is an unbounded 1-cocycle for π.

Proof of Theorem 5.2. Denote by π_1, b_1 and π_2, b_2 the linear and translation parts of the actions α_1 and α_2.

$(ii) \Rightarrow (i)$ There exist non zero $(\pi_1 \oplus \pi_2)(G)$-invariant closed linear subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ such that the projected actions of α_1 and α_2 on $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent. Let $A : \mathcal{K}_1 \to \mathcal{K}_2$ be a continuous affine, invertible map implementing the equivalence. Then the graph of A is a proper closed, invariant, affine subspace of the projected action of $\alpha_1 \oplus \alpha_2$ onto $\mathcal{K}_1 \oplus \mathcal{K}_2$. Hence, $\alpha_1 \oplus \alpha_2$ is reducible, by (A6) from Proposition 2.3.

$(i) \Rightarrow (ii)$ Since $\alpha_1 \oplus \alpha_2$ is reducible, we can find, by (A3) from Proposition 2.3, a non-zero closed linear subspace $\mathcal{K}$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$ which is invariant under $(\pi_1 \oplus \pi_2)(G)$ and such that the projection of $b = b_1 \oplus b_2$ on $\mathcal{K}$ is bounded. Upon conjugating $\alpha = \alpha_1 \oplus \alpha_2$ by a translation, we may assume that the projection of b on $\mathcal{K}$ is 0.

Denote by $P_i : \mathcal{K} \to \mathcal{H}_i$ the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}_i$. We may also assume that $P_i(\mathcal{K})$ is dense in $\mathcal{H}_i$ for $i = 1, 2$; indeed, otherwise we can replace α by its projected action on $\overline{P_1(\mathcal{K})} \oplus \overline{P_2(\mathcal{K})}$.

Next, observe that $\mathcal{K}$ is transverse to the $\mathcal{H}_i$'s. Indeed, if the intersection $\mathcal{K} \cap \mathcal{H}_i$ were non-zero, the projection of b_i on $\mathcal{K} \cap \mathcal{H}_i$ being bounded, this would contradict the irreducibility of α_i. So, P_1 and P_2 are injective. We can therefore consider the densely defined, unbounded, invertible closed operator $S = P_2 P_1^{-1}$ (for background about unbounded operators, see e.g. [25, Chap. 5]). Note that $\mathcal{K}$ being $(\pi_1 \oplus \pi_2)(G)$-invariant, it is immediate that the domain $\mathcal{D}(S)$ of S is $\pi_1(G)$-invariant, that its range is $\pi_2(G)$-invariant and that S intertwines the corresponding two subrepresentations of π_1 and π_2 (on non-closed subspaces!). Now, recall that, for every $g \in G$, the vector $b(g) = b_1(g) \oplus b_2(g)$ is orthogonal to $\mathcal{K}$; hence, we have

$$\langle b_1(g), v \rangle + \langle b_2(g), Sv \rangle = 0 \quad \text{for all } v \in \mathcal{D}(S).$$

This relation implies that

$$|\langle b_2(g), Sv \rangle| = |\langle b_1(g), v \rangle| \leq \|b_1(g)\| \|v\|;$$

hence $b_2(g)$ belongs to the domain of S^* and $b_1(g) = -S^* b_2(g)$ for all $g \in G$. This shows that $-S^\star$ intertwines α_2, projected on the domain of S^*, and α_1. The closed operator $-S^\star$ has a polar decomposition $-S^\star = UT$, where T is a positive self-adjoint unbounded operator on $\mathcal{H}_2$ and U is a linear isometry between $\mathcal{H}_2$ and $\mathcal{H}_1$. Let B be a bounded Borel subset of the spectrum of T with positive measure, and denote by P_B the corresponding spectral projector. Then $-S^\star P_B$ is a bounded operator and provides an equivalence between α_2 projected on $\mathrm{Im}(P_B)$ and α_1 projected on $\mathrm{Im}(S^\star P_B)$. This concludes the proof. □

6. Products and lattices in products

6.1. **Product groups.** The following result about irreducible affine actions of product groups is a consequence of a result of Shalom from [31] combined with Proposition 5.3.

