On the spectral theory of groups of affine transformations of compact nilmanifolds

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Abstract

Let N be a connected and simply connected nilpotent Lie group, Λ a lattice in N, and $\Lambda\backslash N$ the corresponding nilmanifold. Let $\mathrm{Aff}(\Lambda\backslash N)$ be the group of affine transformations of $\Lambda\backslash N$.

We characterize the countable subgroups H of $\mathrm{Aff}(\Lambda\backslash N)$ for which the action of H on $\Lambda\backslash N$ has a spectral gap, that is, such that the associated unitary representation U^0 of H on the space of functions from $L^2(\Lambda\backslash N)$ with zero mean does not weakly contain the trivial representation. Denote by T the maximal torus factor associated to $\Lambda\backslash N$. We show that the action of H on $\Lambda\backslash N$ has a spectral gap if and only if there exists no proper H-invariant subtorus S of T such that the projection of H on $\mathrm{Aut}(T/S)$ has an abelian subgroup of finite index.

We first establish the result in the case where $\Lambda\backslash N$ is a torus. In the case of a general nilmanifold, we study the asymptotic behaviour of matrix coefficients of U^0 using decay properties of metaplectic representations of symplectic groups. The result shows that the existence of a spectral gap for subgroups of $\mathrm{Aff}(\Lambda\backslash N)$ is equivalent to strong ergodicity in the sense of K. Schmidt. Moreover, we show that the action of H on $\Lambda\backslash N$ is ergodic (or strongly mixing) if and only if the corresponding action of H on T is ergodic (or strongly mixing).

1 Introduction

Let H be a countable group acting measurably on a probability space (X,ν) by measure preserving transformations. Let $U : h \mapsto U(h)$ denote the corresponding Koopman representation of H on $L^2(X,\nu)$. We say that the action of H on X has a spectral gap if the restriction U^0 of U to the H-invariant subspace

$$L^2_0(X,\nu) = \{\xi \in L^2(X,\nu) \ : \ \int_X \xi(x)d\nu(x) = 0\}$$

does not have almost invariant vectors, that is, there is no sequence of unit vectors ξ_n in $L^2_0(X,\nu)$ such that $\lim_n \|U^0(h)\xi_n - \xi_n\| = 0$ for all $h \in H$. A useful equivalent condition for the existence of a spectral gap is as follows. Let μ be a probability measure on H such that the support of μ generates H. Let $U^0(\mu)$ be the convolution operator defined on $L^2_0(X,\nu)$ by

$$U^0(\mu)\xi = \sum_{h\in H} \mu(h)U^0(h)\xi, \qquad \xi \in L^2_0(X,\nu).$$

Observe that we have $\|U^0(\mu)\| \leq 1$ and hence $r(U^0(\mu)) \leq 1$ for the spectral radius $r(U^0(\mu))$ of $U^0(\mu)$. Assume that μ is aperiodic, (that is, if $\mathrm{supp}(\mu)$ is not contained in the coset of a proper subgroup of H). Then the action of H on X has a spectral gap if and only if $r(U^0(\mu)) < 1$ and this is equivalent to $\|U^0(\mu)\| < 1$.

Ergodic theoretic applications of the existence of a spectral gap (or of the stable spectral gap; see below for the definition) to random walks (such as the rate of L^2-convergence in the random ergodic theorem, pointwise ergodic theorem, analogues of the law of large numbers and of the central limit theorem, etc) are given in [CoGu11], [CoLe11], [FuSh99], [GoNe10] and [Guiv05]. Another application of the spectral gap property is the uniqueness of ν as H-invariant mean on $L^\infty(X,\nu)$; for this as well as for further applications, see [BeHV08], [Lubo94], [Popa08], [Sarn90].

Recall that a factor (Y,m,H) of the system (X,ν,H) is a probability space (Y,m) equipped with an H-action by measure preserving transformations together with a H-equivariant mesurable mapping $\Phi : X \to Y$ with $\Phi_*(\nu) = m$. Observe that $L^2(Y,m)$ can be identified with a H-invariant closed subspace of $L^2(X,\nu)$.

By a result proved in [JuRo79, Theorem 2.4], no action of a countable amenable group by measure preserving transformations on a non-atomic probability space has a spectral gap. As a consequence, if there exists a non-atomic factor (Y,m,H) of the system (X,ν,H) such that H acts as an amenable group on Y, then the action of H on X has no spectral gap. Our main result (Theorem 1) shows in particular that this is the only obstruction for the existence of a spectral gap when H is a countable group of affine transformations of a compact nilmanifold X.

Let N be a connected and simply connected nilpotent Lie group. Let Λ be a lattice in N; the associated nilmanifold $\Lambda\backslash N$ is known to be compact. The group N acts by right translations on $\Lambda\backslash N$: every $n \in N$ defines a transformation $\rho(n)$ on $\Lambda\backslash N$ given by $\Lambda x \mapsto \Lambda x n$. Denote by $\mathrm{Aut}(N)$ the group of continuous automorphisms of N and by $\mathrm{Aut}(\Lambda\backslash N)$ the subgroup of continuous automorphisms φ of N such that $\varphi(\Lambda) = \Lambda$. The group $\mathrm{Aut}(N)$ is a linear algebraic group defined over $\mathbf{Q}$ and $\mathrm{Aut}(\Lambda\backslash N)$ is a discrete subgroup of $\mathrm{Aut}(N)$. An affine transformation of $\Lambda\backslash N$ is a mapping $\Lambda\backslash N \to \Lambda\backslash N$ of the form $\varphi \circ \rho(n)$ for some $\varphi \in \mathrm{Aut}(\Lambda\backslash N)$ and $n \in N$. The group $\mathrm{Aff}(\Lambda\backslash N)$ of affine transformations of $\Lambda\backslash N$ is the semi-direct product $\mathrm{Aut}(\Lambda\backslash N) \ltimes N$.

Every $g \in \mathrm{Aff}(\Lambda\backslash N)$ preserves the translation invariant probability measure $\nu_{\Lambda\backslash N}$ induced by a Haar measure on N. The action of $\mathrm{Aff}(\Lambda\backslash N)$ on $\Lambda\backslash N$ is a natural generalization of the action of $SL_n(\mathbf{Z}) \ltimes \mathbf{T}^n$ on the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. In fact, let $T = \Lambda[N,N]\backslash N$ be the maximal torus factor of $\Lambda\backslash N$. Then the nilsystem $(\Lambda\backslash N, H)$ can be viewed as the result, starting with T, of a finite sequence of extensions by tori, with induced actions of H on every stage.

Actions of of higher rank lattices by affine transformations on nilmanifolds arise in Zimmer's programme as one of the standard actions for such groups (see the survey [Fish]). The action of a single affine transformation (or a flow of such transformations) on a nilmanifold have been studied by W. Parry from the ergodic, spectral or topological point of view (see [Parr69],[Parr70-a],[Parr70-b]; see also [AuGH63] for the case of translations).

Let V be a finite dimensional real vector space and Δ a lattice in V. As is well-known, $T = V/\Delta$ is a torus and Δ defines a rational structure on V. Let W be a rational linear subspace of V. Then $S = W/(W \cap \Delta)$ is a subtorus of T and we have a torus factor $\overline{T} = T/S$. Let H be a subgroup of $\mathrm{Aff}(T)$ and assume that W is invariant under $p_{\mathrm{a}}(H)$, where $p_{\mathrm{a}} : \mathrm{Aff}(\Lambda\backslash N) \to \mathrm{Aut}(\Lambda\backslash N)$ is the canonical projection. Then H leaves S invariant and the induced action of H on $\overline{T}$ is a factor of the action of H on T. We will say that $\overline{T}$ is an H-invariant factor torus of T. Here is our main result.

Theorem 1 *Let $\Lambda\backslash N$ be a compact nilmanifold with associated maximal torus factor T. Let H be a countable subgroup $\mathrm{Aff}(\Lambda\backslash N)$. The following properties are equivalent:*

(i) The action of H on $\Lambda\backslash N$ has a spectral gap.

(ii) The action of H on T has a spectral gap.

(iii) There exists no non-trivial H-invariant factor torus $\overline{T}$ of T such that the projection of $p_{\mathrm{a}}(H)$ on $\mathrm{Aut}(\overline{T})$ is a virtually abelian group (that is, it contains an abelian subgroup of finite index).

To give an an example, let $T = \mathbf{R}^d/\mathbf{Z}^d$ be the d-dimensional torus. Observe that $\mathrm{Aut}(T)$ can be identified with $GL_d(\mathbf{Z})$. Let H be a subgroup of $\mathrm{Aff}(T) = GL_d(\mathbf{Z}) \ltimes T$. Assume that $p_{\mathrm{a}}(H)$ is not virtually abelian and that $p_{\mathrm{a}}(H)$ acts $\mathbf{Q}$-irreducibly on $\mathbf{R}^d$ (that is, there is no non-trivial $p_{\mathrm{a}}(H)$-invariant rational subspace of $\mathbf{R}^d$). Then the action of H on T has a spectral gap. For more details, see Corollary 6 and Example 7 below.

The result above is new even in the case where $\Lambda\backslash N$ is a torus; see however [FuSh99, Theorem 6.5.ii] for a sufficient condition for the existence of a spectral gap for groups of torus automorphisms. Our results shows, in particular, that the spectral gap property for a countable subgroup H of $\mathrm{Aff}(\Lambda\backslash N)$ is equivalent to the spectral gap property for its automorphism part $p_{\mathrm{a}}(H)$.

The proof of Theorem 1 breaks into two parts. We first establish the result in the case where $\Lambda\backslash N$ is a torus (see Theorem 5 below). Our proof is based here on the existence of appropriate invariant means on finite dimensional vector spaces. A crucial tool will be (a version of) Furstenberg's result on stabilizers of probability measures on projective spaces over local fields. In the case of a general nilmanifold $\Lambda\backslash N$ with associated maximal torus factor T, we show that (ii) implies (i) by studying the asymptotic behaviour of matrix coefficients of the Koopman representation U of H restricted to the orthogonal complement of $L^2(T)$ in $L^2(\Lambda\backslash N)$; for this, we will use decay properties of the metaplectic representation of symplectic groups due to R. Howe and C. C.Moore [HoMo79]. The equivalence of (i) and (ii) was proved in [BeHe10] in the special case of a group of automorphisms of Heisenberg nilmanifolds.

Actions of countable amenable groups on a non-atomic probability space fail to have a property which is weaker than the spectral gap property. Recall that the action of a countable group H by measure preserving transformations on a probability space (X, ν) is said to be *strongly ergodic* in Schmidt's sense (see [Schm80], [Schm81]) if every sequence $(A_n)_n$ of measurable subsets of X which is asymptotically invariant (that is, which is such that $\lim_n \nu(gA_n \triangle A_n) = 0$ for all $g \in H$) is trivial (that is, $\lim_n \nu(A_n)(1-\nu(A_n)) =$

0). It is easy to see that if the action of H on X has a spectral gap, then the action is strongly ergodic (see, for instance, [BeHV08, Proposition 6.3.2]). The converse does not hold in general (see Example (2.7) in [Schm81]). As shown in [Schm81], no action of a countable amenable group by measure preserving transformations on a non-atomic, probability space can be strongly ergodic.

An interesting feature of strong ergodicity (as opposed to the spectral gap property) is that this notion only depends on the equivalence relation on X defined by the partition of X into H-orbits. Our result shows that the existence of a spectral gap for subgroups of $\mathrm{Aff}(\Lambda\backslash N)$ is equivalent to strong ergodicity.

Corollary 2 *The action of a countable subgroup of* $\mathrm{Aff}(\Lambda\backslash N)$ *on a compact nilmanifold* $\Lambda\backslash N$ *has a spectral gap if and only if it is strongly ergodic.*

We suspect that the previous corollary is true for every countable group of affine transformations of the quotient of a Lie group by a lattice. In fact, the following stronger statement could be true. Let G be a connected Lie group and Γ a lattice of G. Let H be a countable subgroup of $\mathrm{Aff}(\Gamma\backslash G)$. Assume that the action of H on $\Gamma\backslash G$ does not have a spectral gap. Is it true that there exists a non-trivial H-invariant factor $\overline{\Gamma}\backslash\overline{G}$ of $\Gamma\backslash G$ such that the closure of the projection of H on $\mathrm{Aff}(\overline{\Gamma}\backslash\overline{G})$ is an amenable group?

As our result shows, this is indeed the case if G is a nilpotent Lie group; it is also the case if G is a simple non-compact Lie group with finite centre (see Theorem 6.10 in [FuSh99]). It is worth mentioning that the corresponding statement in the framework of countable standard equivalence relations has been proved in [JoSc87].

Let again H be a countable group acting by measure preserving transformations on a probability space (X,ν). The following useful strengthening of the spectral gap property has been considered by several authors ([Bekk90], [BeGu06], [FuSh99], [Popa08]). Following [Popa08], let us say that the action of H has a *stable spectral gap* if the diagonal action of H on $(X\times X,\nu\otimes\nu)$ has a spectral gap (see Lemma 3.2 in [Popa08] for the rationale of this terminology). The following result is an immediate consequence of Theorem 1 above and of the corresponding result for groups of torus automorphisms obtained in [FuSh99, Theorem 6.4].

Corollary 3 *If the action of a countable subgroup of* $\mathrm{Aff}(\Lambda\backslash N)$ *on a compact nilmanifold* $\Lambda\backslash N$ *has a spectral gap, then it is has stable spectral gap.*

Next, we turn to the question of the ergodicity or mixing of the action of a (not necessarily countable) subgroup H of $\mathrm{Aff}(\Lambda\backslash N)$ on $\Lambda\backslash N$. As a consequence of our methods, we will see that this reduces to the same question for the action of H on the associated torus.

Recall that an action of a group H on a probability space (X, ν) is weakly mixing if the Koopman representation U of H on $L^2(X, \nu)$ has no finite dimensional subrepresentation, and that the action of of a countable group H is strongly mixing if the matrix coefficients $g \mapsto \langle U(g)\xi, \eta\rangle$ vanish at infinity for all $\xi, \eta \in L^2_0(X, \nu)$.

Theorem 4 *Let H be a group of affine transformations of the compact nilmanilfold $\Lambda\backslash N$. Let T be the maximal T torus factor associated to $\Lambda\backslash N$.*

(i) If the action of H on T is ergodic (or weakly mixing), then its action on $\Lambda\backslash N$ is ergodic (or weakly mixing).

(ii) Assume that H is as subgroup of $\mathrm{Aut}(\Lambda\backslash N)$. *If the action of H on T is strongly mixing, then its action on $\Lambda\backslash N$ is strongly mixing.*

Part (i) of the previous theorem has been independently established in [CoGu11]) with a different method of proof. In the case of a single affine transformation (that is, in the case of $H = \mathbf{Z}$), the result is due to W.Parry (see [Parr69], [Parr70-a]). Also, [CoGu11] gives an example of a group of automorphisms H acting ergodically on a nilmanilfold $\Lambda\backslash N$ for which no single automorphism from H acts ergodically on $\Lambda\backslash N$, showing that the previous theorem does not follow from Parry's result.

Sections 1-7 are devoted to the proof our main result Theorem 1 in the case where $\Lambda\backslash N$ is a torus. The proof of the extension to general nilmanifold is given in Sections 8-14. Theorem 4 is treated in Section 15.

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2 Spectral gap property for groups of affine transformations of a torus: statement of the main result

Let V be a finite dimensional real vector space of dimension $d \geq 1$ and let Δ be a lattice in V. Let T be the torus $T = V/\Delta$. The group of affine transformations of T is the semi-direct product $\mathrm{Aff}(T) = \mathrm{Aut}(T) \ltimes T$.

The aim of this section is to state the following result, which will be proved in the next two sections. Recall that p_{a} denotes the canonical homomorphism $\mathrm{Aff}(T) \to \mathrm{Aut}(T)$.

Theorem 5 *Let H be a countable subgroup of $\mathrm{Aff}(T)$. The following properties are equivalent. The following properties are equivalent:*

(i) The action of H on T does not have a spectral gap.

(ii) There exists a non-trivial H-invariant factor torus $\overline{T}$ such that the projection of $p_{\mathrm{a}}(H)$ on $\mathrm{Aut}(\overline{T})$ is amenable.

(iii) There exists a non-trivial H-invariant factor torus $\overline{T}_0$ such that the projection of $p_{\mathrm{a}}(H)$ on $\mathrm{Aut}(\overline{T}_0)$ is virtually abelian.

The following corollary is an immediate consequence of the implication $(i) \Rightarrow (iii)$ in the previous theorem.

Corollary 6 *Let $T = V/\Delta$ be a torus. Let H be a countable subgroup of $\mathrm{Aff}(T)$ such that $p_{\mathrm{a}}(H) \subset \mathrm{Aut}(T)$ is not virtually abelian. Assume that the action of H on V is $\mathbf{Q}$-irreducible for the rational structure on V defined by Δ. Then the action of H on T has a spectral gap.*

This last result was proved in [FuSh99, Theorem 6.5.ii] for a subgroup H of $\mathrm{Aut}(T)$ under the stronger assumption that the action of H on V is $\mathbf{R}$-irreducible. We give an example of a subgroup H of automorphisms of a 6-dimensional torus $T = V/\Delta$ which acts $\mathbf{Q}$-irreducibly but not $\mathbf{R}$-irreducibly on V and which has a spectral gap on T.

Example 7 Let q be the quadratic form on $\mathbf{R}^3$ given by

$$q(x) = x_1^2 + x_2^2 - \sqrt{2}x_3^2,$$

and let $SO(q, \mathbf{R}) \subset GL_3(\mathbf{R})$ be the orthogonal group of q. Set

$$H = SL_3(\mathbf{Z}[\sqrt{2}] \cap SO(q, \mathbf{R}).$$

Let σ be the non-trivial automorphism of the field $\mathbf{Q}[\sqrt{2}]$. For every $g \in SO(q, \mathbf{R})$, the matrix g^σ, obtained by conjugating each entry of g, preserves the conjugate form q^σ of q under σ. The mapping

$$\mathbf{Q}[\sqrt{2}] \to \mathbf{R} \times \mathbf{R}, \qquad x \mapsto (x, \sigma(x))$$

induces an isomorphism between $\mathbf{Z}[\sqrt{2}]^3$ and a lattice Δ in $\mathbf{R}^3 \times \mathbf{R}^3$. It induces also an isomophism $\gamma \mapsto (\gamma, \gamma^\sigma)$ between H and a lattice Γ in $SO(q, \mathbf{R}) \times SO(q^\sigma, \mathbf{R})$. Moreover, H leaves $\mathbf{Z}[\sqrt{2}]^3$ invariant and Γ leaves Δ invariant. We obtain in this way an action of H on the torus $T = \mathbf{R}^6/\Delta$.

