Operator-algebraic superridigity for $SL_n(\mathbb{Z}),\ n \geq 3$

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Abstract

For $n \geq 3$, let $\Gamma = SL_n(\mathbb{Z})$. We prove the following superridigity result for Γ in the context of operator algebras. Let $L(\Gamma)$ be the von Neumann algebra generated by the left regular representation of Γ. Let M be a finite factor and let $U(M)$ be its unitary group. Let $\pi : \Gamma \to U(M)$ be a group homomorphism such that $\pi(\Gamma)'' = M$. Then either

(i) M is finite dimensional, or

(ii) there exists a subgroup of finite index Λ of Γ such that $\pi|_\Lambda$ extends to a homomorphism $U(L(\Lambda)) \to U(M)$.

This answers, in the special case of $SL_n(\mathbb{Z})$, a question of A. Connes discussed in [Jone00, Page 86]. The result is deduced from a complete description of the tracial states on the full C^*–algebra of Γ.

As another application, we show that the full C^*–algebra of Γ has no faithful tracial state, thus answering a question of E. Kirchberg.

1 Introduction

Two major achievements in the study of discrete subgroups in semi-simple Lie groups are Mostow's ridigity theorem and Margulis' superrigidity theorem. A weak version of the latter is as follows. Let Γ be a lattice in a simple real Lie group G with finite centre and with $\mathbb{R} - \text{rank}(G) \geq 2$. Let H be another simple real Lie group with finite centre, and let $\pi : \Gamma \to H$ be a homomorphism such that $\pi(\Gamma)$ is Zariski-dense in H. Then, either H is compact or there exists a finite index subgroup Λ of Γ such that $\pi|_\Lambda$ extends to a continuous homomorphism $G \to H$. For more general results, see [Marg91] and [Zimm84]. Moreover, as shown by Corlette, the superridity theorem continues to hold for the simple real Lie groups G with $\mathbb{R} - \mathrm{rank}(G) = 1$ which are not locally isomorphic to $SO(n,1)$ or $SU(n,1)$.

In the theory of von Neumann algebras, discrete groups (as well as their actions) always played a prominent rôle. To a discrete group Γ is associated a distinguished von Neumann algebra $L(\Gamma)$, namely the von Neumann algebra generated by the left regular representation λ_Γ of Γ; thus, $L(\Gamma)$ is the closure for the strong operator topology of the linear span of $\{\lambda_\Gamma(\gamma) \ : \ \gamma \in \Gamma\}$ in the algebra $\mathcal{L}(\ell^2(\Gamma))$ of all bounded operators on the Hilbert space $\ell^2(\Gamma)$.

The first rigidity result in the context of operator algebras is the result by A. Connes [Conn80] showing that, for a group Γ with Kazhdan's Property (T), the group of outer automorphisms of $L(\Gamma)$ is countable. A major problem in this area is whether such a group Γ can be reconstructed from its von Neumann algebra $L(\Gamma)$. In recent years, a series of remarkable results concerning this question, with applications to ergodic theory, have been obtained by S. Popa ([Popa06-a], [Popa06-b]; for an account, see [Vaes06]). Other relevant work includes [CoHa89] and [Furm99].

The purpose of this paper is to discuss another kind of rigidity, namely the rigidity of a discrete group in the unitary group of its von Neumann algebra. If Γ is a discrete group, we view Γ as a subgroup of the unitary group $U(L(\Gamma))$ of $L(\Gamma)$, that is, the group of the unitary operators in $L(\Gamma)$. It was suggested by Connes (see [Jone00, Page 86]) that, for Γ as in the statement of Margulis' theorem, a superrigity result should hold in which G above is replaced by $U(L(\Gamma))$ and H by the unitary group $U(M)$ of a type II_1 factor. We prove such a superrigity result in the case $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$.

Recall that a von Neumann algebra M is a factor if the centre of M is reduced to the scalar operators. The von Neumann algebra M is said to be finite if there exists a finite normal faithful trace on M. A finite factor is a type II_1 factor which is infinite dimensional. Recall also that $L(\Gamma)$ is a finite von Neumann algebra. Moreover, $L(\Gamma)$ is a factor if and only if Γ is an ICC-group, that is, if all its conjugacy classes, except $\{e\}$, are infinite. For an account on the theory of von Neumann algebras, see [Dix-vN].