Proposition 6.2. *Let $G = G_1 \times \cdots \times G_n$ be the product of non-trivial, compactly generated, locally compact groups. Let π be a unitary representation of G, not weakly containing the trivial representation, and let α be an affine isometric action of G with linear part π. The following properties are equivalent:*

i) *α is irreducible.*
ii) *$\alpha \simeq \alpha_1 \oplus \cdots \oplus \alpha_n$, where α_i is an irreducible affine action of G factoring through G_i for every $i = 1, \dots n$.*

Proof. $(i) \Rightarrow (ii)$ Set $H_i = G_1 \times \cdots \times G_{i-1} \times \{1\} \times G_{i+1} \times \cdots \times G_n$. Let $b \in Z^1(G, \pi)$ be the cocycle defining α. We appeal to a result of Shalom ([31], Theorem 3.1; this uses the assumption that π does not weakly contain the trivial representation): b is cohomologous to a sum $b_1 + \cdots + b_n$, where b_i is a cocycle factoring through G_i and taking values in the space $\mathcal{H}^{\pi(H_i)}$ of $\pi(H_i)$-fixed vectors. Upon conjugating α by a translation, we may assume that $b = b_1 + \cdots + b_n$. Denote by π_i the subrepresentation of π defined by the $\pi(G)$-invariant space $\mathcal{H}^{\pi(H_i)}$. Since π_i factors through G_i, the only possible common sub-representation of π_i and π_j for $i \neq j$ is the trivial representation, which is ruled out by the fact that π has no non-zero fixed vector. Hence, the spaces $\mathcal{H}^{\pi(H_i)}$ are pairwise orthogonal, so $b = b_1 \oplus \cdots \oplus b_n$. By irreducibility of α, we have $\mathcal{H} = \mathcal{H}^{\pi(H_1)} \oplus \cdots \oplus \mathcal{H}^{\pi(H_n)}$.

Define α_i as the projected action of α on $\mathcal{H}^{\pi(H_i)}$. By construction, $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n$ and α_i factors through G_i; finally α_i is irreducible, by (A6) from Proposition 2.3.

$(ii) \Rightarrow (i)$ Let π_i be the linear part of α_i. As above, the π_i's are pairwise disjoint representations of G, since π_i factors through G_i. So Proposition 5.3 applies, and α is irreducible. □

Corollary 6.3. *Keep notations as in Proposition 6.2. Let π be an irreducible unitary representation of G, not weakly containing the trivial representation. If $H^1(G, \pi) \neq 0$, then π factors through G_i for some $i = 1, \dots, n$.*

Proof. Let $b \in Z^1(G, \pi)$ be a cocycle which is not a coboundary. By Example 1.3, the affine action $\alpha_{\pi,b}$ is irreducible. By Proposition 6.2, we can write $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n$, where α_i factors through G_i. Let π_i be the linear part of α_i, so that $\pi = \pi_1 \oplus \cdots \oplus \pi_n$. By irreducibility of π, only one of the π_i's can be a non-zero representation. □

We note that the assumption that π does not weakly contain the trivial representation is necessary in Proposition 6.2 and Corollary 6.3. To see it, let us introduce, for a discrete group Γ, the "left-right" representation ϑ of $\Gamma \times \Gamma$ on $\ell^2(\Gamma)$, defined by:

$$(\vartheta(g,h)\xi)(x) = \xi(g^{-1}xh), \quad \xi \in \ell^2(\Gamma), g, h, x \in \Gamma.$$

We thank N. Monod for suggesting to look for irreducible affine actions of $\Gamma \times \Gamma$ with linear part ϑ.

Proposition 6.4. *Let Γ be an infinite, countable, amenable ICC group. Then ϑ is the linear part of some irreducible affine action of $\Gamma \times \Gamma$, which can be chosen to have almost fixed points.*

Proof. Since Γ is amenable and infinite, the representation ϑ almost has invariant vectors but no non-zero fixed vector. Hence the space $B^1(\Gamma \times \Gamma, \vartheta)$ is not closed in $Z^1(\Gamma \times \Gamma, \vartheta)$, by [15, Corollaire1] (note that countability is used here). Choose a cocycle b in the closure of B^1 but not in B^1. Then the corresponding affine action $\alpha_{\vartheta,b}$ almost has fixed points. Finally, note that ϑ is an irreducible representation of $\Gamma \times \Gamma$, as Γ is ICC. So $\alpha_{\vartheta,b}$ is irreducible, by Example 1.3. □

This must be contrasted with Theorem 4.31 above, which deals with the left regular representation of an amenable group.

6.5. **A super-rigidity result.** We now reach a super-rigidity result for lattices in a product of locally compact groups.

Theorem 6.6. *Let $G = G_1 \times \cdots \times G_n$ be the product of non-trivial, compactly generated, locally compact groups, and let Γ be a lattice in G, projecting densely to all factors. Assume that either Γ is co-compact, or that every G_i is the group of K_i-points of an almost K_i-simple, K_i-isotropic linear algebraic group over some local field K_i. Let π be a unitary representation of Γ, not weakly containing the trivial representation, and let α be an affine isometric action of Γ with linear part π. The following properties are equivalent:*

i) α is irreducible.

ii) For every $i = 1, \ldots, n$, there exists an irreducible affine action α_i of G, with α_i factoring through G_i, such that $\alpha \simeq (\bigoplus_{i=1}^n \alpha_i)|_\Gamma$.