Since $SO(q^\sigma, \mathbf{R}) \cong SO(3)$ is compact, H is a lattice in $SO(q, \mathbf{R})$. This implies (Borel density theorem) that the Zariski closure of H in $SL_3(\mathbf{R})$ is the simple Lie group $SO(q, \mathbf{R})$, so that the action of H on $\mathbf{R}^3$ is $\mathbf{R}$-irreducible and hence $\mathbf{Q}$-irreducible for the usual rational structure on $\mathbf{R}^3$. It follows that the action of H on $\mathbf{R}^6$ is $\mathbf{Q}$-irreducible for the rational structure defined by the lattice Δ of $\mathbf{R}^6$. Observe that the action of H on $\mathbf{R}^6$ is not $\mathbf{R}$-irreducible since Γ leaves invariant each copy of $\mathbf{R}^3$ in $\mathbf{R}^6 = \mathbf{R}^3 \oplus \mathbf{R}^3$. Moreover, H is not virtually abelian as it is a lattice in $SO(q, \mathbf{R}) \cong SO(2, 1)$. As a consequence of the previous corollary, the action of H on T has a spectral gap.

Concerning the proof of Theorem 5, we will first treat the case of groups of toral automorphisms.

Choosing a basis for the $\mathbf{Z}$-module Δ, we identify V with $\mathbf{R}^d$ and Δ with $\mathbf{Z}^d$. By means of the standard scalar product on $\mathbf{R}^d$, we identify the dual group $\widehat{V}$ of V (that is, the group of unitary characters of V) with V. The dual action of an element $g \in GL(V)$ on $\widehat{V}$ corresponds to the action of $(g^{-1})^t$ on V. Since $T = V/\Delta$, the dual group $\widehat{T}$ can be identified with Δ. Let W be a rational linear subspace of V. The dual group of the quotient V/W corresponds to the orthogonal complement $W^\perp$ of W, which is also a rational linear subspace of V. The dual group of the torus factor $\overline{T} = (V/W)/((W + \Delta)/\Delta)$ corresponds to $W^\perp \cap \Delta$.

The discussion above shows that Theorem 5, in the case of a group of toral automorphisms is equivalent to the following theorem.

Theorem 8 *Let H be a subgroup of $GL_d(\mathbf{Z})$. The following properties are equivalent.*

- *(i) The action of H on $T = \mathbf{R}^d/\mathbf{Z}^d$ does not have a spectral gap.*
- *(ii) There exists a non-trivial rational subspace W of $\mathbf{R}^d$ which is invariant under the subgroup H^t of $GL_d(\mathbf{Z})$ and such that the image of H^t in $GL(W)$ is an amenable group.*
- *(iii) There exists a non-trivial rational subspace W of $\mathbf{R}^d$ which is invariant under H^t and such that the the image of H^t in $GL(W)$ is a virtually abelian group.*

Observe that the implication $(iii) \Longrightarrow (ii)$ is obvious and that the implication $(ii) \Longrightarrow (i)$ follows from the result in [JuRo79] quoted in the introduction. Therefore, it remains to show that (i) implies (ii) and that (ii) implies (iii).

3 A canonical amenable group associated to a linear group

Let V be a finite-dimensional real vector space. (Although we will consider only real vector spaces, the results in this section are valid for vector spaces over any local field.) Let $g \in GL(V)$ and W a g-invariant linear subspace of V. We denote by $g_W \in GL(W)$ the automorphism of W given by the restriction of g to W. If W' is another g-invariant subspace contained in W, we will denote by $g_{W/W'} \in GL(W/W')$ the automorphism of W/W' induced by g. Also, if H is a subgroup of $GL(V)$ and $W' \subset W$ are H-invariant subspaces of V, we will denote by H_W and $H_{W/W'}$ the corresponding subgroups of $GL(W)$ and $GL(W/W')$, respectively.

For a subgroup H of $GL(V)$, we denote by $\overline{H}$ its closure for the usual locally compact topology on $GL(V)$. The aim of this section is to prove the following result.

Proposition 9 *Let H be a subgroup of $GL(V)$. There exists a largest H-invariant linear subspace $V(H)$ of V such that the group $\overline{H_{V(H)}}$ is amenable. More precisely, let $V(H)$ be the subspace of V generated by the union of the H-invariant subspaces $W \subset V$ for which $\overline{H_W}$ is amenable. Then $\overline{H_{V(H)}}$ is amenable.*

A more explicit description of $V(H)$ will be given later (Proposition 15). For the proof of the proposition above, we will need the following elementary lemma.

Lemma 10 *Let H be a closed subgroup of $GL(V)$ and W an H-invariant subspace of V. Then H is amenable if and only if $\overline{H_W}$ and $\overline{H_{V/W}}$ are amenable.*

Proof Since $\overline{H_W}$ and $\overline{H_{V/W}}$ are closures of quotients of H, both are amenable if H is amenable.

Assume that $\overline{H_W}$ and $\overline{H_{V/W}}$ are amenable. Let L be the closed subgroup consisting of the elements $g \in GL(V)$ leaving W invariant and for which g_W belongs to $\overline{H_W}$ and $g_{V/W}$ belongs to $\overline{H_{V/W}}$. The mapping

$$\varphi : L \to \overline{H_W} \times \overline{H_{V/W}}, \qquad g \mapsto (g_W, g_{V/W})$$

is a continuous homomorphism. It is clear that φ is surjective. Moreover, $U = \mathrm{Ker}(\varphi)$ is a unipotent closed subgroup of L. Since $\overline{H_W} \times \overline{H_{V/W}}$ and U are amenable, L is amenable. The closed subgroup H of L is therefore amenable.■

Proof of Proposition 9 We can write $V(H) = \sum_{i=1}^{r} W_i$ as a sum of finitely many H-invariant subspaces $W_1, \ldots, W_r$ of V such that $\overline{H_{W_i}}$ is amenable for every $1 \leq i \leq r$.

We show by induction on $s \in \{1, \ldots, r\}$ that $\overline{H_{W^s}}$ is amenable, where $W^s = \sum_{i=1}^{s} W_i$. The case $s = 1$ being obvious, assume that $\overline{H_{W^s}}$ is amenable for some $s \in \{1, \ldots, r-1\}$. The group

$$GL(W^{s+1}/W^s) = GL((W^s + W_{s+1})/W^s)$$

is canonically isomorphic to $GL(W_{s+1}/(W^s \cap W_{s+1})$ and $\overline{H_{W^{s+1}/W^s}}$ corresponds to $\overline{H_{W_{s+1}/(W^s \cap W_{s+1})}}$ under this isomorphism. Now, $\overline{H_{W_{s+1}/(W^s \cap W_{s+1})}}$ is amenable since $\overline{H_{W_{s+1}}}$ is amenable. Hence, $\overline{H_{W^{s+1}/W^s}}$ is amenable. Moreover, $\overline{H_{W^s}}$ is amenable by the induction hypothesis. The previous lemma implies that $\overline{H_{W^{s+1}}}$ is amenable. ■

4 Invariant means supported by rational subspaces

Let G be a locally compact group. There is a well-known relationship between weak containment properties of the trivial representation 1_G and existence on invariant means on appropriate spaces (see below). We will need to make this relationship more precise in the case where H is a subgroup of toral automorphisms.

By a unitary representation $(\pi, \mathcal{H})$ of G, we will always mean a strongly continuous homomorphism $\pi : G \to U(\mathcal{H})$ from G to the unitary group of a complex Hilbert space $\mathcal{H}$.

Recall that, for every finite measure μ of G, the operator $\pi(\mu) \in \mathcal{B}(\mathcal{H})$ is defined by the integral

$$\pi(\mu)\xi = \int_G \pi(g)\xi d\mu(g) \qquad \text{for all} \quad \xi \in \mathcal{H}.$$

Assume that G is a discrete group and π and ρ are unitary representations of G; then π is weakly contained in ρ if and only if $\|\pi(\mu)\| \leq \|\rho(\mu)\|$ for every finite measure μ on G (see Section 18 in [Dixm69]). Recall also that, given a probability measure μ on G which is aperiodic, the trivial representation 1_G is weakly contained in a unitary representation π if and only if $\|\pi(\mu)\| = 1$ (see [BeHV08, G.4.2]).

Let X be a topological space and $C^b(X)$ the Banach space of all bounded continuous functions on X equipped with the supremum norm. Recall that a mean on X is a linear functional m on $C^b(X)$ such that $m(1_X) = 1$ and such that $m(\varphi) \geq 0$ for every $\varphi \in C^b(X)$ with $\varphi \geq 0$. A mean is automatically continuous. We will often write $m(A)$ instead of $m(1_A)$ for a subset A of X.

Observe that the means on a compact space X are the probability measures on X.

Let H be a group acting on X by homeomorphisms. Then H acts naturally on $C^b(X)$. A mean m on X is H-invariant if $m(h.\varphi) = m(\varphi)$ for all $\varphi \in C^b(X)$ and $h \in H$.

Let Y be another topological space and $f : X \to Y$ a continuous mapping. For every mean m on X, the push-forward $f_*(m)$ of m is the mean on Y defined by $\varphi \mapsto m(\varphi \circ f)$ for $\varphi \in C^b(Y)$.

We will consider invariant means on two kinds of topological spaces:

• X is a set with the discrete topology and endowed with an action of a group H. It is well-known (see Théorème on p. 44 in [Eyma72]) that there exists an H-invariant mean on X if and only if the natural unitary representation U of H on $\ell^2(X)$ almost has invariant vectors (that is, if and only if U weakly contains the trivial representation 1_H of H).

• $X = V \setminus \{0\}$, where V is a finite dimensional real vector space. Let H be a subgroup of $GL(V)$. If m is an H-invariant mean on $V \setminus \{0\}$, then $\pi_*(m)$ is an H-invariant probability measure on the projective space $\mathbf{P}(V)$, where $\pi : V \setminus \{0\} \to \mathbf{P}(V)$ is the canonical projection.

The following result is a version of Furstenberg's celebrated lemma (see [Furs76] or [Zimm84, Corollary 3.2.2]) on stabilizers of probability measures on projective spaces. We will need later (in Section 5) the more precise form we give for this lemma (see also the proof of Theorem 6.5 (ii) in [FuSh99]).

For a subgroup H of $GL(V)$, we denote by $\mathrm{Zc}(H)$ the closure of H in the Zariski topology and by $\mathrm{Zc}(H)^0$ the connected component of $\mathrm{Zc}(H)$ in the Zariski topology. As is well-known, $\mathrm{Zc}(H)^0$ has finite index in $\mathrm{Zc}(H)$.

Lemma 11 *Let H be a closed subgroup of $GL(V)$. Assume that H stabilizes a probability measure ν on $\mathbf{P}(V)$ which is not supported on a proper projective subspace. Then the commutator subgroup $[H^0, H^0]$ of H^0 is relatively compact, where H^0 is the normal subgroup of finite index $H \cap \mathrm{Zc}(H)^0$ of H. In particular, H is amenable.*

Proof We can find finitely many positive measures $(\nu_i)_{1\leq i\leq r}$ on $\mathbf{P}(V)$ with $\nu = \sum_{1\leq i\leq r} \nu_i$ such that $\nu(V_i \cap V_j) = 0$ for $i \neq j$ and such that $\mathrm{supp}(\nu_i) \subset \pi(V_i)$ for every $i \in \{1, \ldots, r\}$, where V_i is a linear subspace of V of minimal dimension with $\nu_i(\pi(V_i)) > 0$. The H-orbit of V_i and hence the H-orbit of ν_i is finite (see Proof of Corollary 3.2.2 in [Zimm84]). Since stabilizers of probability measures on $\mathbf{P}(V)$ are algebraic (see Theorem 3.2.4 in [Zimm84]), it follows that H^0 stabilizes each V_i and each ν_i. Now ν_i, viewed as measure on $\mathbf{P}(V_i)$, is zero on every proper projective subspace of $\mathbf{P}(V_i)$. Hence (see Corollary 3.2.2 in [Zimm84]), the image of the restriction H_i^0 of H^0 to V_i is a relatively compact subgroup of $PGL(V_i)$, for every $i \in \{1, \ldots, r\}$. Since $[H_i^0, H_i^0]$ is contained in $SL(V_i)$, it follows that $\overline{[H_i^0, H_i^0]}$ is compact in $GL(V_i)$. This implies that $\overline{[H^0, H^0]}$ is compact. As $H^0/\overline{[H^0, H^0]}$ is abelian, it follows that H^0 (and hence H) is amenable.■

Remark 12 The conclusion of the previous lemma does not hold in general if we replace H^0 by an arbitrary subgroup of finite index of H. For example, let $V = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ and let $H \subset GL_2(\mathbf{R})$ be the stabilizer of the measure $\nu = (\delta_{\pi(e_1)} + \delta_{\pi(e_2)})/2$ on $\mathbf{P}(V)$. Then $[H, H] = H$ is not bounded; however, H^0 is the subgroup of index two consisting of the diagonal matrices in H and $[H^0, H^0]$ is trivial.

Proposition 13 *Let H be a subgroup of $GL(V)$ and $V(H)$ the largest H-invariant susbpace of V such that $\overline{H_{V(H)}}$ is amenable.*

(i) Assume H stabilizes a mean m on $V \setminus \{0\}$. Then $V(H) \neq \{0\}$.

(ii) Let Δ be a lattice in V and m a mean on $\Delta \setminus \{0\}$. Assume H leaves Δ invariant and stabilizes m. Then $m(V(H) \cap \Delta) = 1$. In particular, the $\mathbf{R}$-linear span of $V(H) \cap \Delta$ is a non-trivial rational subspace of V (for the rational structure defined by Δ).

Proof (i) Let $\pi : V \setminus \{0\} \to \mathbf{P}(V)$ be the canonical projection and $\nu = \pi_*(m)$. Then ν is an H-invariant probability measure on $\mathbf{P}(V)$. Let W the linear span of $\pi^{-1}(\mathrm{supp}(\nu))$. Then W is non-trivial and ν is not supported on a proper projective subspace of $\pi(W)$. It follows from Lemma 11 applied to the closed subgroup $\overline{H_W}$ of $GL(W)$ that $\overline{H_W}$ is amenable. Hence, $V(H) \neq \{0\}$, by the definition of $V(H)$.

(ii) Set $\overline{V} = V/V(H)$. Since $V(H)$ is H-invariant, we have an induced action of H on $\overline{V}$. Denote by $p : V \to \overline{V}$ the canonical projection. We consider the mean $\overline{m} = (p|_\Delta)_*(m)$ on the set $\overline{\Delta} := p(\Delta)$ equipped with the discrete topology. Observe that $\overline{m}$ is H-invariant, since H stabilizes m.

Assume, by contradiction, that $m(V(H) \cap \Delta) < 1$. Then $\overline{m}(\{0\}) = m(V(H) \cap \Delta) < 1$. Setting $\alpha = m(V(H) \cap \Delta)$, we define an H-invariant mean $\overline{m_1}$ on $\overline{\Delta} \setminus \{0\}$ by

$$\overline{m_1}(\varphi) = \frac{1}{1-\alpha}\overline{m}(\varphi) \qquad \text{for all} \quad \varphi \in \ell^\infty(\overline{\Delta} \setminus \{0\}).$$

Let $i_*(\overline{m_1})$ be the mean on $\overline{V} \setminus \{0\}$ induced by the canonical injection $i : \overline{\Delta} \setminus \{0\} \to \overline{V} \setminus \{0\}$. Observe that $i_*(\overline{m_1})$ is H-invariant. Hence, by (i), we have $\overline{V}(H) \neq \{0\}$. This implies that $V(H)$ is a proper subspace of the vector space $W := p^{-1}(\overline{V}(H))$. On the other hand, $\overline{H_W}$ is amenable, by Lemma 10. This contradicts the definition of $V(H)$. ■

At this point, we can give the proof of the fact that (i) implies (ii) in Theorem 5 (or, equivalently, in Theorem 8) in the case of group of automorphisms.

Proof of $(i) \Longrightarrow (ii)$ in Theorem 8

Let H be a countable subgroup of $GL_d(\mathbf{Z})$. Assume that the action of H on $T = \mathbf{R}^d/\mathbf{Z}^d$ does not have a spectral gap. Then the unitary representation of the transposed subgroup H^t on $\ell^2(\mathbf{Z}^d \setminus \{0\})$ weakly contains the trivial representation 1_{H^t}. Hence, there exists an H^t-invariant mean on $\mathbf{Z}^d \setminus \{0\}$. By Proposition 13, the linear span W of $V(H^t) \cap \mathbf{Z}^d$ is a non-trivial rational subspace of $\mathbf{R}^d$. Morever, $H^t_W = \overline{H^t_W}$ is amenable. ■

5 Proof of $(ii) \Longrightarrow (iii)$ in Theorem 8

For the proof of $(ii) \Longrightarrow (iii)$ in Theorem 8, we will need a precise description of the subspace $V(H)$ associated to a subgroup H of $GL(V)$ and introduced in Proposition 9. For this, we will use the following result which appears as Lemma 1 and Lemma 2 in [CoGu74]. Since the arguments in [CoGu74] are slightly incomplete, we give the proof of this lemma.

Lemma 14 *Let V be finite-dimensional real vector space and let H be a subgroup of $GL(V)$ such that the action of H on V is completely reducible.*

(i) Assume that the eigenvalues of every element in H all have modulus 1. Then H is relatively compact.

(ii) Assume that there exists an integer $N \geq 1$ such that the eigenvalues of every element in H are all N-th roots of unity. Then H is finite.

Proof By hypothesis, we can decompose V into a direct sum $V = \oplus_{1\leq i\leq r} V_i$ of irreducible H-invariant subspaces V_i. Let $V^{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of V. The action of H on each V_i extends to a representation of H on $V_i^{\mathbf{C}}$ which either is irreducible or decomposes as a direct sum of two irreducible (mutually conjugate) representations of H. It suffices therefore to prove the following

Claim: Let H be a subgroup of $GL_d(\mathbf{C})$ acting irreducibly on $\mathbf{C}^d$. Then the conclusion (i) and (ii) hold.