Proof. $(ii) \Rightarrow (i)$ follows by induction over n, combining Proposition 6.2 with Theorem 4.2 (and appealing to Remark 4.3 in the non-co-compact case).

$(i) \Rightarrow (ii)$ Let $b \in Z^1(\Gamma, \pi)$ be the cocycle defining α. By a result of Shalom ([31], Corollary 4.2, using the assumption that π does not weakly contain the trivial representation): b is cohomologous to a sum $b_1 + \cdots + b_n$, where b_i takes values in a $\pi(\Gamma)$-invariant subspace $\mathcal{H}_i \subset \mathcal{H}$; moreover, denoting by σ_i the restriction of π to $\mathcal{H}_i$, the affine action α_{σ_i,b_i} extends continuously to an affine action α_i of G that factors through an action of G_i.

As in the proof of Proposition 6.2, conjugating α by a translation we may assume $b = b_1 + \cdots + b_n$, from which we deduce $\alpha = (\alpha_1 \oplus \cdots \oplus \alpha_n)|_\Gamma$. Since α_{σ_i,b_i} is a projected action of α, it is an irreducible action of Γ. Finally, since $\alpha_i|_\Gamma = \alpha_{\sigma_i,b_i}$ and the projection of Γ to G_i is dense, α_i is an irreducible action of G. □

Corollary 6.7. *Keep notations as in Theorem 6.6. Let π be a unitary irreducible representation of Γ, not containing weakly the trivial representation. If $H^1(\Gamma, \pi) \neq 0$, then for some $i = 1, \ldots, n$ the representation π extends to a unitary irreducible representation σ_i of G factoring through G_i. Moreover the restriction map $H^1(G, \sigma_i) \to H^1(\Gamma, \pi)$ is an isomorphism.*

Proof. The first statement is obtained from Theorem 6.6 exactly as the same way as Corollary 6.3 was obtained from Proposition 6.2. It also shows surjectivity of the restriction map $H^1(G, \sigma_i) \to H^1(\Gamma, \pi)$. Injectivity follows immediately from density of the projection of Γ in G_i. □

Example 6.8.

i) Let p be a prime number. The group $PSL_2(\mathbf{Q}_p)$ has a unique unitary irreducible representation σ with non-vanishing H^1 (it is the representation on the first L^2-cohomology of the Bruhat-Tits tree); similarly $PSL_2(\mathbf{R})$ has two unitary irreducible representations π_+, π_- with non-vanishing H^1 (these are the representations on square-integrable holomorphic and anti-holomorphic 1-forms on the Poincaré disk); for all this, see [5]. Viewing $\Gamma_p := PSL_2(\mathbf{Z}[\frac{1}{p}])$ as a lattice in $PSL_2(\mathbf{Q}_p) \times PSL_2(\mathbf{R})$, we see from Corollary 6.7 that Γ_p has exactly three irreducible unitary representations, not weakly containing the trivial representation, with non-vanishing H^1, namely the restrictions of σ, π_+, π_- to Γ_p.

   Similarly, viewing $\Lambda_p := PSL_2(\mathbf{Z}[\sqrt{p}])$ as a lattice in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{R})$, we see that Λ_p has exactly four unitary irreducible representations, not weakly containing the trivial representation, with non-vanishing H^1: namely, $\pi_+|_{\Lambda_p}$, $\pi_-|_{\Lambda_p}$, $\pi_+ \circ \tau$, $\pi_- \circ \tau$, where $\tau : a + b\sqrt{p} \mapsto a - b\sqrt{p}$ is the non-trivial element of the Galois group $\mathrm{Gal}(\mathbf{Q}(\sqrt{p})/\mathbf{Q})$.

ii) Consider the quadratic form Q in 5 variables, defined over $\mathbf{Q}(\sqrt{2})$:

$$Q(x) = x_1^2 + x_2^2 + x_3^2 + \sqrt{2}x_4^2 - x_5^2.$$

   Set $\Gamma = SO_0(Q)(\mathbf{Z}[\sqrt{2}])$, and view it as a lattice in $G = SO_0(Q)(\mathbf{R}) \times SO_0(\tau Q)(\mathbf{R})$, where as above τ denotes the non-trivial element of the Galois group $\mathrm{Gal}(\mathbf{Q}(\sqrt{2})/\mathbf{Q})$. As a Lie group G is isomorphic to $SO_0(4,1) \times SO_0(3,2)$, the latter factor having property (T), the former not. Actually it is known (see [5]) that $SO_0(4,1)$ has a unique irreducible unitary representation π with non-zero H^1. By Corollary 6.7, the group Γ has a unique irreducible unitary representation, not weakly containing the trivial representation, with non-zero H^1: it is $\pi|_\Gamma$.