For every $h \in H$, we consider the linear functional φ_h on the algebra $M_d(\mathbf{C})$ of complex $(d \times d)$-matrices defined by $\varphi_h(x) = \mathrm{Tr}(hx)$. Since H acts irreducibly, it follows from Burnside theorem that the algebra generated by H coincides with $M_d(\mathbf{C})$. Hence, there exists a basis $\{h_1, \ldots, h_{d^2}\}$ of the vector space $M_d(\mathbf{C})$ contained in H. Then $\{\varphi_{h_1}, \ldots, \varphi_{h_{d^2}}\}$ is a basis of the dual space of $M_d(\mathbf{C})$.

Assume that the eigenvalues of every element in H all have modulus 1. Then the φ_{h_i}'s are bounded on H by d. It follows that the matrix coefficients of the elements in H are bounded. Hence, H is relatively compact subset of $M_d(\mathbf{C})$.

Assume that, for a fixed $N \geq 1$, the eigenvalues of every element in H are N-th roots of unity.. Then the φ_{h_i}'s take only a finite set of values on H. It follows that H is finite subset of $M_d(\mathbf{C})$.■

Proposition 15 *Let V be a finite-dimensional real vector space and H a subgroup H of $GL(V)$. Set $H^0 = H \cap \mathrm{Zc}(H)^0$. Let V^1 be the largest H-invariant linear subspace of V such that, for every $h \in [H^0, H^0]$, the eigenvalues of the restriction of h to V^1 all have modulus 1. Then $V(H) = V^1$.*

Proof Let us first show that $V(H) \subset V^1$. Since $\overline{H_{V(H)}}$ is amenable, there exists an H-invariant probability measure ν on $\mathbf{P}(V(H)) \subset \mathbf{P}(V)$. Let W be the smallest H-invariant subspace such that ν is supported on $\mathbf{P}(W)$. It follows from Lemma 11 that $[H^0, H^0]$ acts isometrically on W, with respect to an appropriate norm on W. We can apply the same argument to the group $\overline{H_{V(H)/W}}$ acting on the quotient space $V(H)/W$. Hence, by induction, we obtain a flag

$$\{0\} = W_0 \subset W = W_1 \subset W_2 \subset \cdots \subset W_r = V(H)$$

of H-invariant subspaces such that $[H^0, H^0]$ acts isometrically on each quotient W_{i+1}/W_i. It follows from this that the eigenvalues of the restriction to $V(H)$ of any element $h \in [H^0, H^0]$ have all modulus 1. Hence, $V(H) \subset V^1$.

To show that $V^1 \subset V(H)$, we have to prove that $\overline{H_{V^1}}$ is amenable. Recall that that H/H^0 is finite and observe that $\overline{H^0_{V^1}}/\overline{[H^0_{V^1}, H^0_{V^1}]}$ is abelian. Hence, it suffices to show that $\overline{[H^0_{V^1}, H^0_{V^1}]}$ is amenable.

Let

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_r = V^1$$

be a Jordan-Hölder sequence for the $[H^0_{V^1}, H^0_{V^1}]$-module V^1, that is, every W_i is an $[H^0_{V^1}, H^0_{V^1}]$-invariant subspace of V^1 and $[H^0_{V^1}, H^0_{V^1}]$ acts irreducibly on every quotient W_{i+1}/W_i. By Lemma 14.i, the image of $[H^0, H^0]$ in $GL(W_{i+1}/W_i)$ is relatively compact for every $i \in \{0, \ldots, r-1\}$.

Let N be the unipotent subgroup of $GL(V^1)$ consisting of the elements in $GL(V^1)$ which act trivially on every quotient W_{i+1}/W_i.

We can choose a scalar product on V^1 such that, denoting by $W_i^\perp$ the orthogonal complement of W_i in W_{i+1}, every $h \in [H^0, H^0]$ can be written in the form $h = kh_0$, where $h_0 \in N$ and where k leaves $W_i^\perp$ invariant and acts isometrically on $W_i^\perp$ for every $i \in \{0, \ldots, r-1\}$, This shows that $\overline{[H^0_{V^1}, H^0_{V^1}]}$ can be embedded as a closed subgroup of $K \ltimes N \subset GL(V^1)$, where K is the product of the the orthogonal groups of the $W_i^\perp$'s. Since $K \ltimes N$ is amenable, the same is true for $\overline{[H^0_{V^1}, H^0_{V^1}]}$. ■

We will need need the following corollary of (the proof of) the previous proposition .

Corollary 16 *Let Γ be a subgroup of $GL_d(\mathbf{Z})$. Assume that the eigenvalues of every $\gamma \in \Gamma$ all have modulus 1. Then Γ contains a unique maximal unipotent subgroup Γ^0 of finite index. In particular, Γ^0 is a characteristic subgroup of Γ.*

Proof As in the proof of the previous proposition, we consider a Jordan-Hölder sequence for the Γ-module $\mathbf{R}^d$

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_r = \mathbf{R}^d$$

and let N be the subgroup of all $g \in GL(V)$ which act trivially on every W_{i+1}/W_i. We choose a scalar product on $\mathbf{R}^d$ such that Γ embeds as a subgroup of the semi-direct product $K \ltimes N$ for $K = \prod_{i=1}^d O(W_i^\perp)$, where $W_i^\perp$ is the orthogonal complement of W_i in W_{i+1}.

Let $\gamma \in \Gamma$. For every $l \geq 1$, the l-th powers of the eigenvalues of γ are roots of the same monic polynomial with integer coefficients and of degree d. Since the eigenvalues of γ are all of modulus 1, the coefficients of this polynomial are bounded by a number only depending on d. By a standard argument (see e.g. the proof of Lemma 11.6 in [StTa87]), it follows that all the eigenvalues of γ are roots of unity of a fixed order N which only depends on d.

Let $\overline{\Gamma}$ be the projection of Γ in K. The action of $\overline{\Gamma}$ is completely reducible, since the $W_i^\perp$'s are irreducible, and it follows from Lemma 14.ii that $\overline{\Gamma}$ is finite. Hence, $\Gamma \cap N$ is a unipotent normal subgroup of finite index in Γ.

We have therefore proved that Γ contains a unipotent normal subgroup of finite index. We claim that $\Gamma^0 := \Gamma \cap \mathrm{Zc}(\Gamma)^0$ is the unique maximal unipotent normal subgroup of finite index in Γ.

Indeed, let Γ_1 be a unipotent normal subgroup of finite index in Γ. Set $U := \mathrm{Zc}(\Gamma_1)$. Observe that the connected component of U coincides with $\mathrm{Zc}(\Gamma)^0$, since Γ_1 has finite index in Γ. On the other hand, as is well-known, U is connected since it is a unipotent algebraic group. (Indeed, the Zariski closure of the subgroup generated by a unipotent element $u \in GL(\mathbf{R}^d)$ contains the one-parameter subgroup through u; see e.g. 15.1. Lemma C in [Hum81].) It follows that $\mathrm{Zc}(\Gamma)^0 = U$ is unipotent. Moreover, since $\Gamma_1 \subset U$, we have $\Gamma_1 \subset \Gamma^0$ and the claim is proved. ■

We can now complete the proof of Theorem 8.

Proof of $(ii) \Longrightarrow (iii)$ **in Theorem 8**

Let $T = V/\Delta$ be a torus and H a subgroup of $\mathrm{Aut}(T) \subset GL(V)$. Assume that there exists a non-trivial rational subspace W of V which is H-invariant and such that such that the restriction H_W of H to W is an amenable group. In particular, we have $W \subset V(H)$.

Set $H^0 = H \cap \mathrm{Zc}(H)^0$. By Proposition 15, for every $h \in [H^0, H^0]$, all the eigenvalues of the restriction of h to W have modulus 1. Since W is rational, by the choice of a convenient basis of W, we can assume that $\Gamma := [H^0, H^0]_W$ is a subgroup of $GL_d(\mathbf{Z})$, where $d = \dim W$. It follows from Corollary 16 that Γ contains a unipotent subgroup Γ^0 of finite index which is moreover characteristic. Let W_1 be the space of the Γ^0-fixed vectors in W. Then W_1 is a rational and non-trivial linear subspace of W. Moreover, W_1 is H-invariant, since Γ^0 is characteristic.

We claim that H_{W_1} is virtually abelian. For this, it suffices to show that $G := H^0_{W_1} \subset GL(W_1)$ is virtually abelian. Observe first that $[G, G] = \Gamma_{W_1}$ is finite, since it is a quotient of the finite group Γ/Γ^0. Since $[\mathrm{Zc}(G), \mathrm{Zc}(G)] \subset \mathrm{Zc}([G, G])$, it follows that $[\mathrm{Zc}(G), \mathrm{Zc}(G)]$ is finite. On the other hand, the group $[\mathrm{Zc}(G)^0, \mathrm{Zc}(G)^0]$ is connected (see e.g. Proposition 17.2 in [Hum81]). Hence, $\mathrm{Zc}(G)^0$ is abelian. The subgroup $G \cap \mathrm{Zc}(G)^0$ has finite index in G and is abelian. ■

6 Herz's majoration principle for induced representations

Unitary representations of a separable locally compact group G induced by a closed subgroup H will appear several times in the sequel. We review their definition when the homogeneous space $H\backslash G$ has G-invariant measure. This will always be the case in the situations we will encounter. (Induced representation are still defined in the general case, after appropriate change; see [Mack76] or [BeHV08].)

Let ν be non-zero G-invariant measure on $H\backslash G$. Let $(\sigma, \mathcal{K})$ be a unitary representation of H. We will use the following model for the induced representation $\mathrm{Ind}_H^G \sigma$. Choose a measurable section $s : H\backslash G \to G$ for the canonical projection $G \to H\backslash G$. Let $c : (H\backslash G) \times G \to H$ be the corresponding cocycle defined by

$$s(x)g = c(x, g)s(xg) \qquad \text{for all} \quad x \in H\backslash G,\ g \in G.$$

The Hilbert space of $\mathrm{Ind}_H^G \sigma$ is the space $L^2(H\backslash G, \mathcal{K})$ of all square-integrable measurable mappings $\xi : H\backslash G \to \mathcal{K}$ and the action of G on $L^2(H\backslash G, \mathcal{K})$ is given by

$$(\mathrm{Ind}_H^G \sigma)(g)\xi(x) = \sigma(c(x, g))\xi(xg), \qquad g \in G,\ \xi \in L^2(H\backslash G, \mathcal{K}),\ x \in G/H.$$

In the sequel, we will use several times a well-known strengthening of Herz's majoration principle from [Herz70] concerning norms of convolution operators under an induced representation. For an even more general version, see [Anan03, 2.3.1]. For the convenience of the reader, we give the short proof.

Proposition 17 ***(Herz's majoration principle)*** *Let H be a closed subgroup of G such that $H\backslash G$ has a G-invariant Borel measure ν and let $(\sigma, \mathcal{K})$ be a unitary representation of H. For every probability measure μ on the Borel subsets of G, we have*

$$\|(\mathrm{Ind}_H^G \sigma)(\mu)\| \leq \|\rho_{G/H}(\mu)\|,$$

where $\lambda_{G/H}$ is the natural representation of G on $L^2(G/H)$.

Proof Let $c : H\backslash G \to H$ be the cocycle defined by a Borel section of $H\backslash G \to G$. For $\xi \in L^2(H\backslash G, \mathcal{K}, \nu)$, define φ in the Hilbert space $L^2(H\backslash G, \nu)$, of $\mathrm{Ind}_H^G \sigma$ by $\varphi(x) = \|\xi(x)\|$ and observe that $\|\varphi\| = \|\xi\|$. Using Jensen's inequality, we have

$$\begin{aligned}
\|(\mathrm{Ind}_H^G \sigma)(\mu)\xi\|^2 &= \int_{H\backslash G} \|(\mathrm{Ind}_H^G(\mu)\xi(x))\|^2 d\nu(x) \\
&= \int_{H\backslash G} \| \int_G \sigma(c(x,g))\xi(xg) d\mu(g)\|^2 d\nu(x) \\
&\leq \int_{H\backslash G} \int_G \|\sigma(c(x,g))\xi(xg)\|^2 d\mu(g) d\nu(x) \\
&= \int_{H\backslash G} \int_G \|\xi(xg)\|^2 d\mu(g) d\nu(x) \\
&= \|(\mathrm{Ind}_H^G 1_H)(\mu)\varphi\|^2.
\end{aligned}$$

Since $\mathrm{Ind}_H^G 1_H$ is equivalent to $\lambda_{G/H}$, the claim follows. ■

We will also need (in Section 10) a precise description of the kernel of an induced representation.

Lemma 18 *With the notation as in the previous proposition, let $\pi = \mathrm{Ind}_H^G \sigma$. Then $\mathrm{Ker}(\pi) = \bigcap_{g \in G} g\mathrm{Ker}(\sigma)g^{-1}$, that is, $\mathrm{Ker}(\pi)$ coincides the largest normal subgroup of G contained in $\mathrm{Ker}\sigma$.*

Proof

Let $c : H\backslash G \times G \to H$ be the cocycle corresponding to a measurable section $s : H\backslash G \to G$ with $s(H) = e$. Let $a \in \mathrm{Ker}(\pi)$. Then, for every $\xi \in L^2(H\backslash G, \mathcal{K})$, we have

$$\sigma(c(x,a))\xi(xa) = \xi(x) \qquad \text{for all} \quad x \in H\backslash G.$$

Taking for ξ mappings supported on a neighbourhood of Ha, we see that $a \in H$. Hence $c(H,a) = a$. Taking for ξ continuous mappings with $\xi(H) \neq 0$ and evaluating at H, we obtain that $a \in \mathrm{Ker}(\sigma)$. Since $\mathrm{Ker}(\pi)$ is normal in G, it follows that $gag^{-1} \in \mathrm{Ker}(\sigma)$ for all $g \in G$.

Conversely, let $a \in G$ be such that $gag^{-1} \in \mathrm{Ker}(\sigma)$ for all $g \in G$. Since

$$s(x)a = (s(x)as(x)^{-1})s(x),$$

we have $c(x,a) = s(x)as(x)^{-1}$ for all $x \in H\backslash G$. Hence, for every $\xi \in L^2(H\backslash G, \mathcal{K})$ and $x \in H\backslash G$, we have

$$(\pi(a)\xi)(x) = \sigma(c(x,a))\xi(xa) = \sigma(s(x)as(x)^{-1})\xi(x) = \xi(x).$$

This shows that $a \in \mathrm{Ker}(\pi)$ and the claim is proved. ■

7 Proof of Theorem 5

Let $T = V/\Delta$ be a torus and H a countable subgroup of $\mathrm{Aff}(T) = \mathrm{Aut}(T) \ltimes T$. The implication $(iii) \Longrightarrow (ii)$ is obvious and the implication $(ii) \Longrightarrow (i)$ follows from [JuRo79]. The fact that (ii) implies (iii) has been proved in Theorem 8. Therefore, it remains to show that (i) implies (ii). Again by Theorem 8, it suffices to show that if the action of H on T has no spectral gap, then the same is true for the action of $p_{\mathrm{a}}(H)$ on T, where p_{a} is the projection from $\mathrm{Aff}(T)$ to $\mathrm{Aut}(T)$. This will be an immediate consequence of the next proposition.

For a probability measure μ on $\mathrm{Aff}(T)$, we denote by $p_{\mathrm{a}}(\mu)$ the probability measure on $\mathrm{Aut}(T)$ which is the image of μ under p_{a}. Let U_0 be the Koopman representation of $\mathrm{Aff}(T)$ on $L^2_0(T)$.

Proposition 19 *For every probability measure μ on $\mathrm{Aff}(T)$, we have*

$$\|U_0(\mu)\| \leq \|U_0(p_{\mathrm{a}}(\mu))\|.$$

Proof Set $\Gamma = \mathrm{Aut}(T)$. Let $\widehat{T} \cong \mathbf{Z}^d$ be the dual group of T. The Fourier transform sets up a unitary equivalence between U_0 and the representation V of $\mathrm{Aff}(T)$ on $\ell^2\left(\widehat{T} \setminus \{1_T\}\right)$ given by

$$(*) \qquad V(\gamma, a)\chi = \chi(a)\chi^\gamma \qquad \text{for all} \quad \chi \in \widehat{T} \setminus \{1_T\}, \gamma \in \Gamma,\ a \in T,$$

where $\chi^\gamma \in \widehat{T}$ is defined by $\chi^\gamma(x) = \chi(\gamma^{-1}(x))$.

Choose a set of representatives S for the Γ-orbits in $\widehat{T} \setminus \{1_T\}$. Then $\ell^2\left(\widehat{T} \setminus \{1_T\}\right)$ decomposes as the direct sum of $\mathrm{Aff}(T)$-invariant subspaces

$$\ell^2\left(\widehat{T} \setminus \{1_T\}\right) = \bigoplus_{\chi \in S} \ell^2(\mathcal{O}_\chi),$$

where $\mathcal{O}_\chi$ is the orbit of $\chi \in S$ under Γ.

It follows from Formula $(*)$ above that the restriction V_χ of V to $\ell^2(\mathcal{O}_\chi)$ is equivalent to the induced representation $\mathrm{Ind}_{\Gamma_\chi \ltimes T}^{\Gamma \ltimes T} \widetilde{\chi}$, where Γ_χ is the stabilizer of χ in Γ and where $\widetilde{\chi}$ is the extension of χ to $\Gamma_\chi \ltimes T$ given by

$$\widetilde{\chi}(\gamma, a) = \chi(a) \qquad \text{for all} \quad \gamma \in \Gamma_\chi,\ a \in T.$$

The proposition will be proved if we can show that, for all $\chi \in S$, we have

$$(**) \qquad \|V_\chi(\mu)\| \leq \|V_\chi(p_{\mathrm{a}}(\mu))\|.$$

Now, the restriction of V_χ to Γ is equivalent to the natural representation of Γ in $\ell^2(\mathcal{O}_\chi)$, which is the induced representation $\mathrm{Ind}_{\Gamma_\chi \ltimes T}^{\Gamma \ltimes T} 1_\Gamma$. Observe that $\mathrm{Ind}_{\Gamma_\chi \ltimes T}^{\Gamma \ltimes T} 1_\Gamma$ is equivalent to $\left(\mathrm{Ind}_{\Gamma_\chi}^{\Gamma} 1_\Gamma\right) \circ p_{\mathrm{a}}$. Hence, Inequality $(**)$ follows from Herz's majoration principle (Proposition 17) and the proof of Theorem 5 is complete. ■

The following corollary gives a more precise information about the spectral structure of the Koopman representation associated to the action on T of a countable subgroup of $\mathrm{Aff}(T)$.