7. On the first L^2-Betti number of a locally compact group

Let G be a unimodular, locally compact group with Haar measure dg. Recall that a unitary irreducible representation $(\sigma, \mathcal{H}_\sigma)$ of G is *square-integrable* if

$$\int_G |\langle\sigma(g)\xi|\xi\rangle|^2 dg < \infty \text{ for all } \xi \in \mathcal{H}_\sigma.$$

This is the case if and only if σ is a subrepresentation of the left regular representation $(\lambda_G, L^2(G))$ of G. Indeed, there exists a constant $d_\sigma > 0$, called the *formal dimension* of σ, such that the orthogonality relations hold

$$\int_G \langle\sigma(g)\xi|\eta\rangle\overline{\langle\sigma(g)\xi'|\eta'\rangle}dg = d_\sigma^{-1}\langle\xi|\xi'\rangle\overline{\langle\eta|\eta'\rangle} \text{ for all } \xi, \xi', \eta, \eta' \in \mathcal{H}_\sigma$$

For every unit vector $\xi_0 \in \mathcal{H}_\sigma$, the G-equivariant map $L : \mathcal{H}_\sigma \to L^2(G)$ given by $L\eta(g) = \sqrt{d_\sigma}\langle \pi(g^{-1})\eta|\xi_0\rangle$, is isometric and identifies $\mathcal{H}_\sigma$ with a $\lambda_G(G)$-invariant closed subspace of $L^2(G)$.

We denote by $\hat{G}_d$ the *discrete series* of G, i.e. the set of equivalence classes of square-integrable representations. Let Γ be a lattice in G.

Fix $\sigma \in \widehat{G}_d$ with Hilbert space $\mathcal{H}_\sigma$. The restriction of σ to Γ extends to $L(\Gamma)$ so that $\mathcal{H}_\sigma$ is a Hilbert module over $L(\Gamma)$. As such, $\mathcal{H}_\sigma$ has a von Neumann dimension $\dim_{L(\Gamma)}\mathcal{H}_\sigma$ (see Subsection 4.24). This dimension is given by Atiyah-Schmid's formula from [1] (see also [13, Theorem 3.3.2]):

$$\dim_{L(\Gamma)} \mathcal{H}_\sigma = d_\sigma \mathrm{covol}(\Gamma).$$

As in Subsection 4.24, set

$$\beta^1_{(2)}(\Gamma) = \dim_{R(\Gamma)} H^1_{(2)}(\Gamma),$$

Theorem 7.1. *Let G be separable, compactly generated, locally compact group containing a finitely generated lattice Γ satisfying condition (4) from Remark 4.3 (e.g., Γ co-compact). Assume that G is not amenable. Then*

$$\beta^1_{(2)}(\Gamma) \geq \mathrm{covol}(\Gamma) \sum_{\sigma\in\hat{G}_d} d_\sigma \cdot \dim_{\mathbf{C}} H^1(G,\sigma).$$

Proof. It is enough to prove that, for every finite subset F of $\hat{G}_d$ and integers k_σ with $k_\sigma \leq \dim_{\mathbf{C}} H^1(G,\sigma)$ for $\sigma \in F$, we have

$$\beta^1_{(2)}(\Gamma) \geq \mathrm{covol}(\Gamma) \sum_{\sigma\in F} k_\sigma d_\sigma.$$

Choose 1-cocycles $b_1, \ldots, b_{k_\sigma}$ such that the classes $[b_1], \ldots, [b_{k_\sigma}]$ are linearly independent in $H^1(G,\sigma)$ and form the affine isometric action

$$\alpha = \bigoplus_{\sigma\in F}(\oplus_{i=1}^{k_\sigma}\alpha_{\sigma,b_i});$$

Propositions 5.4 and 5.3 implies that the affine action α is irreducible.

By Theorem 4.2, the restriction $\alpha|_\Gamma$ is irreducible. Moreover, Γ is non amenable as G is non amenable. Hence, by Corollary 4.26 combined with the Atiyah-Schmid formula from above, we have

$$\beta^1_{(2)}(\Gamma) \geq \sum_{\sigma\in F} k_\sigma \dim_{L(\Gamma)}\mathcal{H}_\sigma = \mathrm{covol}(\Gamma) \sum_{\sigma\in F} k_\sigma d_\sigma. \tag{5}$$ □

Let G be a second countable, locally compact unimodular group with Haar measure dg. Denote by $L(G)$ the group von Neumann algebra of G; it carries a semi-finite trace ψ defined on the positive cone of $L(G)$ by $\psi(x^*x) = \int_G |f(g)|^2\, dg$, where x is left convolution by $f \in L^2(G)$; note that ψ depends on the choice of the Haar measure on G.

In two papers [26, 16], Petersen and Kyed-Petersen-Vaes extended the classical definition of L^2-Betti numbers for discrete groups [7] to that more general framework, by setting

$$\beta^n_{(2)}(G) := \dim_{(L(G),\psi)} H^n(G, \lambda_G)$$

where λ_G denotes the left regular representation on $L^2(G)$, and $\dim_{(L(G),\psi)}$ denotes the von Neumann dimension of $L(G)$-modules with respect to the semi-finite trace ψ. They established a number of important results; in particular $\beta^1_{(2)}(G) < \infty$ as soon as G is compactly generated, and

$$\beta^n_{(2)}(G) = \frac{\beta^n_{(2)}(\Gamma)}{\mathrm{covol}(\Gamma)}$$

for every lattice Γ in G.

Recall that a locally compact group which contains a lattice is unimodular.

Theorem 7.2. *Let G be a second countable, compactly generated, locally compact group. Assume that G contains a finitely generated lattice satisfying condition (4) from Remark 4.3 (e.g. a co-compact lattice). Then*

$$\beta^1_{(2)}(G) \geq \sum_{\sigma \in \hat{G}_d} d_\sigma \cdot \dim_{\mathbf{C}} H^1(G, \sigma).$$

Proof. When G is not amenable, the inequality is a direct consequence of Theorem 7.1 and the formula linking $\beta^1_{(2)}(G)$ and $\beta^n_{(2)}(\Gamma)$ from [26, 16].

So we may assume that G is amenable. We claim that both sides of the equality are zero. The vanishing of $\beta^1_{(2)}(G)$ follows from Theorem C in [16].

Now we check that $H^1(G, \sigma) = 0$ for every $\sigma \in \hat{G}_d$. By (2.10) in [16], the vanishing of $\beta^1_{(2)}(G)$ implies that the reduced first cohomology group $\overline{H}^1(G, \lambda_G)$ is trivial. Since σ is a subrepresentation of λ_G, we get $\overline{H}^1(G, \sigma) = 0$.

Assume first that σ is not the trivial representation 1_G. Since σ is square-integrable, it defines a closed point in the dual $\hat{G}$. So, σ does not weakly contain 1_G and hence $B^1(G, \sigma)$ is closed in $Z^1(G, \sigma)$, by [15, Théorème 1]; therefore $H^1(G, \sigma) = 0$.

On the other hand, if σ is the trivial representation 1_G, then G must be compact and therefore $H^1(G, 1_G) = \mathrm{Hom}(G, \mathbf{C}) = 0$. □

Remark 7.3. The proof of Theorem 7.2 shows that the conclusion of Theorem 7.1 holds also in the case where G is amenable.

Corollary 7.4. *Keep the assumptions of Theorem 7.2. If $\beta^1_{(2)}(G) = 0$, then $H^1(G, \sigma) = 0$ for all $\sigma \in \hat{G}_d$.*

Corollary 7.5. *Let $X_{k,\ell}$ be the (k, ℓ)-biregular tree ($k = \ell$ being allowed). Let G be a closed non-compact subgroup of $\mathrm{Aut}(X_{k,\ell})$, acting transitively on the boundary $\partial X_{k,\ell}$ and with two orbits on vertices of $X_{k,\ell}$. Normalize Haar measure on G so that edge stabilizers have measure 1. Let σ_0 be the unique irreducible, square-integrable representation of G with non-vanishing H^1 (see [22]). Then $1 - \frac{1}{k} - \frac{1}{\ell} \geq d_{\sigma_0}$.*

Proof. First, G contains a co-compact lattice (by Theorem 3.10 in [2]), so we may apply Theorem 7.2:

$$\beta^1_{(2)}(G) \geq d_{\sigma_0} \dim_{\mathbf{C}} H^1(G, \sigma_0).$$

Second, $\beta^1_{(2)}(G) = 1 - \frac{1}{k} - \frac{1}{\ell}$ by Corollary 5.18 in [26]. Third, $\dim_{\mathbf{C}} H^1(G, \sigma_0) = 1$ by the main Theorem in [22]. $\square$

Remark 7.6. Theorem 7.2 served as motivation for the main result in [27]: for G a type I, unimodular, separable, locally compact group:

$$\beta^n_{(2)}(G) = \sum_{\sigma \in \hat{G}_d} d_\sigma \cdot \dim_{\mathbf{C}} H^n(G, \sigma) + \int_{\hat{G} \setminus \hat{G}_d} \dim_{\mathbf{C}} \overline{H}^n(G, \omega)\, d\mu(\omega),$$

where μ is the Plancherel measure on the dual $\hat{G}$ of G, and $\overline{H}^n$ denotes reduced n-cohomology. The proof, of operator-algebraic nature, is completely different. Observe the different sets of assumptions: type I in [27], existence of a suitable lattice in Theorem 7.2 above.