Corollary 20 *Let H be a a countable subgroup of* $\mathrm{Aff}(T)$ *and* $\Gamma = p_{\mathrm{a}}(H)$. *There exists a Γ-invariant torus factor $\overline{T}$ of T such that the projection of H in* $\mathrm{Aff}(\overline{T})$ *is an amenable group and which is the largest one with this property: every other Γ-invariant torus factor S of T for which the projection of H in* $\mathrm{Aff}(S)$ *is amenable is a factor of $\overline{T}$. Moreover, the torus factor $\overline{T}$ has the following properties:*

(i) the projection of Γ *on* $\mathrm{Aut}(\overline{T})$ *is a virtually polycyclic group;*

(ii) the restriction to $L^2(\overline{T})^\perp$ *of the Koopman representation of* H *does not weakly contain the trivial representation* 1_H.

Proof As for the proof of Theorem 5, we proceed by duality, using Fourier analysis and identifying V and Δ with their dual groups.

Let $V_{\mathrm{rat}}(\Gamma)$ be the subspace generated by the union of Γ-invariant rational subspaces W of V for which Γ_W is amenable. Then $V_{\mathrm{rat}}(\Gamma)$ is a Γ-invariant rational subspace and, by Proposition 9, $\Gamma_{V_{\mathrm{rat}}(\Gamma)}$ is amenable.

We claim that the natural unitary representation of Γ on $\ell^2(\Delta\backslash(V_{\mathrm{rat}}(\Gamma)\cap\Delta))$ does not weakly contain 1_Γ. Indeed, assume by contradiction that this is not the case. Then there exists a Γ-invariant mean m on $\Delta\setminus(V_{\mathrm{rat}}(\Gamma)\cap\Delta))$. We consider the vector space $\overline{V}=V/V_{\mathrm{rat}}(\Gamma)$ with the lattice $\overline{\Delta}=p(\Delta)$, where $p:V\to\overline{V}$ is the canonical projection. Then $p_*(m)$ is a Γ-invariant mean on $\overline{\Delta}\setminus\{0\}$. Hence, by Proposition 13, there exists a non-trivial Γ-invariant rational $\overline{W}$ subspace of $\overline{V}$ such that the image of Γ in $GL(\overline{W})$ is amenable. Then $W=p^{-1}(\overline{W})$ is a Γ-invariant rational subspace of V for which Γ_W is amenable. This is a contradiction since $V_{\mathrm{rat}}(\Gamma)$ is a proper subspace of W.

Let $\Gamma^0=\Gamma\cap \mathrm{Zc}(\Gamma)^0$. By Proposition 15, the eigenvalues of the restriction of every element in $[\Gamma^0,\Gamma^0]$ to $V_{\mathrm{rat}}(\Gamma)$ are all of modulus 1. Hence, by Corollary 16, the image of $[\Gamma^0,\Gamma^0]$ in $GL(V_{\mathrm{rat}}(\Gamma))$ is virtually nilpotent. It follows that $\Gamma_{V_{\mathrm{rat}}(\Gamma)}$ is virtually polycyclic.■

8 Some basic facts on Kirillov's theory and on decay of matrix coefficients of unitary representations

We first recall some basic facts from Kirillov's theory of unitary representations of nilpotent Lie groups.

For a locally compact second countable group G, the unitary dual $\widehat{G}$ of G is the set of classes (for unitary equivalence) of irreducible unitary representations of G.

Let N be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Kirillov's theory provides a parametrization of $\widehat{N}$ in terms of the co-adjoint orbits in the dual space $\mathfrak{n}^*=\mathrm{Hom}_{\mathbf{R}}(\mathfrak{n},\mathbf{R})$ of $\mathfrak{n}$. We will review the basic features of this theory.

Fix $l \in \mathfrak{n}^*$. There exists a polarization $\mathfrak{m}$ for l, that is, a Lie subalgebra $\mathfrak{m}$ such that $l([\mathfrak{m}, \mathfrak{m}]) = 0$ and which is of maximal dimension; the codimension of $\mathfrak{m}$ is $\frac{1}{2}\dim(\mathrm{Ad}^*(N)l)$, where $\mathrm{Ad}^*(N)l$ is the orbit of l under the co-adjoint representation Ad^* of N. The induced representation $\mathrm{Ind}_M^N \chi_l$ is irreducible, where $M = \exp(\mathfrak{m})$ and χ_l is the unitary character of M defined by

$$\chi_l(\exp X) = e^{2\pi i l(X)}, \qquad X \in \mathfrak{m}.$$

The unitary equivalence class of $\mathrm{Ind}_M^N \chi_l$ only depends on the co-adjoint orbit $\mathrm{Ad}^*(N)l$ of l. We obtain in this way a mapping

$$\mathfrak{n}^*/\mathrm{Ad}^*(N) \to \widehat{N}, \qquad \mathcal{O} \mapsto \pi_{\mathcal{O}}$$

called the Kirillov mapping, from the orbit space $\mathfrak{n}^*/\mathrm{Ad}^*(N)$ of the co-adjoint representation to the unitary dual $\widehat{N}$ of N The Kirillov mapping is in fact a bijection. For all of this, see [Kiri62] or [CoGr89].

We have to recall a few general facts about decay of matrix coefficients of unitary group representations, following [HoMo79] and [Howe82].

Let $(\pi, \mathcal{H})$ be a unitary representation of the locally compact group G. The projective kernel of π is the normal subgroup P_π of G defined by

$$P_\pi = \{g \in G \ : \ \pi(g) = \lambda_\pi(g)I \text{ for some } \lambda_\pi(g) \in \mathbf{C}\}.$$

Observe that the mapping $g \mapsto \lambda_\pi(g)$ defines a unitary character λ_π of P_π. Observe also that, for $\xi, \eta \in \mathcal{H}$, the absolute value of the matrix coefficient

$$C^\pi_{\xi,\eta} : g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is constant on cosets modulo P_π. For a real number p with $1 \leq p < +\infty$, the representation π is said to be strongly L^p modulo P_π, if there is dense subspace $D \subset \mathcal{H}$. such that, for every $\xi, \eta \in D$, the function $|C^\pi_{\xi,\eta}|$ belongs to $L^p(G/P_\pi)$. Observe that then π is strongly L^q modulo P_π for any $q > p$, since $C^\pi_{\xi,\eta}$ is bounded.

Moreover, if π is strongly L^2 modulo P_π, then π is contained in an infinite multiple of $\mathrm{Ind}_{P_\pi}^G \lambda_\pi$ (this can be shown by a straightforward adaptation of Proposition 1.2.3 in Chapter V of [HoTa92]).

We will also use the notion of a projective representation. Recall that a mapping $\pi : G \to U(\mathcal{H})$ from G to the unitary group of the Hilbert space $\mathcal{H}$ is a projective representation of G if the following holds:

- $\pi(e) = I$,
- for all $g_1, g_2 \in G$, there exists $c(g_1, g_2) \in \mathbf{C}$ such that

$$\pi(g_1 g_2) = c(g_1, g_2)\pi(g_1)\pi(g_2),$$

- the function $g \mapsto \langle \pi(g)\xi, \eta \rangle$ is measurable for all $\xi, \eta \in \mathcal{H}$.

The mapping $c : G \times G \to \mathbf{S}^1$ is a 2-cocycle with values in the unit cercle $\mathbf{S}^1$. The projective kernel of π is defined in the same way as for an ordinary representation. Every projective unitary representation of G can be lifted to an ordinary unitary representation of a central extension of G (for all this, see [Mack76] or [Mack58]).

9 Decay of extensions of irreducible representations of nilpotent Lie groups

Let N be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$.

The group $\mathrm{Aut}(N)$ of continuous automorphisms of N can be identified with the group $\mathrm{Aut}(\mathfrak{n})$ of automorphisms of the Lie algebra $\mathfrak{n}$ of N, by means of the mapping $\varphi \mapsto d_e\varphi$, where $d_e\varphi : \mathfrak{n} \to \mathfrak{n}$ is the differential of $\varphi \in \mathrm{Aut}(N)$ at the group unit. In this way, $\mathrm{Aut}(N)$ becomes an algebraic subgroup of $GL(\mathfrak{n})$. Therefore, the group $\mathrm{Aff}(N) = \mathrm{Aut}(N) \ltimes N$ of affine transformations of N is also an algebraic group over $\mathbf{R}$.

Set $G := \mathrm{Aff}(N)$. In the following, we view N as a normal subgroup of G. The group G acts by inner automorphisms on N and hence by automorphisms on $\mathfrak{n}, \mathfrak{n}^*$, and $\widehat{N}$; observe that, for $g \in G$ and $l \in \mathfrak{n}^*$, we have

$$(\mathrm{Ad}^*(n)l)^g = \mathrm{Ad}^*(gng^{-1})(l^g) \qquad \text{for all} \quad n \in N.$$

This shows that g permutes the orbits of the co-adjoint representation, mapping the orbit of l onto the orbit of l^g. Let $\pi \in \widehat{N}$ with corresponding co-adjoint orbit $\mathcal{O}$. The representation $\pi^g \in \widehat{N}$, defined by $\pi^g(n) = \pi(gng^{-1})$, corresponds to the orbit $\mathcal{O}^g$.

For a co-adjoint orbit $\mathcal{O}$ in $\mathfrak{n}^*$, we denote by $G_{\mathcal{O}}$ the stabilizer of $\mathcal{O}$ in G. Similarly,

$$G_\pi = \{g \in G \ : \ \pi^g \text{ is equivalent to } \pi\}$$

is the stabilizer in G of $\pi \in \widehat{N}$. Observe that, if π is the representation corresponding to the co-adjoint orbit $\mathcal{O}$ in Kirillov's picture, then $G_\pi = G_{\mathcal{O}}$. Observe also that N is contained in G_π.

The following elementary fact will be crucial for the sequel.

Proposition 21 *Let π be an irreducible unitary representation of N. The stabilizer G_π of π is an algebraic subgroup of G. Moreover, for every l in the co-adjoint orbit corresponding to π, we have $G_\pi = G_l N$ where G_l is the stabilizer of l in G*

Proof The co-adjoint orbit $\mathcal{O}$ associated to π is an algebraic subvariety of $\mathfrak{n}^*$ (see Theorem 3.1.4 in [CoGr89]). It follows that $G_\pi = G_{\mathcal{O}}$ is an algebraic subgroup of G. Moreover, since N acts transitively on $\mathcal{O}$, it is clear that $G_{\mathcal{O}} = G_l N$ for every $l \in \mathcal{O}$. ■

Let π be an irreducible unitary representation of N, with Hilbert space $\mathcal{H}$. It is a well-known part of Mackey's theory of unitary representations of group extensions that there exists a projective unitary representation $\widetilde{\pi}$ of G_π on $\mathcal{H}$ which extends π. Indeed, for every $g \in G_\pi$, there exists a unitary operator $\widetilde{\pi}(g)$ on $\mathcal{H}$ such that

$$\pi(g(n)) = \widetilde{\pi}(g)\pi(n)\widetilde{\pi}(g)^{-1} \qquad \text{for all} \quad n \in N.$$

One can choose $\widetilde{\pi}(g)$ such that $g \mapsto \widetilde{\pi}(g)$ is a projective representation unitary representation of G_π which extends π (see Theorem 8.2 in [Mack58]).

The following proposition, which will play a central rôle in our proofs, is a consequence of arguments from [HoMo79] concerning decay properties of unitary representations of algebraic groups.

Proposition 22 *Let π be an irreducible unitary representation of N on $\mathcal{H}$ and let $\widetilde{\pi}$ be a projective unitary representation of G_π which extends π. There exists a real number $p \geq 1$, only depending on the dimension of G, such that $\widetilde{\pi}$ is strongly L^p modulo its projective kernel.*

Proof Since π is irreducible, $\widetilde{\pi}(g)$ is uniquely determined up to a scalar multiple of the identity operator I for every $g \in G_\pi$. In particular, all projective unitary representations of G_π which extend π have the same projective kernel.

We will need to give an explicit construction of a projective representation of G_π extending π. This representation will lift to an ordinary representation of a two-fold cover of G_π.

We denote by $\mathcal{O}$ the co-adjoint orbit associated to π and we fix throughout the proof a linear functional l in $\mathcal{O}$.

Set $H = \mathrm{Aut}(N)$ so that $G = H \ltimes N$. Let H_l be the stabilizer of l in H. As shown in Proposition 21, G_π is an algebraic subgroup of G and $G_\pi = H_l N$. It is clear that H_l is also an algebraic subgroup of G. Let U_l be the unipotent radical of H_l. Then $U = U_l N$ is the unipotent radical of G_π.

•*First step:* We claim that π can be extended to an ordinary unitary representation σ of U.

Indeed, let $\mathfrak{u}_l$ be the Lie algebra of U_l. We extend l to a linear functional $\widetilde{l}$ on the Lie algebra $\mathfrak{u} = \mathfrak{u}_l \oplus \mathfrak{n}$ of U by defining $\widetilde{l}(X) = 0$ for all $X \in \mathfrak{u}_l$.

Let $\mathfrak{m} \subset \mathfrak{n}$ be a polarization for l. We claim that $\widetilde{\mathfrak{m}} := \mathfrak{u}_l \oplus \mathfrak{m}$ is a polarization for $\widetilde{l}$. Indeed, we have $\widetilde{l}([\widetilde{\mathfrak{m}}, \widetilde{\mathfrak{m}}]) = 0$ since $[X, Y] \in \mathfrak{n}$ and $(\exp X)l = l$ for all $X \in \mathfrak{u}_l$ and $Y \in \mathfrak{m}$. Moreover, the codimension of $\widetilde{\mathfrak{m}}$ in $\mathfrak{u}$ coincides with the codimension of $\mathfrak{m}$ in $\mathfrak{n}$ and the dimension of the co-adjoint orbit of $\widetilde{l}$ under $\mathrm{Ad}^*(U)$ coincides with the dimension of $\mathrm{Ad}^*(N)l$. Since the codimension of $\mathfrak{m}$ in $\mathfrak{n}^*$ is $\frac{1}{2}\dim(\mathrm{Ad}^*(N)l)$, it follows that the codimension of $\widetilde{\mathfrak{m}}$ in $\mathfrak{u}^*$ is $\frac{1}{2}\dim(\mathrm{Ad}^*(U)\widetilde{l})$. Hence, $\widetilde{\mathfrak{m}}$ is a polarization for $\widetilde{l}$.

Recall that π is unitarily equivalent to the induced representation $\mathrm{Ind}_M^N \chi_l$, where $M = \exp(\mathfrak{m})$ and χ_l is the unitary character of M defined by

$$\chi_l(\exp X) = e^{2\pi i l(X)} \qquad \text{for all} \quad X \in \mathfrak{m}.$$

Let $\widetilde{M}$ be the closed subgroup of U corresponding to $\widetilde{\mathfrak{m}}$. The unitary character $\chi_{\widetilde{l}}$ of $\widetilde{M}$ given by $\widetilde{l}$ coincides with χ_l on M. Since a fundamental domain for $M\backslash N$ is also a fundamental domain for $\widetilde{M}\backslash U$, we see that $\mathrm{Ind}_{\widetilde{M}}^U \chi_{\widetilde{l}}$ can be realized on the Hilbert space of $\mathrm{Ind}_M^N \chi_l$ and that $\sigma := \mathrm{Ind}_{\widetilde{M}}^U \chi_{\widetilde{l}}$ extends $\pi = \mathrm{Ind}_M^N \chi_l$.

•*Second step:* We claim that $G_\sigma = G_\pi$.

It is obvious that $G_\sigma \subset G_\pi$. Let $H_l = RU_l$ be a Levi decomposition of H_l, where R is a reductive subgroup of G_l. In order to show that $G_\pi \subset G_\sigma$, it suffices to prove that $R \subset G_\sigma$, since $G_\pi = RU$. Now, R leaves $\mathfrak{u}_l$ and $\mathfrak{n}$ invariant and fixes l. Hence, R fixes the extension $\widetilde{l}$ of l defined above and the claim follows.

•*Coda:* As a result, upon replacing N by U, we can assume that N is the unipotent radical of G_π. Since the connected component of G_π has finite index, we can also assume that G_π is connected.

As shown above, we have a Levi decomposition $G_\pi = RN$ with R a reductive subgroup contained in G_l. According to [Howe73], we can find in N algebraic subgroups $K_1 \subset P_1 \subset N_1$ with the following properties:

- K_1, P_1, and N_1 are normalized by R;
- K_1 and P_1 are normal in N_1 and N_1/K_1 is a Heisenberg group with centre P_1/K_1;
- there exists a unitary character λ of P_1/K_1 such that π is equivalent to the induced representation $\mathrm{Ind}_{N_1}^{N}\pi_1$, where π_1 is the lift to N_1 of the unique irreducible representation of the Heisenberg group N_1/K_1 with central character λ.

The action of R on N_1/K_1 defines a homomorphism from R to the symplectic group $Sp(N_1/P_1)$ of the vector space N_1/P_1; as a result, we have a homomorphism $\varphi : RN_1 \to Sp(N_1/P_1) \ltimes (N_1/K_1)$. The representation π_1 of N_1/K_1 extends to a projective representation ω of $Sp(N_1/P_1) \ltimes (N_1/K_1)$, called the metaplectic (or oscillator, or Shale-Weil) representation; more precisely, there exists a two-fold cover $\widetilde{Sp}$ of $Sp(N_1/P_1)$ and a unitary representation ω of $\widetilde{Sp} \ltimes (N_1/K_1)$ on the Hilbert space of π_1 which extends π_1.

We can lift φ to a homomorphism $\widetilde{\varphi} : \widetilde{R}N_1 \to \widetilde{Sp} \ltimes (N_1/K_1)$ for a two-fold cover $\widetilde{R}$ of R. Then $\rho := \omega \circ \widetilde{\varphi}$ is a unitary representation of $\widetilde{R}N_1$ on the Hilbert space of π_1 which extends π_1.

Set $\widetilde{\pi} := \mathrm{Ind}_{\widetilde{R}N_1}^{\widetilde{R}N}\rho$. Then $\widetilde{\pi}$ is a unitary representation of the two-fold cover $\widetilde{G}_\pi := \widetilde{R}N$ of $G_\pi = RN$; moreover, $\widetilde{\pi}$ extends π, since π is equivalent to $\mathrm{Ind}_{N_1}^{N}\pi_1$, and ρ extends π_1.

Observe that $\widetilde{G}_\pi$ is in general not an algebraic group. Let $p : \widetilde{G}_\pi \to G_\pi$ be the covering map. Let us say that a connected subgroup H of $\widetilde{G}_\pi$ is reductive if $p(H)$ is a reductive subgroup of G_π. We claim that $\widetilde{G}_\pi$ has no non-trivial reductive normal subgroup. Indeed, let H be a reductive normal subgroup of $\widetilde{G}_\pi$. Since $G_\pi = RN$ is a Levi decomposition of G_π, the normal subgroup $p(H)$ of G_π is conjugate to a subgroup of R and therefore $p(H) \subset R$. Hence, $p(H)$ centralizes N. It follows that $p(H)$ is trivial since $p(H) \subset \mathrm{Aut}(N)$.

Now, the same arguments as those on pages 87–93 in [HoMo79] show that there exists an integer k such that the k-fold tensor power $\widetilde{\pi}^{\otimes k}$ of π is square integrable modulo the projective kernel $P_{\widetilde{\pi}}$ of $\widetilde{\pi}$. For instance, let us check how the first step in [HoMo79] towards this claim carries over to our situation. For an integer k, we are interested in the tensor power $\widetilde{\pi}^{\otimes k}$. In order to apply Mackey's tensor product theorem (see [Mack76, Theorem 3.6]), we have to show that $(\widetilde{R}N_1)^k$ and the diagonal subgroup $\Delta\widetilde{G}_\pi$ of $\widetilde{G}_\pi^k$ are regularly related. Now, the quotient space $\widetilde{G}_\pi^k/(\widetilde{R}N_1)^k$ is can be canonically identified with $G_\pi^k/(RN_1)^k$, and the action of $\Delta\widetilde{G}_\pi$ on $\widetilde{G}_\pi^k/(\widetilde{R}N_1)^k$ corresponds, via the covering mapping $p : \widetilde{G}_\pi \to G_\pi$, to the action of ΔG_π on $G_\pi^k/(RN_1)^k$. Since ΔG_π of G_π^k are algebraic subgroups of G_π^k, the claim follows. ■

Remark 23 According to [HoMo79, p.93], a crude bound for the number p in Proposition 22 is

$$p \leq (\dim(G_\pi) + 1)^2.$$

The generalized metaplectic representation $\widetilde{\pi}$ which appears in the proof above has been studied by several authors (see [Dufl72], [Howe73], [Lion79]).

10 Rational unitary representations of a nilpotent Lie group

As in the previous section, let N be a connected and simply connected nilpotent Lie group and

$$G := \mathrm{Aff}(N) = \mathrm{Aut}(N) \ltimes N.$$

Let π be an irreducible unitary representation of N and G_π the stabilizer of π in G. Let $\widetilde{\pi}$ be a projective unitary representation of G_π extending π. In the following proposition, we describe the projective kernel $P_{\widetilde{\pi}}$ of $\widetilde{\pi}$.

Proposition 24 *Let L_π be the connected component of* $\mathrm{Ker}(\pi)$. *Set $\overline{N} = N/L_\pi$ and let $p : N \to \overline{N}$ be the canonical projection. For $g = (h, n) \in G_\pi$ with $h \in$* $\mathrm{Aut}(N)$ *and $n \in N$, the following are conditions are equivalent:*

(i) $g \in P_{\widetilde{\pi}}$;

(ii) h leaves L_π invariant and the automorphism of $\overline{N}$ induced by h coincides with the inner automorphism $\mathrm{Ad}(p(n)^{-1})$.

Proof Assume that $g = (h, n) \in P_{\widetilde{\pi}}$. By definition of $P_{\widetilde{\pi}}$, we have $\widetilde{\pi}(h) = \lambda_\pi(g)\pi(n^{-1})$. It follows that, for every $x \in N$

$$\pi(h(x)) = \widetilde{\pi}(h)\pi(x)\widetilde{\pi}(h)^{-1} = \pi(n^{-1})\pi(x)\pi(n) = \pi(n^{-1}xn),$$

that is,

$$h(x)n^{-1}x^{-1}n \in \mathrm{Ker}(\pi) \qquad \text{for all} \quad x \in N.$$

Since N is connected, this is equivalent to

$$h(x)n^{-1}x^{-1}n \in L_\pi \qquad \text{for all} \quad x \in N.$$

As L_π is normal in N, this shows that L_π is invariant under h and that the automorphism induced by h on $\overline{N}$ is $\mathrm{Ad}(p(n)^{-1})$.

Conversely, suppose that L_π is invariant under h and that the automomorphism $\overline{h}$ induced by h on $\overline{N}$ coincides with $\mathrm{Ad}(p(n)^{-1})$. Observe that π factorizes to a representation σ of $\overline{N}$. Let $\widetilde{\sigma}$ be an extension of σ to the stabilizer of σ in $\mathrm{Aut}(\overline{N}) \ltimes \overline{N}$. Then

$$\widetilde{\sigma}(\overline{h})\sigma(p(x))\widetilde{\sigma}(\overline{h})^{-1} = \sigma(p(n))^{-1}\sigma(p(x))\sigma(p(n)) \qquad \text{for all} \quad x \in N,$$

that is, $\sigma(p(n))\widetilde{\sigma}(\overline{h})$ commutes with $\sigma(p(x))$ for all $x \in N$. Since π is irreducible, it follows that $\sigma(p(n))\widetilde{\sigma}(\overline{h})$ and hence $\pi(n)\widetilde{\pi}(h)$ is a scalar operator. This means that $g = (h, n) \in P_{\widetilde{\pi}}$.■

Next, we review some well-known facts about rational structures on $\mathfrak{n}$ (see [CoGr89], [Ragh72]).

Recall first that a lattice Γ in a locally compact group G is a discrete subgroup such that the translation invariant measure induced by a Haar measure on G on the homogeneous space $\Gamma\backslash G$ is finite.

The Lie algebra $\mathfrak{n}$ (or the corresponding nilpotent Lie group $N = \exp(\mathfrak{n})$) has a *rational structure* if there is a Lie algebra $\mathfrak{n}_{\mathbf{Q}}$ over $\mathbf{Q}$ such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. If $\mathfrak{n}$ has a rational structure given by $\mathfrak{n}_{\mathbf{Q}}$, then N contains a cocompact lattice Λ such that $\log \Lambda \subset \mathfrak{n}_{\mathbf{Q}}$. Conversely, if N contains a lattice Λ, then Λ is cocompact and $\mathfrak{n}$ has a rational structure given by $\mathfrak{n}_{\mathbf{Q}} = \mathbf{Q} - \mathrm{span}(\log \Lambda)$.

Assume from now on that N has a rational structure $\mathfrak{n}_{\mathbf{Q}}$ and let Λ be a lattice inducing this rational structure. We say that a $\mathbf{R}$-subspace $\mathfrak{h}$ of $\mathfrak{n}$ is rational if $\mathfrak{h} = \mathbf{R} - \mathrm{span}(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{Q}})$. All subalgebras in the ascending or ascending series as well as the centre of $\mathfrak{n}$ are rational. A connected closed subgroup H of N is said to be rational if the corresponding subalgebra Lie algebra $\mathfrak{h}$ is rational. This is equivalent to the fact that $H \cap \Lambda$ is a lattice in H.

Let H be a rational connected normal closed subgroup of N with Lie algebra $\mathfrak{h}$ Then N/H has a canonical rational structure $(\mathfrak{n}/\mathfrak{h})_{\mathbf{Q}}$ induced by the lattice $\Lambda H/H$ of N/H.

There is a unique rational structure $\mathfrak{n}^*_{\mathbf{Q}}$ on the dual space $\mathfrak{n}^*$ defined as follows: a functional $l \in \mathfrak{n}^*$ belongs to $\mathfrak{n}^*_{\mathbf{Q}}$ if and only if $l(X) \in \mathbf{Q}$ for all $X \in \mathfrak{n}_{\mathbf{Q}}$.

An important role will be played later (in Section 12) by irreducible unitary representations of N which are rational in the sense of the following definition.

Definition 25 An irreducible unitary representation π of N is *rational* if its co-adjoint orbit $\mathcal{O}_\pi$ is rational, that is, if $\mathcal{O}_\pi \cap \mathfrak{n}^*_{\mathbf{Q}} \neq \emptyset$.

We fix for the rest of this section a rational irreducible unitary representation π of N.

We first establish the rationality of the kernel of π.

Proposition 26 *The connected component L_π of* $\mathrm{Ker}(\pi)$ *is a rational normal subgroup of N. As a consequence, $\overline{\Lambda} = \Lambda L_\pi / L_\pi$ is a lattice in N/L_π.*

Proof Since π is rational, the corresponding co-adjoint orbit in $\mathfrak{n}^*$ contains a functional $l \in \mathfrak{n}^*_{\mathbf{Q}}$. The representation π is unitarily equivalent to $\mathrm{Ind}_M^G \chi_l$, where $\mathfrak{m}$ is a polarization for l, $M = \exp(\mathfrak{m})$, and χ_l is the unitary character of M corresponding to l.

Recall from Lemma 18 that $\mathrm{Ker}(\pi)$ coincides with the largest normal subgroup of N contained in $\mathrm{Ker}(\chi_l)$. For the ideal $\mathfrak{l}$ corresponding to $\mathrm{Ker}(\pi)$, we have therefore

$$\mathfrak{l} = \bigcap_{n \in N} \mathrm{Ker}(\mathrm{Ad}^*(n)l) = \bigcap_{X \in \mathfrak{n}_{\mathbf{Q}}} \mathrm{Ker}(\mathrm{Ad}^*(\exp X)l).$$

Since $\mathrm{Ker}(\mathrm{Ad}^*(\exp X)l)$ is rational for all $X \in \mathfrak{n}_{\mathbf{Q}}$, it follows that $\mathfrak{l}$ is rational. Thus, the connected component L_π of $\mathrm{Ker}(\pi)$ is rational, by definition. ■

The set $\mathrm{Aut}(\Lambda\backslash N)$ consisting of the automorphisms $\gamma \in \mathrm{Aut}(N)$ with $\gamma(\Lambda) = \Lambda$ is a discrete subgroup of the algebraic group $\mathrm{Aut}(N)$.

Let G_π be the stabilizer of π in G and $\widetilde{\pi}$ a projective unitary representation of G_π extending π. Set

$$\Gamma_\pi = G_\pi \cap \mathrm{Aut}(\Lambda\backslash N).$$

The projective kernel $P_{\widetilde{\pi}}$ of $\widetilde{\pi}$ was determined in Proposition 24 . We will need to have a precise description of $P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$.

As before, let L_π be the connected component of $\mathrm{Ker}(\pi)$, $\overline{N} = N/L_\pi$, $p : N \to \overline{N}$ the canonical projection, and $\overline{\Lambda} = p(\Lambda)$. Observe that $g(L_\pi) = L_\pi$ for all $g \in G_\pi \cap \mathrm{Aut}(N)$. Consider the induced continuous homomorphism

$$\varphi : G_\pi \to \mathrm{Aff}(\overline{N}) = \mathrm{Aut}(\overline{N}) \ltimes \overline{N}.$$

Proposition 27 *Let* $\mathrm{Norm}(\overline{\Lambda})$ *be the normalizer of* $\overline{\Lambda}$ *in* $\overline{N}$.

(i) We have

$$P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N) = \varphi^{-1}\left(\{(\mathrm{Ad}(x), x^{-1}) \ : \ x \in \mathrm{Norm}(\overline{\Lambda})\}\right).$$

(ii) Let $\Delta := \{(\mathrm{Ad}(x), x^{-1}z) \ : x \in \overline{\Lambda}, z \in Z(\overline{N})\}$, *where* $Z(\overline{N})$ *is the centre of* $\overline{N}$. *Then* $\varphi^{-1}(\Delta) \cap (\Gamma_\pi \ltimes N)$ *is a subgroup of finite index in* $P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$.

Proof (i) By Proposition 24, we have

$$P_{\widetilde{\pi}} = \varphi^{-1}\left(\{(\mathrm{Ad}(x), x^{-1}) \ : x \in \overline{N}\}\right).$$

Let $g = (\gamma, n) \in P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$. Then $\varphi(g) = (\mathrm{Ad}(x), x^{-1})$ for some $x \in \overline{N}$. Since $\gamma(\Lambda) = \Lambda$, we have $\mathrm{Ad}(x)(\overline{\Lambda}) = \overline{\Lambda}$, that is, $x \in \mathrm{Norm}(\overline{\Lambda})$. Conversely, it is obvious that, if $g = (\mathrm{Ad}(x), x^{-1})$ for some $x \in \mathrm{Norm}(\overline{\Lambda})$, then $g \in P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$.

(ii) In view of (i), it suffices to prove that the subgroup $\overline{\Lambda} Z(\overline{N})$ has finite index in $\mathrm{Norm}(\overline{\Lambda})$.

To show this, recall that $\overline{\Lambda}$ is a cocompact lattice in $\overline{N}$ (Proposition 26). Let $\mathrm{Norm}(\overline{\Lambda})_0$ be the connected component of $\mathrm{Norm}(\overline{\Lambda})$. Since $\mathrm{Norm}(\overline{\Lambda})_0$ normalizes $\overline{\Lambda}$ and since $\overline{\Lambda}$ is discrete, $\mathrm{Norm}(\overline{\Lambda})_0$ lies in the centralizer of every element of $\overline{\Lambda}$. As $\overline{\Lambda}$ is Zariski dense in $\overline{N}$ (see e.g. Theorem 2.1 in [Ragh72]), it follows that $\mathrm{Norm}(\overline{\Lambda})_0 = Z(\overline{N})$. Since the projection of $\overline{\Lambda}$ has finite covolume in the discrete group $\mathrm{Norm}(\overline{\Lambda})/\mathrm{Norm}(\overline{\Lambda})_0$, the claim follows.■

The next proposition will allow us to deduce decay properties of representations of G_π restricted to $\Gamma_\pi \ltimes N$.

Proposition 28 *The subgroup* $(\Gamma_\pi \ltimes N)P_{\widetilde{\pi}}$ *is closed in* G_π.

Proof Using Proposition 24, we see that

$$P_{\widetilde{\pi}} N = \varphi^{-1}\left(\mathrm{Ad}(\overline{N}) \ltimes \overline{N}\right)$$

and hence

$$(\Gamma_\pi \ltimes N)P_{\widetilde{\pi}} = \varphi^{-1}\left((\varphi(\Gamma_\pi)\mathrm{Ad}(\overline{N})) \ltimes \overline{N}\right).$$

It therefore suffices to show that $\varphi(\Gamma_\pi)\mathrm{Ad}(\overline{N})$ is closed in $\mathrm{Aut}(\overline{N})$.

Observe that, for every $\gamma \in \Gamma_\pi$, we have $\gamma(\Lambda) = \Lambda$ (since $\Gamma_\pi \subset \mathrm{Aut}(\Lambda\backslash N)$) and hence $\varphi(\Gamma_\pi) \subset \mathrm{Aut}(\overline{\Lambda}\backslash\overline{N})$.

Let $(\gamma_i)_i$ and $(x_i)_i$ be sequences in Γ_π and in $\mathrm{Ad}(\overline{N})$ such that

$$\lim_i \varphi(\gamma_i)x_i = g \in \mathrm{Aut}(\overline{N}).$$

Since $\mathrm{Ad}(\overline{\Lambda})$ is a cocompact lattice in $\mathrm{Ad}(\overline{N})$, there exists a compact subset D of $\mathrm{Ad}(\overline{N})$ such that $x_i = \delta_i d_i$ for some $\delta_i \in \mathrm{Ad}(\overline{\Lambda})$ and $d_i \in D$. As D is compact, we can assume that $\lim d_i = d \in \mathrm{Ad}(\overline{N})$ exists. Then $\lim_i \varphi(\gamma_i)\delta_i = gd^{-1}$. Now,

$$\mathrm{Ad}(\overline{\Lambda}) = \varphi(\mathrm{Ad}(\Lambda)) \subset \varphi(\Gamma_\pi)$$

and $\varphi(\Gamma_\pi)$ is a subgroup of the discrete group $\mathrm{Aut}(\overline{\Lambda}\backslash\overline{N})$. It follows that $gd^{-1} \in \varphi(\Gamma_\pi)$, that is, $g \in \varphi(\Gamma_\pi)\mathrm{Ad}(\overline{N})$. Hence, $\varphi(\Gamma_\pi)\mathrm{Ad}(\overline{N})$ is closed in $\overline{N}$. ■

Corollary 29 *Let $\Delta = \{(\mathrm{Ad}(x), x^{-1}z) \ : x \in \overline{\Lambda}, z \in Z(\overline{N})\}$ and $\varphi : G_\pi \to \mathrm{Aff}(\overline{N})$ the canonical projection, where $\overline{N} = N/L_\pi$. The restriction of $\widetilde{\pi}$ to $\Gamma_\pi \ltimes N$ is strongly L^p modulo $\varphi^{-1}(\Delta) \cap (\Gamma_\pi \ltimes N)$ for the real number p appearing in Proposition 22.*

Proof We know from Proposition 27 that $\varphi^{-1}(\Delta) \cap (\Gamma_\pi \ltimes N)$ has finite index in $P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$. Hence, it suffices to prove that the restriction of $\widetilde{\pi}$ to $\Gamma_\pi \ltimes N$ is strongly L^p modulo $P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$.