For infinite discrete groups, Theorem 7.2 just gives $\beta^1_{(2)}(G) \geq 0$, since $\hat{G}_d$ is empty in this case. On the other hand, the computations in [27] show that equality may occur either in Theorem 7.2 or in Corollary 7.5, with the right hand side being non-zero: this is the case for $PSL_2(\mathbf{R})$, $PSL_2(\mathbf{Q}_p)$ and for $\mathrm{Aut}(X_{k,\ell})$. Actually it follows from [27] that equality holds in Corollary 7.5, under the extra assumption that G is type I. It is an open question whether a group satisfying the assumptions of Corollary 7.5 must be type I.

8. Comparison with other forms of irreducibility

In [8], the authors study orbits of affine isometric actions, and make the following definitions.

Definition 8.1.

i) An affine isometric action α of a group G on a complex or real Hilbert space $\mathcal{H}$ has enveloping orbits if the closed convex hull of every orbit is equal to $\mathcal{H}$.
ii) A unitary or orthogonal representation π of G is strongly cohomological if $H^1(G, \sigma) \neq 0$ for every non-zero sub-representation σ of π.

It is observed in Lemma 4.3 of [8] that the linear part of an action with enveloping orbits is strongly cohomological. We notice that irreducibility lies in between having enveloping orbits and having a strongly cohomological linear part.

Proposition 8.2. *Let π be a unitary or orthogonal representation of G. Every of following properties implies the next one:*

(i) There exists an affine isometric action with linear part π and with enveloping orbits.
(ii) There exists an irreducible affine isometric action with linear part π.
(iii) π is strongly cohomological.

Proof. $(i) \Rightarrow (ii)$ follows from the definitions that, if an affine isometric action has enveloping orbits, then it is irreducible.

$(ii) \Rightarrow (iii)$ follows from $(A1) \Rightarrow (A3)$ in Proposition 2.3. □

Let us check that none of the converse implications in Proposition 8.2 holds.

Example 8.3. Let $F = \mathbf{F}_2$ be the free group on 2 generators. As observed in Remark 3.5 of [8], every representation of G is strongly cohomological. Now let π be the trivial representation of G on a Hilbert space with dimension > 2. There is no irreducible affine isometric action with linear part π.

The following example, suggested by Y. Cornulier, shows that the converse of $(i) \Rightarrow (ii)$ in Proposition 8.2 does not hold in infinite dimension. We denote by $\mathrm{Sym}(\mathbf{N})$ the full symmetric group of $\mathbf{N}$ (viewed as a discrete group), and by $C_2^{(\mathbf{N})}$ the direct sum of countably many copies of the cyclic group C_2 of order 2. Note that $\mathrm{Sym}(\mathbf{N})$ acts on $C_2^{(\mathbf{N})}$ by permutation of the indices.

Proposition 8.4. *Let G be the semi-direct product $C_2^{(\mathbf{N})} \rtimes \mathrm{Sym}(\mathbf{N})$. Then G admits an irreducible orthogonal representation $(\pi, \mathcal{H})$ and an unbounded 1-cocycle $b \in Z^1(G, \pi)$ such that, for a dense set of vectors $w \in \mathcal{H}$, the function $g \mapsto \langle b(g)|w\rangle$ is bounded on G (so that $\overline{\mathrm{conv}(b(G))} \neq \mathcal{H}$).*

Proof. We identify C_2 with the multiplicative group $\{\pm 1\}$, and $C_2^{(\mathbf{N})}$ with the group of finitely supported functions $\mathbf{N} \to \{\pm 1\}$. Let $\mathcal{F}$ be the space of all real-valued sequences on $\mathbf{N}$, and $\mathcal{H} = \ell^2$ be the subspace of square-summable sequences. Then $C_2^{(\mathbf{N})}$ acts on $\mathcal{F}$ by pointwise multiplication, and $\mathrm{Sym}(\mathbf{N})$ acts on $\mathcal{F}$ by permutation of the indices. Let σ be the corresponding linear representation of G on $\mathcal{F}$. The subspace $\mathcal{H}$ is invariant, and we denote by π the restriction of σ to $\mathcal{H}$. The proof of the proposition will be carried out in four steps.

(i) Clearly, the only $\sigma(G)$-fixed vector in $\mathcal{F}$ is 0.