By Proposition 28, $(\Gamma_\pi \ltimes N)P_{\widetilde{\pi}}$ is closed in G_π. Therefore, $(\Gamma_\pi \ltimes N)P_{\widetilde{\pi}}/P_{\widetilde{\pi}}$ is homeomorphic as a $(\Gamma_\pi \ltimes N)$-space to $(\Gamma_\pi \ltimes N)/(P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N))$. It follows from Proposition 22 (see the proof of Proposition 6.2 in [HoMo79]) that the restriction of $\widetilde{\pi}$ to $\Gamma_\pi \ltimes N$ is strongly L^p modulo $P_{\widetilde{\pi}} \cap (\Gamma_\pi \ltimes N)$. ■

11 A general estimate for norms of convolution operators

Let G be a locally compact group. For a unitary representation $(\pi, \mathcal{H})$ of G, the contragredient (or conjugate) representation $\overline{\pi}$ acts on the conjugate

Hilbert space $\overline{\mathcal{H}}$. Recall that, for an integer $k \geq 1$, the k-fold tensor product $\pi^{\otimes k}$ of π is a unitary representation of G acting on the tensor product Hilbert space $\mathcal{H}^{\otimes k}$.

We will need in a crucial way the following estimate which appears in the proof of Theorem 1 in [Nevo98].

Proposition 30 *Let μ be a probability measure on the Borel subsets of G. Let $(\pi, \mathcal{H})$ be a unitary representation of G. For every integer $k \geq 1$, we have*

$$\|\pi(\mu)\| \leq \|\left(\pi \otimes \overline{\pi}\right)^{\otimes k}(\mu)\|^{1/2k},$$

Proof Denote by $\check{\mu}$ the probability measure on G defined by $\check{\mu}(A) = \mu(A^{-1})$ for every Borel subset A of G.

Using Jensen's inequality, we have for every vector $\xi \in \mathcal{H}$,

$$\begin{aligned}
\|\pi(\mu)\xi\|^{4k} &= |\langle \pi(\check{\mu} * \mu)\xi, \xi\rangle|^{2k} \\
&= \left|\int_G \langle \pi(g)\xi, \xi\rangle d(\check{\mu} * \mu)(g)\right|^{2k} \\
&\leq \int_G \langle|\pi(g)\xi, \xi\rangle|^{2k} d(\check{\mu} * \mu)(g) \\
&= \int_G |\langle(\pi \otimes \overline{\pi})(g)(\xi \otimes \xi), \xi \otimes \xi\rangle|^{k} d(\check{\mu} * \mu)(g) \\
&= \int_G \langle(\pi \otimes \overline{\pi})^{\otimes k}(g)(\xi \otimes \xi)^{\otimes k}, (\xi \otimes \xi)^{\otimes k}\rangle d(\check{\mu} * \mu)(g) \\
&= |\langle(\pi \otimes \overline{\pi})^{\otimes k}(\check{\mu} * \mu)(\xi \otimes \xi)^{\otimes k}, (\xi \otimes \xi)^{\otimes k}\rangle| \\
&= \|(\pi \otimes \overline{\pi})^{\otimes k}(\mu)(\xi \otimes \xi)^{\otimes k}\|^{2}.
\end{aligned}$$

and the claim follows. ■

12 Analysis of the Koopman representation of the affine group of a nilmanifold

Let N be a connected and simply connected nilpotent Lie group, Λ a lattice in N. There is a unique translation invariant probability measure $\nu_{\Lambda\backslash N}$ on $\Lambda\backslash N$ and it is induced by a Haar measure on N. This measure is also invariant under $\mathrm{Aut}(\Lambda\backslash N)$.

We fix throughout this section a subgroup Γ of $\mathrm{Aut}(\Lambda\backslash N)$. The Koopman representation U of $\Gamma \ltimes N$ associated to the action of $\Gamma \ltimes N$ on $\Lambda\backslash N$ is given by

$$U(\gamma, n)\xi(x) = \xi(\gamma^{-1}(x)n) \qquad \gamma \in \Gamma,\ n \in N,\ \xi \in L^2(\Lambda\backslash N),\ x \in \Lambda\backslash N.$$

In particular, we have

$$(1) \qquad U(\gamma^{-1})U(n)U(\gamma) = U(\gamma^{-1}(n)) \qquad \text{for all} \quad \gamma \in \Gamma,\ n \in N.$$

Recall that $T = \Lambda[N, N]\backslash N$ is the maximal factor torus associated to $\Lambda\backslash N$. The action of $\mathrm{Aff}(\Lambda\backslash N)$ on $\Lambda\backslash N$ induces an action of $\mathrm{Aff}(\Lambda\backslash N)$ on T. We identify $L^2(T)$ with a closed subspace of $L^2(\Lambda\backslash N)$.

More generally, let L be a connected closed subgroup of N which is both rational and invariant under Γ. Then $\Lambda\cap L$ is a lattice in L and $\overline{\Lambda} = \Lambda L/L$ is a lattice in $\overline{N} = N/L$. There is an induced action of $\Gamma\ltimes N$ on the subnilmanifold $L/(\Lambda \cap L)$ and on the factor nilmanifold $\overline{\Lambda}\backslash\overline{N}$. The canonical mapping $p : \Lambda\backslash N \mapsto \overline{\Lambda}\backslash\overline{N}$ is $\Gamma \ltimes N$-equivariant and presents $\Lambda\backslash N$ as a fibre bundle over $\overline{\Lambda}\backslash\overline{N}$ with fibres diffeomorphic to $L/(\Lambda \cap L)$. The Hilbert space $L^2(\overline{\Lambda}\backslash\overline{N})$ can be identified, as $\Gamma \ltimes N$-representation, with the $\Gamma \ltimes N$-invariant closed subspace of $L^2(\Lambda\backslash N)$ consisting of the square-integrable functions on $\Lambda\backslash N$ which are constant on the fibres of p.

We write

$$L^2(\Lambda\backslash N) = L^2(T) \oplus \mathcal{H},$$

where $\mathcal{H}$ is the orthogonal complement of $L^2(T)$ on $L^2(\Lambda\backslash N)$, and observe that $\mathcal{H}$ is invariant under $\mathrm{Aff}(\Lambda\backslash N)$.

We are going to show that the restriction of U to $\mathcal{H}$ has a canonical decomposition into a direct sum of induced representations from the stabilizers in $\Gamma \ltimes N$ of certain representations $\pi \in \widehat{N}$; this decomposition can be viewed as generalization of the decomposition of $L^2(T)$ which appears in the proof of Proposition 19.

Since Λ is cocompact in N, we can consider the decomposition of $\mathcal{H}$ into its N- isotypical components: we have

$$\mathcal{H} = \bigoplus_{\pi\in\Sigma} \mathcal{H}_\pi,$$

where Σ is a certain set of infinite-dimensional pairwise non-equivalent irreducible unitary representations of N; for every $\pi \in \Sigma$, the space $\mathcal{H}_\pi$ is the union of the closed $U(N)$-invariant subspaces $\mathcal{K}$ of $\mathcal{H}$ for which the corresponding representation of N in $\mathcal{K}$ is equivalent to π. According to [Moor65, Corollary2], every $\pi \in \Sigma$ is rational in the sense of Section 10. Every $\mathcal{H}_\pi$ is a direct sum of finitely many irreducible unitary representations; therefore, the restriction of $U(N)$ to $\mathcal{H}_\pi$ is unitarily equivalent to a tensor product $\pi \otimes I$ acting on $\mathcal{K}_\pi \otimes \mathcal{L}_\pi$, where $\mathcal{K}_\pi$ is the Hilbert space of π and where $\mathcal{L}_\pi$ is a finite dimensional Hilbert space. (For a precise computation of the dimension of $\mathcal{L}_\pi$, see [Howe71] and [Rich71]; the fact that $\mathcal{L}_\pi$ is finite-dimensional will not be relevant for our arguments.)

Let γ be a fixed automorphism in Γ. Let U^γ be the conjugate representation of U by γ, that is, $U^\gamma(g) = U(\gamma^{-1}(g))$ for all $g \in G$. On the one hand, for every $\pi \in \Sigma$, the subspace $\mathcal{H}_{\pi^{\gamma^{-1}}}$ is the isotypical component of $U^\gamma|_N$ corresponding to π. On the other hand, relation (1) shows that $U(\gamma^{-1})$ provides a unitary equivalence between $U|_N$ and $U^\gamma|_N$. It follows that

$$U(\gamma^{-1})(\mathcal{H}_\pi) = \mathcal{H}_{\pi^{\gamma^{-1}}} \qquad \text{for all} \quad \gamma \in \Gamma$$

In summary, we see that Γ permutes the $\mathcal{H}_\pi$'s among themselves according to its action on $\widehat{N}$.

Write $\Sigma = \bigcup_{i \in I} \Sigma_i$, where the Σ_i's are the Γ-orbits in Σ, and set

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{\pi \in \Sigma_i} \mathcal{H}_\pi.$$

Every $\mathcal{H}_{\Sigma_i}$ is invariant under $\Gamma_i \ltimes N$ and we have an orthogonal decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i}.$$

Fix $i \in I$. Choose a representation π_i in Σ_i and set $\mathcal{H}_i = \mathcal{H}_{\pi_i}$. Let Γ_i denote the stabilizer of π_i in Γ. The space $\mathcal{H}_i$ is invariant under $\Gamma_i \ltimes N$. Let V_i be the corresponding representation of $\Gamma_i \ltimes N$ on $\mathcal{H}_i$.

Choose a set S_i of representatives for the cosets in

$$\Gamma/\Gamma_i = (\Gamma \ltimes N)/(\Gamma_i \ltimes N)$$

with $e \in S_i$. Then $\Sigma_i = \{\pi_i^s \ : \ s \in S_i\}$ and the Hilbert space $\mathcal{H}_{\Sigma_i}$ is the sum of mutually orthogonal spaces:

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{s \in S_i} \mathcal{H}_i^s.$$

Moreover, $\mathcal{H}_i^s$ is the image under $U(s)$ of $\mathcal{H}_i$ for every $s \in S_i$. This exactly means that the restriction U_i of U to $\mathcal{H}_{\Sigma_i}$ of the Koopman representation U of $\Gamma \ltimes N$ is equivalent to the induced representation $\mathrm{Ind}_{\Gamma_i \ltimes N}^{\Gamma \ltimes N} V_i$.

As we have seen above, we can assume that $\mathcal{H}_i$ is the tensor product

$$\mathcal{H}_i = \mathcal{K}_i \otimes \mathcal{L}_i$$

of the Hilbert space $\mathcal{K}_i$ of π_i with a finite dimensional Hilbert space $\mathcal{L}_i$, in such a way that

$$(2) \qquad V_i(n) = \pi_i(n) \otimes I_{\mathcal{L}_i} \qquad \text{for all} \quad n \in N.$$

Let $g \in \Gamma_i \ltimes N$. By (1) and (2) above, we have

$$(3) \qquad V_i(g)\,(\pi_i(n) \otimes I_{\mathcal{L}_i})\,V_i(g)^{-1} = \pi_i(gng^{-1}) \otimes I_{\mathcal{L}_i} \qquad \text{for all} \quad n \in N.$$

On the other hand, let G_i be the stabilizer of π_i in $\mathrm{Aff}(N)$; then π_i extends to an irreducible projective representation $\widetilde{\pi}_i$ of G_i (see the remark just before Proposition 22). Since

$$\widetilde{\pi}_i(g)\pi_i(n)\ \widetilde{\pi}_i(g^{-1}) = \pi_i(gng^{-1}) \qquad \text{for all} \quad n \in N,$$

it follows from (3) that the operator $(\widetilde{\pi}_i(g^{-1}) \otimes I_{\mathcal{L}_i})\,V_i(g)$ commutes with $\pi_i(n) \otimes I_{\mathcal{L}_i}$ for all $n \in N$. Since π_i is irreducible, there exists a unitary operator $W_i(g)$ on $\mathcal{L}_i$ such that

$$V_i(g) = \widetilde{\pi}_i(g) \otimes W_i(g).$$

It is clear that W_i is a projective unitary representation of $\Gamma_i \ltimes N$, since V_i is a unitary representation of $\Gamma_i \ltimes N$.

13 Proof of Theorem 1: first step

We summarize the discussion from the previous section. We have a first orthogonal decomposition into $\mathrm{Aff}(\Lambda\backslash N)$-invariant subspaces

$$L^2(\Lambda\backslash N) = L^2(T) \oplus \mathcal{H},$$

where T is the maximal torus factor of $\Lambda\backslash N$. Let Γ be a subgroup of $\mathrm{Aut}(\Lambda\backslash N)$. There exists a sequence of Γ-invariant sets $(\Sigma_i)_{i\in I}$ of rational infinite dimensional unitary irreducible representations of N such that we have a decomposition into mutually orthogonal $\Gamma \ltimes N$-invariant subspaces

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_{\Sigma_i}$$

with the following property: for every i, the representation U_i of $\Gamma \ltimes N$ defined on $\mathcal{H}_{\Sigma_i}$ is equivalent to

$$\mathrm{Ind}_{\Gamma_i \ltimes N}^{\Gamma \ltimes N} \left(\widetilde{\pi}_i \otimes W_i\right),$$

where π_i is a representation from Σ_i, where $\widetilde{\pi}_i$ is the restriction to $\Gamma_i \ltimes N$ of an extension of π_i to the stabilizer G_i of π_i in $G = \mathrm{Aff}(N)$, and where W_i is some finite dimensional projective unitary representation of $\Gamma_i \ltimes N$.

We need to recall the decomposition of the representation U_{tor} of Γ on $L^2_0(T)$ from Section 7. Let $\widehat{T} \cong \mathbf{Z}^d$ be the dual group of T and let S be a set of representatives for the Γ-orbits in $\widehat{T} \setminus \{1_T\}$. Then

$$U_{\mathrm{tor}} \cong \bigoplus_{\chi\in S} \lambda_{\Gamma/\Gamma_\chi}, \tag{4}$$

where Γ_χ is the stabilizer of χ in Γ and $\lambda_{\Gamma/\Gamma_\chi}$ is the natural representation of Γ on $\ell^2(\Gamma/\Gamma_\chi)$.

In the following result, we establish a link between the restrictions to $\mathcal{H}$ and to $L^2_0(T)$ of the Koopman representation of Γ. This result, which is a consequence of the discussion above and of results from Section 10, is a major step in our proof of Theorem 1.

Recall that p_{a} denotes the canonical projection $\mathrm{Aff}(\Lambda\backslash N) \to \mathrm{Aut}(\Lambda\backslash N)$. For a probability measure μ on $\mathrm{Aff}(\Lambda\backslash N)$, let $p_{\mathrm{a}}(\mu)$ be the probability measure on $\mathrm{Aut}(\Lambda\backslash N)$ which is the image of μ under p_{a}.

Proposition 31 *There exists an integer $k \geq 1$ only depending on $\dim N$ with the following property. Let Γ be a subgroup of $\mathrm{Aut}(\Lambda\backslash N)$ which stabilizes some $\pi \in \widehat{N}$ appearing in the decomposition $\mathcal{H} = \bigoplus_{\pi\in\Sigma} \mathcal{H}_\pi$ of $\mathcal{H}$ into isotypical components under N. For every probability measure μ on $\Gamma \ltimes N$, we have*

$$\|U_\pi(\mu))\| \leq \|U_{\mathrm{tor}}(p_{\mathrm{a}}(\mu))\|^{1/2k},$$

where U_π and U_{tor} are the restrictions of the Koopman representation of $\Gamma \ltimes N$ to $\mathcal{H}_\pi$ and $L^2_0(T)$ respectively.

Proof Let G_π be the stabilizer of π in $G = \mathrm{Aff}(N)$. Let $\widetilde{\pi}$ a projective representation of G_π extending π.

As we have seen above, U_π is equivalent to $(\widetilde{\pi}|_{\Gamma\ltimes N}) \otimes W$ for some finite dimensional projective unitary representation W of $\Gamma \ltimes N$. Let P denote the projective kernel of U_π. Observe that $P = P_1 \cap P_2$, where P_1 and P_2 are the projective kernels of $\widetilde{\pi}|_{\Gamma\ltimes N}$ and W.

Denote by L_π the connected component of $\mathrm{Ker}(\pi)$ and $\overline{N} = N/L_\pi$. As in Section 10, let $\varphi : G_\pi \to \mathrm{Aff}(\overline{N})$ be the corresponding homomorphism and

$$\Delta = \{(\mathrm{Ad}(x), x^{-1}z) \ : x \in \overline{\Lambda}, z \in Z(\overline{N})\},$$

where $\overline{\Lambda}$ is the lattice $\Lambda L_\pi/L_\pi$ in $\overline{N}$ and $Z(\overline{N})$ the centre of $\overline{N}$. Then

$$Q := \varphi^{-1}(\Delta) \cap (\Gamma_\pi \ltimes N)$$

is a subgroup of finite index of P_1 (Proposition 27). By Corollary 29, there exists a real number $p \geq 1$ only depending on the dimension of $\mathrm{Aut}(N) \ltimes N$ such that $\widetilde{\pi}_{|\Gamma_\pi\ltimes N}$ is strongly L^p modulo Q.

We claim that Q is contained in P. Indeed, for $g \in Q$, we have

$$\varphi(g) = (\mathrm{Ad}(x), x^{-1}z)$$

for some $x \in \overline{\Lambda}$ and $z \in Z(\overline{N})$. Hence $\varphi(g)$ acts as the right translation by z on $L^2(\overline{\Lambda}\backslash\overline{N})$. Observe that $\mathcal{H}_\pi$ is contained in $L^2(\overline{\Lambda}\backslash\overline{N})$ and that g acts as $\varphi(g)$ on $\mathcal{H}_\pi$. Since N acts as a multiple of the irreducible representation π on $\mathcal{H}_\pi$, it follows that $g \in P$ and the claim is proved

As a consequence, we see that Q is a subgroup of finite index in P. Observe that Q is also contained in P_2. It follows that $U_\pi = (\widetilde{\pi}|_{\Gamma\ltimes N}) \otimes W$ is strongly L^p modulo Q and hence U_π is strongly L^p modulo P.