(ii) The representation π is irreducible. Actually $\pi|_{\mathrm{Sym}(\mathbf{N})}$ is already irreducible. Indeed, by transitivity of the action of $\mathrm{Sym}(\mathbf{N})$ on $\mathbf{N}$, we can identify (in a $\mathrm{Sym}(\mathbf{N})$-equivariant way) $\mathbf{N}$ with $\mathrm{Sym}(\mathbf{N})/\mathrm{Sym}(\mathbf{N})_0$, where $\mathrm{Sym}(\mathbf{N})_0$ is the stabilizer of 0 in $\mathrm{Sym}(\mathbf{N})$. So π is equivalent to the quasi-regular representation on $\ell^2(\mathrm{Sym}(\mathbf{N})/\mathrm{Sym}(\mathbf{N})_0)$. Now observe that $\mathrm{Sym}(\mathbf{N})_0$ is equal to its commensurator in $\mathrm{Sym}(\mathbf{N})$; indeed, for $g \in \mathrm{Sym}(\mathbf{N}) \backslash \mathrm{Sym}(\mathbf{N})_0$, the subgroup $\mathrm{Sym}(\mathbf{N})_0 \cap g\mathrm{Sym}(\mathbf{N})_0 g^{-1}$ is the stabilizer of $g(0)$ in $\mathrm{Sym}(\mathbf{N})_0$, so it has infinite index as $\mathrm{Sym}(\mathbf{N})_0$ acts transitively on $\mathbf{N} \backslash \{0\}$. Irreducibility then follows from Mackey's classical criterion for irreducibility of induced representations from self-commensurating subgroups [20].

(iii) Let $v = (1, 1, 1, \dots)$ be a constant sequence in $\mathcal{F}$. Form the affine action $t_v \circ \sigma \circ t_{-v}$. The associated 1-cocycle is $b(g) = v - \sigma(g)v$, which is 0 if $g \in \mathrm{Sym}(\mathbf{N})$ and has finite support if $g \in C_2^{(\mathbf{N})}$. In particular, this affine action preserves $\mathcal{H}$ and induces on it an affine action α. Since v is the only fixed point of $t_v \circ \sigma \circ t_{-v}$ (as seen above) and $v \notin \mathcal{H}$, we see that α has no fixed point, i.e. b is unbounded as a map $G \to \mathcal{H}$. Note also that α is irreducible, since π is.

(iv) Observe that $b(G)$ is the set of sequences consisting of 0's and 2's, with finitely many 2's. View ℓ^1 as a dense subspace of ℓ^2. For $w \in \ell^1$ and $g \in G$, we have $|\langle b(g)|w\rangle| = |\sum_{n=0}^{\infty} b(g)_n w_n| \leq 2\|w\|_1$. □

It turns out that, in Proposition 8.2, the converse of $(ii) \Rightarrow (i)$ holds in *finite* dimension.

Proposition 8.5. *Let α be an affine isometric action of a group G on $\mathbf{R}^n$. If α is irreducible, then α has enveloping orbits.*

Proof. We first observe that the result trivially holds for $n = 1$, since by irreducibility $\alpha(G)$ must contain a non-zero translation. Now, proceeding by contradiction, let n be the smallest integer such that there exists an irreducible affine isometric action α on $\mathbf{R}^n$, with the property that for some orbit $\alpha(G)x_0$, the closed convex set $C := \overline{\mathrm{conv}(\alpha(G)x_0)}$ is not equal to $\mathbf{R}^n$. Then C is contained in some closed affine half-space $\{x \in \mathbf{R}^n : \langle x|w\rangle \leq a\}$, for some unit vector $w \in \mathbf{R}^n$ and some $a \in \mathbf{R}$. As C is unbounded, it contains some half-line $D = x_0 + \mathbf{R}^+.v_0$, where v_0 is some unit vector, such that $\langle w|v_0\rangle \leq 0$. Since $\alpha(g)D \subset C$ for every $g \in G$, we have similarly $\langle w|\pi(g)v_0\rangle \leq 0$ for every $g \in G$. Now two cases may occur:

- $\langle w|\pi(g)v_0\rangle < 0$ for some $g \in G$. Let K be the closure of $\pi(G)$ in the orthogonal group $O(n)$. So K is a compact group, with normalized Haar measure dk. Set $v = \int_K k.v_0\, dk$: then $v \neq 0$ since $\langle w|v\rangle = \int_K \langle w|k.v_0\rangle\, dk < 0$ (as the integrand is < 0 on a neighborhood of $\pi(g)$). So v is a non-zero $\pi(G)$-fixed vector. Let then α_0 be the projected action on the 1-dimensional subspace $V = \mathbf{R}.v$; the action α_0 is irreducible by Proposition 2.3. On the other hand, the projection of $\alpha(G)x_0$ is contained in a half-line of V, contradicting the result for $n = 1$.
- $\langle w|\pi(g)v_0\rangle = 0$ for every $g \in G$. Let then V_0 be the $\pi(G)$-invariant subspace spanned by the $\pi(g)v_0$'s, let V_1 be the orthogonal of V_0, and let π_1 be the restriction of π to V_1. Let α_1 be the projected action on V_1. By Proposition 2.3, α_1 is irreducible, so it has enveloping orbits by minimality of n. On the other hand the projection of $\alpha(G)x_0$ on V_1 is contained in a closed affine half-space, a contradiction. □

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References

[1] M. Atiyah and W. Schmid. *A geometric construction of the discrete series for semisimple Lie groups.* Invent. Math. 42, 1–62 (1977).
[2] H. Bass and A. Lubotzky. *Tree lattices* Birkhäuser, 2001

[3] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's property (T).* Cambridge University Press, 2008.