Let k be an integer with $k \geq p/4$. Then the tensor power $\left(U_\pi \otimes \overline{U_\pi}\right)^{\otimes k}$ is strongly L^2 modulo P. Hence, as discussed in Section 8, $\left(U_\pi \otimes \overline{U_\pi}\right)^{\otimes k}$ is contained in an infinite multiple of the induced representation $\mathrm{Ind}_P^{\Gamma\ltimes N}\lambda_\pi$, for the associated unitary character λ_π of P. It follows that, for every probability measure μ on $\Gamma \ltimes N$, we have

$$\| \left(U_\pi \otimes \overline{U_\pi}\right)^{\otimes k}(\mu)\| \leq \| \left(\mathrm{Ind}_P^{\Gamma\ltimes N}\lambda_\pi\right)(\mu)\|$$

and hence, using Proposition 30,

$$\|U_\pi(\mu)\| \leq \| \left(\mathrm{Ind}_P^{\Gamma\ltimes N}\lambda_\pi\right)(\mu)\|^{1/2k}.$$

On the other hand, observe that $PN = p_{\mathrm{a}}^{-1}(p_{\mathrm{a}}(P))$ is closed in $\mathrm{Aff}(\Lambda\backslash N)$, as $\mathrm{Aut}(\Lambda\backslash N)$ is discrete. Since, by induction by stages,

$$\mathrm{Ind}_P^{\Gamma\ltimes N}\lambda_\pi = \mathrm{Ind}_{PN}^{\Gamma\ltimes N}\left(\mathrm{Ind}_P^{PN}\lambda_\pi\right),$$

we have, using by Herz's majoration principle (Proposition 17),

$$\|\left(\mathrm{Ind}_P^{\Gamma\ltimes N}\lambda_\pi\right)(\mu)\| \leq \|\lambda_{(\Gamma\ltimes N)/PN}(\mu)\|.$$

Now, $\lambda_{(\Gamma\ltimes N)/PN} = \left(\lambda_{\Gamma/p_{\mathrm{a}}(P)}\right)\circ p_{\mathrm{a}}$ and hence

$$\|\lambda_{(\Gamma\ltimes N)/PN}(\mu)\| = \|\lambda_{\Gamma/p_{\mathrm{a}}(P)}(p_{\mathrm{a}}(\mu))\|.$$

As a consequence, the proposition will be proved if we establish the following inequality

$$\|\lambda_{\Gamma/p_{\mathrm{a}}(P)}(p_{\mathrm{a}}(\mu))\| \leq \|U_{\mathrm{tor}}(p_{\mathrm{a}}(\mu))\|. \tag{5}$$

To show this, recall (see (4) above) that U_{tor} is equivalent to the direct sum $\bigoplus_{\chi\in S}\lambda_{\Gamma/\Gamma_\chi}$, where S is set of representatives for the Γ-orbits in $\widehat{T}\setminus\{1_T\}$. As a consequence, Inequality (5) will be proved if we can show that there exists $\chi\in\widehat{T}\setminus\{1_T\}$ such that

$$\|\lambda_{\Gamma/p_{\mathrm{a}}(P)}(p_{\mathrm{a}}(\mu))\| \leq \|\lambda_{\Gamma/\Gamma_\chi}(\mu)\|.$$

By Herz's majoration principle again, it suffices to show that exists $\chi\in\widehat{T}$ with $\chi\neq 1_T$ such that $p_{\mathrm{a}}(P)\subset\Gamma_\chi$. For this, recall that, for every $g\in P\subset P_1$, there exists $x\in\overline{N}$ such that $\gamma = p_{\mathrm{a}}(g)$ acts as $\mathrm{Ad}(x)$ on $\overline{N}$ (Proposition 27). For every unitary character χ of $\overline{N}$, we have

$$\chi(\varphi(\gamma)(y)) = \chi(xyx^{-1}) = \chi(y) \qquad \text{for all} \quad y\in\overline{N}.$$

Thus, $p_{\mathrm{a}}(P)$ fixes every unitary character of $\overline{N}$.

Observe that $\overline{N}$ is non-trivial, since $\pi\neq 1_N$. Choose a non-trivial unitary character of $\overline{N}$ which is constant on the cosets of $\overline{\Lambda}$ and denote again by χ its lift to N. Then $\chi\in\widehat{T}\setminus\{1_T\}$ and χ is fixed by $p_{\mathrm{a}}(P)$. ■

Remark 32 With Remark 23, we see that a rough estimate for the integer k appearing in the statement of Proposition 31 is

$$k \leq \frac{1}{4}\left(\dim\left(\mathrm{Aut}(N)\ltimes N\right)+1\right)^2+1 \leq \frac{1}{4}((\dim(N))^3+1)^2+1.$$

Example 33 Let $N = H_{2n+1}(\mathbf{R})$ be the $(2n+1)$-dimensional Heisenberg group (over $\mathbf{R}$) and let Λ be a lattice in N. Then $\mathrm{Aut}(\Lambda\backslash N)$ contains a subgroup of finite index Γ consisting of automorphisms which fix every infinite dimensional representation $\pi \in \widehat{N}$ (see [Foll89]). Let H be a countable subgroup of $\mathrm{Aff}(\Lambda\backslash N)$. Assume that the action of H on $\Lambda\backslash N$ does not have a spectral gap. It follows from Proposition 31 that there is a subgroup H_1 of finite index in H, such that the action of $p_{\mathrm{a}}(H_1)$ on T does not have a spectral gap. Therefore, using Theorem 5, the action of H_1 and hence of the action of H on T does not have a spectral gap. This result generalizes Theorem 3 in [BeHe10] to groups of affine transformations of Heisenberg nilmanifolds.

14 Proof of Theorem 1: completion of the proof

We are now in position to give the proof of Theorem 1. In view of Theorem 5, we only need show that (ii) implies (i).

Let H be a countable subgroup of $\mathrm{Aff}(\Lambda\backslash N)$. Assume, by contraposition, that the action of H on $\Lambda\backslash N$ does not have a spectral gap. We have to prove that the action of H on T does not have a spectral gap.

Set $\Gamma = p_{\mathrm{a}}(H)$. By Theorem 5, it suffices to prove that the action on T of some subgroup of finite index in Γ does not have a spectral gap. Let $U^{\mathcal{H}}$ be the representation of $\mathrm{Aff}(\Lambda\backslash N)$ on the orthogonal complement $\mathcal{H}$ of $L^2(T)$ in $L^2(\Lambda\backslash N)$ and U_{tor} the representation on $L^2_0(T)$. Our theorem will be proved if we can show the following

Claim: Let μ be an aperiodic measure on H. Assume that $\|U^{\mathcal{H}}(\mu)\| = 1$. Then there exists a subgroup Δ of finite index in Γ and an aperiodic probability measure ν on Δ such that $\|U_{\mathrm{tor}}(p_{\mathrm{a}}(\nu))\| = 1$.

To prove this claim, we proceed by induction on the dimension of the Zariski closure $\mathrm{Zc}(\Gamma)$ of Γ in $\mathrm{Aut}(N)$.

If $\dim \mathrm{Zc}(\Gamma) = 0$, then Γ is finite and there is nothing to prove.

Assume that $\dim \mathrm{Zc}(\Gamma) \geq 1$ and that the claim above is proved for every countable subgroup of H_1 of $\mathrm{Aff}(\Lambda\backslash N)$ for which $\dim \mathrm{Zc}(p_{\mathrm{a}}(H_1)) < \dim \mathrm{Zc}(\Gamma)$.

Recall from Sections 12 and 13 that, as $\Gamma \ltimes N$-representation, $U^{\mathcal{H}}$ is equivalent to a direct sum

$$\bigoplus_{i\in I} \mathrm{Ind}_{\Gamma_i\ltimes N}^{\Gamma\ltimes N} V_i,$$

where Γ_i is the stabilizer in Γ of a rational representation $\pi_i \in \widehat{N}$ and V_i is a unitary representation of $\Gamma_i \ltimes N$.

Let $I_{\rm fin} \subset I$ be the set of all $i \in I$ such that Γ_i has finite index in Γ and set $I_\infty = I \setminus I_{\rm fin}$. Let

$$U_{\rm fin} = \bigoplus_{i\in I_{\rm fin}} \mathrm{Ind}_{\Gamma_i\ltimes N}^{\Gamma\ltimes N} V_i \qquad \text{and} \qquad U_\infty = \bigoplus_{i\in I_\infty} \mathrm{Ind}_{\Gamma_i\ltimes N}^{\Gamma\ltimes N} V_i$$

and denote by $\mathcal{H}_{\rm fin}$ and $\mathcal{H}_\infty$ the corresponding subspaces of $\mathcal{H}$ defined respectively by $U_{\rm fin}$ and U_∞. Since $\|U^{\mathcal{H}}(\mu)\| = 1$, two cases can occur.

• *First case:* we have $\|U_\infty(\mu)\| = 1$. By Herz's majoration principle, we have

$$\left\|\left(\mathrm{Ind}_{\Gamma_i\ltimes N}^{\Gamma\ltimes N} V_i\right)(\mu)\right\| \leq \left\|\lambda_{(\Gamma\ltimes N)/(\Gamma_i\ltimes N)}(\mu)\right\|$$

for every $i \in I_{\rm fin}$. Since $\lambda_{(\Gamma\ltimes N)/(\Gamma_i\ltimes N)} = \lambda_{\Gamma/\Gamma_i} \circ p_{\rm a}$, it follows that

$$\left\|\bigoplus_{i\in I_\infty} \lambda_{\Gamma/\Gamma_i}(p_{\rm a}(\mu))\right\| = 1.$$

Let $\varepsilon > 0$. We can choose $i \in I_\infty$ such that

$$\|\lambda_{\Gamma/\Gamma_i}(p_{\rm a}(\mu))\| \geq 1 - \varepsilon \tag{6}$$

We claim that $\dim \mathrm{Zc}(\Gamma_i) < \dim \mathrm{Zc}(\Gamma)$. Indeed, otherwise $\mathrm{Zc}(\Gamma_i)$ and $\mathrm{Zc}(\Gamma)$ would have the same connected component C^0, since $\mathrm{Zc}(\Gamma_i) \subset \mathrm{Zc}(\Gamma)$. As the stabilizer of π_i in $\mathrm{Aut}(N)$ is Zariski closed (Proposition 21), C^0 would stabilize π_i. Therefore, $\Gamma \cap C^0$ would be contained in Γ_i. But $\Gamma \cap C^0$ has finite index in Γ. Hence, Γ_i would have a finite index in Γ and this would be a contradiction, since $i \in I_\infty$.

Let μ_i be a probability measure with support equal to $(\Gamma_i \ltimes N) \cap H$. Then $(\mu_i + \mu)/2$ is an aperiodic probability measure on H. Since $\|U^{\mathcal{H}}(\mu)\| = 1$, we also have $\|U^{\mathcal{H}}((\mu_i + \mu)/2)\| = 1$. Therefore, $\|U^{\mathcal{H}}(\mu_i)\| = 1$. Since $\dim \mathrm{Zc}(\Gamma_i) < \dim \mathrm{Zc}(\Gamma)$, it follows from the induction hypothesis that $\|U_{\rm tor}(\mu_i)\| = 1$. Then, by Theorem 5, we also have $\|U_{\rm tor}(p_{\rm a}(\mu_i))\| = 1$.

On the other hand, recall from (4) that, replacing Γ by Γ_i, the Γ_i-representation $U_{\rm tor}$ decomposes into a direct sum

$$U_{\rm tor} \cong \bigoplus_{\chi \in S} \lambda_{\Gamma_i/(\Gamma_\chi \cap \Gamma_i)}.$$

As a consequence, we have

$$\left\| \bigoplus_{\chi \in S} (\lambda_{\Gamma_i/(\Gamma_\chi \cap \Gamma_i)})(p_{\rm a}(\mu_i)) \right\| = 1.$$

Observe that $p_{\rm a}(\mu_i)$ is an aperiodic probability measure on Γ_i (in fact, the support of $p_{\rm a}(\mu_i)$ is Γ_i). It follows that the Γ_i-representation $\bigoplus_{\chi \in S} \lambda_{\Gamma_i/(\Gamma_\chi \cap \Gamma_i)}$ weakly contains the trivial representation 1_{Γ_i}. Since

$$\mathrm{Ind}_{\Gamma_i}^{\Gamma} 1_{\Gamma_i} = \lambda_{\Gamma/\Gamma_i} \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma_i}^{\Gamma} \lambda_{\Gamma_i/(\Gamma_\chi \cap \Gamma_i)} = \lambda_{\Gamma/(\Gamma_\chi \cap \Gamma_i)}$$

it follows, by continuity of induction (see Proposition F.3.5 in [BeHV08]), that the Γ-representation $\bigoplus_{\chi \in S} \lambda_{\Gamma/(\Gamma_\chi \cap \Gamma_i)}$ weakly contains $\lambda_{\Gamma/\Gamma_i}$. As a consequence, we have

$$\|\lambda_{\Gamma/\Gamma_i}(p_{\rm a}(\mu))\| \leq \left\| \bigoplus_{\chi \in S} (\lambda_{\Gamma/(\Gamma_\chi \cap \Gamma_i)})(p_{\rm a}(\mu)) \right\|.$$

Observe that, by Herz's majoration principle again, we have

$$\|\lambda_{\Gamma/(\Gamma_\chi \cap \Gamma_i}(p_{\rm a}(\mu))\| \leq \|\lambda_{\Gamma/\Gamma_\chi}(p_{\rm a}(\mu))\|.$$

Hence

$$\begin{aligned} \|\lambda_{\Gamma/\Gamma_i}(p_{\rm a}(\mu))\| &\leq \left\| \bigoplus_{\chi \in S} \lambda_{\Gamma/\Gamma_\chi}(p_{\rm a}(\mu)) \right\| \\ &= \|U_{\rm tor}(p_{\rm a}(\mu))\|. \end{aligned}$$

Using Inequality (6), it follows that

$$\|U_{\rm tor}(p_{\rm a}(\mu))\| \geq 1 - \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we obtain that $\|U_{\rm tor}(p_{\rm a}(\mu))\| = 1$.

• *Second case:* we have $\|U_{\text{fin}}(\mu)\| = 1$. By the Noetherian property of the Zariski topology on $\text{Aut}(N)$, we can find finitely many indices $i_1, \ldots, i_r$ in I_{fin} such that

$$\text{Zc}(\Gamma_{i_1}) \cap \cdots \cap \text{Zc}(\Gamma_{i_r}) = \bigcap_{i \in I_{\text{fin}}} \text{Zc}(\Gamma_i).$$

Since stabilizers of irreducible representations of N are algebraic (Proposition 21), the subgroup $\Delta := \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_r}$ stabilizes π_i for every $i \in I_{\text{fin}}$. Moreover, Δ has finite index in Γ, since every Γ_i has finite index in Γ.

From Sections 12 and 13, we have a decomposition of $\mathcal{H}_{\text{fin}}$ into $\Delta \ltimes N$-invariant subspaces

$$\mathcal{H}_{\text{fin}} = \bigoplus_{i \in I_{\text{fin}}} \mathcal{H}_i,$$

where $\mathcal{H}_i$ is the isotypical component corresponding to π_i under the action of N. Let ν be a probability measure with support equal to $(\Delta \ltimes N) \cap H$. Considering as above the aperiodic measure $(\mu+\nu)/2$ on H, we have $\|U_{\text{fin}}(\nu))\| = 1$, since $\|U_{\text{fin}}(\mu)\| = 1$.

On the other hand, by Proposition 31, there exists an integer $k \geq 1$, which is independent of i, such that

$$\|U_i(\nu))\| \leq \|U_{\text{tor}}(p_{\text{a}}(\nu))\|^{1/2k} \qquad \text{for all} \quad i \in I_{\text{fin}}$$

where U_i is the representation of $\Delta \ltimes N$ on $\mathcal{H}_i$. As a consequence, we have

$$\|U_{\text{fin}}(\nu))\| \leq \|U_{\text{tor}}(p_{\text{a}}(\nu))\|^{1/2k}$$

and it follows that $\|U_{\text{tor}}(p_{\text{a}}(\nu))\| = 1$. Since the support of $p_{\text{a}}(\nu)$ is the subgroup Δ of finite index in Γ, this completes the proof of Theorem 1. ■

Remark 34 The proof of Theorem 1 we gave above is not effective: it does not give, for a probability measure μ on $\text{Aut}(\Lambda \backslash N)$, a bound for the norm of μ under $U^{\mathcal{H}}$ in terms of the norm of μ under U_{tor} and/or other "known" representations of the group generated by μ, such as the regular representation. In the following example, such an explicit bound is given. The crucial tool we use is Mackey's tensor product theorem This approach succeeds here because of the special features of the example and we could not use it to get explicit bounds in the most general case.

Example 35 Let $\mathfrak{n} = \mathfrak{n}_{3,2}$ be the free 2-step nilpotent Lie algebra on 3 generators and let $N = N_{3,2}$ be the corresponding connected and simply-connected nilpotent Lie group. As is well-known, $\mathfrak{n}$ is a 6-dimensional Lie algebra which can be realized as follows. Set $V_1 = V_2 = \mathbf{R}^3$ and define a Lie bracket on the vector space $\mathfrak{n} = V_1 \oplus V_2$ by

$$[(X_1, Y_1), (X_2, Y_2)] = (0, 2(X_1 \wedge X_2)) \qquad \text{for all} \quad X_1, X_2, Y_1, Y_2 \in \mathbf{R}^3,$$

where $X_1 \wedge X_2$ denotes the usual cross-product on $\mathbf{R}^3$. (The factor 2 appears here just for computational ease.) The centre of $\mathfrak{n}$ is V_2 and the Lie group N is $V_1 \oplus V_2$ with the product

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + y_2 + x_1 \wedge x_2) \qquad \text{for all} \quad x_1, x_2, y_1, y_2 \in \mathbf{R}^3,$$

so that the exponential mapping $\exp : \mathfrak{n} \to N$ is the identity.

Observe that, for a matrix $A \in GL_3(\mathbf{R})$, we have

$$A(X \wedge Y) = (\det A)(A^t)^{-1}(X \wedge Y) \qquad \text{for all} \quad X, Y \in \mathbf{R}^3.$$

The automorphism group $\mathrm{Aut}(N)$ of N is the subgroup of $GL_6(\mathbf{R})$ of matrices $g_{A,B}$ of the form

$$g_{A,B} = \begin{pmatrix} A & 0 \\ B & (\det A)(A^t)^{-1} \end{pmatrix}$$

with $A \in GL_3(\mathbf{R})$ and $B \in M_3(\mathbf{R})$, so that $\mathrm{Aut}(N)$ is isomorphic to the semi-direct product $GL_3(\mathbf{R}) \ltimes M_3(\mathbf{R})$ for the action of $GL_3(\mathbf{R})$ by left multiplication on the vector space $M_3(\mathbf{R})$ of 3×3-real matrices.