[4] B. Bekka and A. Valette. *Group cohomology, harmonic functions and the first l^2-Betti number.* Potential analysis 6, 313–326 (1997).

[5] A. Borel and N. R. Wallach. *Continuous cohomology, discrete subgroups and representations of reductive groups.* Princeton Univ. Press 1980.

[6] P.-E. Caprace and B. Rémy. *Non-distortion of twin building lattices.* Geom. Dedicata. 147, 397–408 (2010).

[7] J. Cheeger and M. Gromov. *L_2-cohomology and group cohomology.* Topology 25, 189–215 (1986).

[8] Y. Cornulier, R. Tessera and A. Valette. *Isometric group actions on Hilbert spaces: structure of orbits.* Canadian J. Math. 60, 1001–1009 (2008).

[9] J. Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann).* Gauthier-Villars, 1969.

[10] G. Forst. *The Lévy-Hinčin representation of negative definite functions.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 34, 313–318 (1976).

[11] D. Gaboriau. *Invariants ℓ^2 de relations d'équivalence et de groupes.* Publ. Math. Inst. Hautes Etudes Sci. 95, 93–150 (2002).

[12] H. Garland and M.S. Raghunathan. *Fundamental domains for lattices in $\mathbf{R}$-rank 1 semisimple Lie groups.* Ann. of Math. 92, 279–326 (1970).

[13] F. M Goodman, P. de la Harpe, and V. Jones. *Coxeter graphs and towers of algebras.* Springer, 1989

[14] A. Gournay and P.-N. Jolissaint. *Functions conditionally of negative type on groups acting on regular trees*, Preprint, Oct. 2014.

[15] A. Guichardet. *Sur la cohomologie des groupes topologiques, II.* Bull. Sc. Math. 96, 305–332 (1972).

[16] D. Kyed, H.D. Petersen and S. Vaes. *L^2-Betti numbers of locally compact groups and their cross section equivalence relations.* Preprint, arXiv:1302.6753.

[17] N. Louvet, Y. Stalder, and A. Valette. *Fonctions conditionnellement de type négatif, représentations irréductibles et propriété (T).* L'Enseign. Math. 50, 239-266 (2004).

[18] A. Lubotzky, A. Zuk. *On Property (τ).* Forthcoming book, 2005.

[19] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory.* Springer-Verlag, 2002.

[20] G.W. Mackey. *The theory of unitary group representations.* The University of Chicago Press, 1976.

[21] G.A. Margulis. *Discrete subgroups of semisimple Lie groups.* Springer-Verlag, 1991.

[22] C. Nebbia, *Cohomology for groups of isometries of regular trees,* Expositiones Mathematicae 30, 1–10 (2012).

[23] Y.A. Neretin. *Notes on affine isometric actions of discrete groups.* Analysis on infinite-dimensional Lie groups and algebras (Marseille, 1997), 274–320, World Sci. Publ., River Edge, NJ, 1998.

[24] G.K. Pedersen. *C^*-algebras and their automorphism groups.* Academic Press, 1979.

[25] G.K. Pedersen. *Analysis now.* Springer-Verlag, 1989.

[26] H.D. Petersen. *L^2-Betti Numbers of Locally Compact Groups.* Thesis, U. Copenhagen, arXiv:1104.3294

[27] H.D. Petersen and A. Valette. *L^2-Betti numbers and Plancherel measure.* Preprint, arXiv:1307.0379, to appear in J.Funct. Anal.

[28] J. Peterson and A. Thom. *Group cocycles and the ring of affiliated operators.* Invent. Math. 185, 561–592 (2011).

[29] A. Robert. *Introduction to the representation theory of locally compact groups.* London Math. Soc. LNS 80, Cambridge University Press 1983.

[30] J-P. Serre. *Arbres, amalgames,* SL_2. Astérisque **46**, SMF, 1977.

[31] Y. Shalom. *Rigidity of commensurators and irreducible lattices.* Invent. Math.141, 1–54 (2000).

[32] A.M. Vershik and S.I. Karpushev. *Cohomology of groups in unitary representations, the neighbourhood of the identity and conditionally positive definite functions.* Math. USSR Sbornik, 47, 513–526 (1984).

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