We will identify $\mathfrak{n}$ with $\mathfrak{n}^*$ by means of the standard scalar product $(X, Y) \mapsto \langle X|Y \rangle$ on $\mathbf{R}^6$. For (x, y) and (X_0, Y_0) in $V_1 \oplus V_2$, we compute that $\mathrm{Ad}^*(x, y)(X_0, Y_0) = (X_0 + x \wedge Y_0, Y_0)$. It follows that the coadjoint orbit of $(X_0, 0)$ is $\{(X_0, 0)\}$ and, for $Y_0 \neq 0$, we have

$$\begin{aligned} \mathrm{Ad}^*(N)(X_0, Y_0) &= \left\{(X_0 + x \wedge Y_0, Y_0) \ : \ x \in \mathbf{R}^3\right\} \\ &= \left\{(X_0 + Y, Y_0) \ : \ Y \in (\mathbf{R}Y_0)^\perp\right\} \\ &= \left\{(\lambda_0 Y_0 + Y, Y_0) \ : \ Y \in (\mathbf{R}Y_0)^\perp\right\}. \end{aligned}$$

for $\lambda_0 = \langle X_0|Y_0 \rangle / \|Y_0\|^2$. The orbits which are not reduced to singletons are therefore the two-dimensional affine planes

$$\mathcal{O}_{\lambda_0, Y_0} = \left\{(\lambda_0 Y_0 + Y, Y_0) \ : \ Y \in (\mathbf{R}Y_0)^\perp\right\},$$

parametrized by $(\lambda_0, Y_0) \in \mathbf{R} \times (\mathbf{R}^3 \setminus \{0\})$.

The subgroup $\Lambda = \mathbf{Z}^3 \oplus \mathbf{Z}^3$ is a lattice in N. The group $\mathrm{Aut}(\Lambda \backslash N)$ is the subgroup of $\mathrm{Aut}(N)$ of automorphisms $g_{A,B}$ as above given by matrices $A \in GL_3(\mathbf{Z})$ and $B \in M_3(\mathbf{Z})$.

Fix $(\lambda_0, Y_0) \in \mathbf{R} \times (\mathbf{R}^3 \setminus \{0\})$. The irreducible unitary representation π_{λ_0,Y_0} of N corresponding to the coadjoint orbit $\mathcal{O}_{\lambda_0,Y_0}$ appears in the decomposition of $L^2(\Lambda \backslash N)$ into N-isotypical components if and only if $\mathcal{O}_{\lambda_0,Y_0} \cap (\mathbf{Z}^3 \oplus \mathbf{Z}^3) \neq \emptyset$. This is the case if and only if $Y_0 \in \mathbf{Z}^3 \setminus \{0\}$ and $\lambda_0 \in \|Y_0\|^{-2} \Delta_{Y_0}$, where Δ_{Y_0} is the subgroup of $\mathbf{Z}$ consisting of the integers m for which $mY_0 \in (\mathbf{R}Y_0)^\perp + \|Y_0\|^2 \mathbf{Z}^3$.

Let Γ be a subgroup of $\mathrm{Aut}(\Lambda \backslash N)$. For simplicity, we assume that Γ consists only of automorphisms $g_{A,0}$ with $A \in SL_3(\mathbf{Z})$. We identify Γ with a subgroup of $SL_3(\mathbf{Z})$. For $A \in SL_3(\mathbf{Z})$, we have

$$A(\mathcal{O}_{\lambda_0,Y_0}) = \mathcal{O}_{\beta_0,(A^t)^{-1}(Y_0)} \qquad \text{for} \qquad \beta_0 = \lambda_0 \|Y_0\|^2 / \|(A^t)^{-1}(Y_0)\|^2.$$

The stabilizer Γ_{λ_0,Y_0} of $\mathcal{O}_{\lambda_0,Y_0}$ (which is the stabilizer of π_{λ_0,Y_0}) in Γ is therefore

$$\Gamma_{\lambda_0,Y_0} = \{A \in \Gamma \ : \ A^t Y_0 = Y_0\},$$

and is isomorphic to a subgroup of the semi-direct product $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$.

Let $\mathcal{H}_{\lambda_0,Y_0}$ be the isotypical component of $L^2(\Lambda \backslash N)$ associated to π_{λ_0,Y_0} and U_{λ_0,Y_0} the corresponding representation of Γ (see Section 12); we know that U_{λ_0,Y_0} is equivalent to $\mathrm{Ind}_{\Gamma_{\lambda_0,Y_0}}^{\Gamma} V_{\lambda_0,Y_0}$ for a representation V_{λ_0,Y_0} of Γ_{λ_0,Y_0} which is strongly L^p modulo its projective kernel P_{λ_0,Y_0} for some real number $p \geq 1$.

The projective kernel P_{λ_0,Y_0} of V_{λ_0,Y_0} coincides with the subgroup of Γ of all automorphisms which fixes every point $(X, Y) \in \mathcal{O}_{\lambda_0,Y_0}$; hence, $P_{\lambda_0,Y_0} = \{I\}$ if $\lambda_0 = 0$ and

$$P_{\lambda_0,Y_0} = \{A \in \Gamma \ : \ A^t Y_0 = Y_0 \quad \text{and} \quad AY = Y \quad \text{for all} \quad Y \in (\mathbf{R}Y_0)^\perp\}$$

if $\lambda_0 \neq 0$.

Every π_{λ_0,Y_0} factorizes to a representation of a quotient of N of dimension 3 or 4, which is isomorphic to the Heisenberg group H_3 or to the direct product $H_3 \oplus \mathbf{R}$. It follows that the representation V_{λ_0,Y_0} of Γ_{λ_0,Y_0} is strongly $L^{6+\varepsilon}$ modulo P_{λ_0,Y_0} for every $\varepsilon > 0$ (see [BeHe10] and [HoMo79]).

Set $\Gamma_0 = \Gamma_{\lambda_0,Y_0}, V = V_{\lambda_0,Y_0}$, and $U = U_{\lambda_0,Y_0}$. We claim that $U^{\otimes 4}$ is weakly contained in the regular representation λ_Γ of Γ on $\ell^2(\Gamma)$.

Indeed, by Mackey's tensor product theorem, $U^{\otimes 4}$ is weakly equivalent to the direct sum

$$\bigoplus_{\gamma_1,\gamma_2,\gamma_3\in\Gamma} \mathrm{Ind}^{\Gamma}_{\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}} \left(V\otimes V^{\gamma_1}\otimes V^{\gamma_2}\otimes V^{\gamma_3}\right),$$

where $V\otimes V^{\gamma_1}\otimes V^{\gamma_2}\otimes V^{\gamma_3}$ is the tensor product of the restrictions of $V, V^{\gamma_1}, V^{\gamma_2}$ and V^{γ_3} to $\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}$. Fix $\gamma_1,\gamma_2,\gamma_3\in\Gamma$. Observe that $\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}$ is the subgroup of elements $\gamma\in\Gamma$ such that γ^t fixes $Y_0, \gamma_1^t(Y_0), \gamma_2^t(Y_0)$ and $\gamma_3^t(Y_0)$. Set

$$U_{\gamma_1,\gamma_2\gamma_3} = \mathrm{Ind}^{\Gamma}_{\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}} \left(V\otimes V^{\gamma_1}\otimes V^{\gamma_2}\otimes V^{\gamma_3}\right).$$

Two cases can occur.

• *First case:* There exists some $i\in\{1,2,3\}$ such that $\gamma_i^t(Y_0)$ is not a multiple of Y_0. Then every element $\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}$ fixes pointwise a plane in $\mathbf{R}^3$; it follows that $\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}$ is abelian and hence amenable. Therefore $U_{\gamma_1,\gamma_2\gamma_3}$ is weakly contained in λ_Γ.

• *Second case:* Every $\gamma_i^t(Y_0)$ is a multiple of Y_0, that is, every γ_i belongs to the subgroup $H=\{\gamma\in\Gamma\ :\ \gamma^t(Y_0)\in\{\pm Y_0\}\}$. Observe that Γ_0 is a subgroup of H of index at most 2. It can be checked that the subgroup $P=P_{\lambda_0,Y_0}$, which is normal in Γ_0, is normal in H. It follows that the restriction of V^{γ_i} to $\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}$ is strongly $L^{6+\varepsilon}$ modulo P for every $i\in\{1,2,3\}$. Hence, $V\otimes V^{\gamma_1}\otimes V^{\gamma_2}\otimes V^{\gamma_3}$ is strongly L^2 modulo P and hence contained in a multiple of $\mathrm{Ind}_P^{\Gamma_0\cap\Gamma_0^{\gamma_1}\cap\Gamma_0^{\gamma_2}\cap\Gamma_0^{\gamma_3}}\lambda$. Since P is amenable, it follows that $U_{\gamma_1,\gamma_2,\gamma_3}$ is weakly contained in λ_Γ. As a consequence, we see that $U^{\otimes 4}$ is weakly contained in λ_Γ.

Let μ be a probability measure on Γ. It follows from what we have seen that

$$\|U^{\mathcal{H}}(\mu)\| \le \|\lambda_\Gamma(\mu)\|^{1/4},$$

where $U^{\mathcal{H}}$ is the Koopman representation of Γ on $\mathcal{H}=L^2(T)^\perp$. As a consequence, we have

$$\|U^0(\mu)\| \le \max\{\|\lambda_\Gamma(\mu)\|^{1/4}, \|U_{\mathrm{tor}}(\mu)\|\},$$

where U^0 and U_{tor} are the Koopman representations of Γ on $L^2_0(\Lambda\backslash N)$ and $L^2_0(T)$. The same estimate was established in [BeHe10, Corollary 3] in the case where N is the Heisenberg group H_3.

15 Proof of Theorem 4

Let H be a subgroup of $\mathrm{Aff}(\Lambda\backslash N)$. The following elementary proposition shows that ergodicity of H on T is inherited by every subgroup of finite index in H.

Proposition 36 *Let H be a subgroup of $\mathrm{Aff}(T)$ and H_1 a subgroup of finite index in H. Assume that $L^2_0(T)$ contains a non-zero H_1-invariant function. Then $L^2_0(T)$ contains a non-zero H-invariant function.*

Proof By standard arguments involving Fourier series, there exists a unitary character χ in $\widehat{T}\setminus\{1_T\}$ with a finite orbit under $p_{\mathrm{a}}(H_1)$ and such that $H_2 := H_1 \cap p_{\mathrm{a}}^{-1}(\Gamma_\chi)$ fixes χ, where Γ_χ is the stabilizer of χ in $\mathrm{Aut}(T)$. Then H_2 has finite index in H and

$$\sum_{s\in H/H_2} U_{\mathrm{tor}}(s)\chi$$

is a non-zero H-invariant function in $L^2_0(T)$.■

Proof of (i) in Theorem 4

As is well-known, the action of a group H on a probability space (X,ν) is weakly mixing if and only if the diagonal action of H on $(X\times X, \nu\otimes\nu)$ is ergodic. Since $T\times T$ is the maximal factor torus of $(\Lambda\backslash N)\times(\Lambda\backslash N)$, we only have to prove the statement about ergodicity.

So, let H be a (not necessarily countable) subgroup of $\mathrm{Aff}(\Lambda\backslash N)$ acting ergodically on T. We have to prove that H acts ergodically on $\Lambda\backslash N$. We can assume that N is not abelian, otherwise there is nothing to prove.

Set $\Gamma = p_{\mathrm{a}}(H)$. Recall from Sections 12 and 13 that we have orthogonal decompositions into $\Gamma\ltimes N$ -invariant subspaces $L^2(\Lambda\backslash N) = L^2(T)\oplus\mathcal{H}$ and

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i},$$

such that the representation U_i of $\Gamma\ltimes N$ on $\mathcal{H}_{\Sigma_i}$ is equivalent to an induced representation $\mathrm{Ind}_{\Gamma_{\pi_i}\ltimes N}^{\Gamma\ltimes N} V_i$, where Γ_{π_i} is the stabilizer in Γ of some $\pi_i\in\Sigma_i$. In view of the previous proposition, it suffices to prove the following

Claim: Assume that, for some i, the subspace $\mathcal{H}_{\Sigma_i}$ contains a non-zero H-invariant function. Then $L^2_0(T)$ contains a non-zero H_1-invariant function for some subgroup H_1 of finite index in H.

To show this, set $\pi = \pi_i$, $\Sigma_\pi = \Sigma_i$, $U_\pi = U_i$, and $V_\pi = V_i$. Let S be a set of representatives for the cosets in

$$\Gamma/\Gamma_\pi \cong (\Gamma \ltimes N)/(\Gamma_\pi \ltimes N)$$

with $e \in S$. Then, by the definition of an induced representation, $\mathcal{H}_{\Sigma_\pi}$ is an orthogonal sum

$$\mathcal{H}_{\Sigma_\pi} = \bigoplus_{s\in S} \mathcal{K}^s,$$

where $\mathcal{K}$ carries the $\Gamma_\pi \ltimes N$-representation V_π and where $\mathcal{K}^s = U_\pi(s)\mathcal{K}$. It follows from this that there exists a non-zero function in $\mathcal{K}$ which is invariant under $H \cap (\Gamma_\pi \ltimes N)$ and that Γ_π has finite index in Γ.

Upon replacing H by the subgroup of finite index $H \cap (\Gamma_\pi \ltimes N)$, we can assume that H is contained in $\Gamma_\pi \ltimes N$.

Let L_π be the connected component of $\mathrm{Ker}(\pi)$ and $\overline{N} = N/L_\pi$. Observe that $\overline{N}$ is not abelian, since π is not a unitary character of N. As seen in Section 10, the action of $\Gamma_\pi \ltimes N$ on $\mathcal{H}_\pi$ factorizes through the quotient nilmanifold $\overline{\Lambda}\backslash\overline{N}$. Hence, we can assume that L_π is trivial.

By the proof of Proposition 31, there exists a real number $p \geq 1$ such that the representation V_π of $\Gamma_\pi \ltimes N$ is strongly L^p modulo Δ, where Δ is the normal subgroup

$$\Delta = \{(\mathrm{Ad}(x), x^{-1}z) \ : x \in \Lambda, z \in Z(N)\}.$$

We claim that $H \cap \Delta$ has finite index in H.

Indeed, let $R = \overline{H\Delta}$ be the closure of $H\Delta$ in $\Gamma_\pi \ltimes N$. Then the restriction of V_π to R is strongly L^p modulo Δ.

Observe that $(\mathrm{Ad}(x), x^{-1}z) \in \Delta$ acts as multiplication with $\lambda_\pi(z)$ on $\mathcal{H}_\pi$, where λ_π is the central character of π. Let ξ a non-zero $V_\pi(H)$-invariant function in $\mathcal{K}$. The function $x \mapsto |\langle V_\pi(x)\xi, \xi\rangle|$ is non-zero, belongs to $L^p(R/\Delta)$, and is R invariant. It follows that R/Δ is a compact group.

Let R_0 be the connected component of R. Since R is a Lie group, R_0 is open in R. It follows that $R_0\Delta/\Delta$ is an open (and hence closed) subgroup of R/Δ. Since R/Δ is compact, we conclude that $R_0\Delta/\Delta \cong R_0/(R_0 \cap \Delta)$ is a subgroup of finite index in R/Δ.

On the other hand, observe that $R_0 \subset N$, since $R \subset \Gamma_\pi \ltimes N$ and since Γ_π is discrete. Observe also that

$$R_0 \cap \Delta = R_0 \cap Z(N),$$

since $Z(N)$ is connected (as N is simply connected). It follows that $R_0 \cap \Delta$ is a connected subgroup of the nilpotent simply connected Lie group R_0. But $R_0/(R_0 \cap \Delta)$ is compact. Hence, $R_0/(R_0 \cap \Delta)$ is trivial. As a consequence, we see that R/Δ is finite. This shows that $H \cap \Delta$ has finite index in H. Therefore, upon replacing H by $H \cap \Delta$, we can assume that $H \subset \Delta$.

The centre $Z(N)$ being a rational subgroup of N, the subgroup $\overline{\Lambda} = \Lambda Z(N)$ of the nilpotent Lie group $\overline{N} = N/Z(N)$ is a lattice. Observe that $\overline{N}$ is non-trivial, since N is non-abelian. The group Δ acts trivially on the factor nilmanifold $\overline{\Lambda}\backslash\overline{N}$ and hence on the associated torus $\overline{T}$. Since $\overline{T}$ is a Δ-invariant factor torus of T, it follows that the action of H on T is not ergodic.■

Proof of (ii) in Theorem 4

Let H be a subgroup of $\mathrm{Aut}(\Lambda\backslash N)$ with a strongly mixing action on T. We have to prove that the action of H on $\Lambda\backslash N$ is strongly mixing.

With the notation as in the proof of Part (i) above, the Koopman representation U of H on $\mathcal{H}$ decomposes as a direct sum $U \cong \oplus_i U_i$, where U_i equivalent to an induced representation $\mathrm{Ind}_{H_{\pi_i}}^{H} V_i$. It suffices to prove that, for every i, the matrix coefficients of U_i belong to $c_0(H)$. This will follow if we show that the matrix coefficients of V_i belong to $c_0(H_{\pi_i})$.

Set $\pi = \pi_i$ and $V_\pi = V_i$. Let L_π be the connected component of $\mathrm{Ker}(\pi)$ and $\overline{\Lambda}\backslash\overline{N}$ the corresponding H_π-invariant factor nilmanifold. Since H_π is contained in $\mathrm{Aut}(\Lambda\backslash N)$, the projective kernel P of V_π coincides with the kernel of the homomorphism $\varphi : H_\pi \to \mathrm{Aut}(\overline{\Lambda}\backslash\overline{N})$, by Proposition 27.

We claim that $P = \mathrm{Ker}(\varphi)$ is finite. Indeed, otherwise the matrix coefficients of the Koopman representation of H_π on the maximal factor torus $\overline{T}$ of $\overline{\Lambda}\backslash\overline{N}$ would not belong to $c_0(H_\pi)$ and this would imply that the action of H_π and hence of H on T is not strongly mixing.

Since P is finite, V_π is strongly L^p for some $p \geq 1$. It follows that the matrix coefficients of V_π belong to $c_0(H_\pi)$. This finishes the proof of Theorem 4.■

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