Theorem 1 *Let $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. Let M be a finite factor and let $U(M)$ its unitary group. Let $\pi : \Gamma \to U(M)$ be a group homomorphism.*

Assume that the linear span of $\pi(\Gamma)$ in dense in M for the strong operator topology. Then either

(i) M is finite dimensional, that is, M is isomorphic to a matrix algebra $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, in which case π factorizes to a multiple of an irreducible representation of some congruence quotient $SL_n(\mathbb{Z}/N\mathbb{Z})$ for $N \in \mathbb{N}$, or

(ii) there exists a subgroup of finite index Λ of Γ such that $\pi|_\Lambda$ extends to a normal homomorphism $L(\Lambda) \to M$ of von Neumann algebras. In particular, $\pi|_\Lambda$ extends to a group homomorphism $U(L(\Lambda)) \to U(M)$.

Let C be the centre of $SL_n(\mathbb{Z})$. Observe that C is trivial for odd n and $C = \{\pm I\}$ for even n. If, in the statement of the theorem above, we take instead $\Gamma = PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/C$, then $L(\Gamma)$ is a factor and the conclusion (ii) holds for $\Lambda = \Gamma$.

The method of proof of Theorem 1 can be adapted to establish the same result when Γ is the symplectic group $Sp_{2n}(\mathbb{Z})$ for $n \geq 2$; it works presumably for the group of integral points of any Chevalley group of rank ≥ 2. No such result can be true for the modular group $SL_2(\mathbb{Z})$; see Remark 4 below.

Remark 2 (i) Let Γ be a countable ICC–group with Kazhdan's Property (T). It was shown in [CoJo85] that $L(\Gamma)$ cannot be a subfactor of $L(F_2)$, where F_2 is a non-abelian free group. This, combined with Theorem 1, shows that every representation of $PSL_n(\mathbb{Z})$ for $n \geq 3$ into $U(L(F_2))$ decomposes as a direct sum of finite dimensional representations. This is a special case of a result of G. Robertson [Robe93] valid for all groups with Property (T). For a related work, see [Vale97].
(ii) Let M be a finite factor. The unitary group $U(M)$ has as centre a copy of the circle group S^1, namely the unitary scalar operators. It was shown in [Harp79] that the projective unitary group $U(M)/S^1$ of M is a simple group.

The result in Theorem 1 amounts to the classification of the *characters* of Γ (see Section 7), that is, the functions $\varphi : \Gamma \to \mathbb{C}$ with the following properties:

- φ is central, that is, $\varphi(\gamma x \gamma^{-1}) = \varphi(x)$ for all $\gamma, x \in \Gamma$,
- φ is positive definite, that is, $\sum_{i=1}^n \overline{c_j} c_i \varphi(\gamma_j^{-1}\gamma_i) \geq 0$ for all $\gamma_1, \dots, \gamma_n \in \Gamma$ and $c_1, \dots, c_n \in \mathbb{C}$,

- φ is normalized, that is, $\varphi(e) = 1$,
- φ is indecomposable, that is, φ cannot be written in a non-trivial way as a convex combination of two central positive definite normalized functions.

There are two obvious examples of characters of a group Γ. First of all, the normalized character (in the usual sense) of an irreducible finite dimensional unitary representations of Γ is a character of Γ in the above sense. For $\Gamma = SL_n(\mathbb{Z})$, $n \geq 3$, it is well–known that every such representation factorizes through some congruence quotient $SL_n(\mathbb{Z}/N\mathbb{Z})$ for an integer N; this a consequence of the solution of the congruence subgroup problem (see [BaMS67], [Menn65]; see also [Stei85]). Much is known about the characters of the finite groups $SL_n(\mathbb{Z}/N\mathbb{Z})$; see [Zele81].

Let C be the centre of the group Γ. Assume that all conjugacy classes, except those of the elements from C, are infinite. Then, for every unitary character χ of the abelian group C, the trivial extension $\widetilde{\chi}$ of χ to Γ, defined by $\widetilde{\chi} = 0$ on $\Gamma \setminus C$, is a character of Γ. In particular, if Γ is ICC, then δ_e, the Dirac function at e, is a character of Γ. When n is even, all conjugacy classes of $PSL_n(\mathbb{Z})$, except $\{I\}$ and $\{-I\}$, are infinite.

Our main result says that $SL_n(\mathbb{Z})$ for $n \geq 3$ has no characters other than the obvious ones described above.

Theorem 3 *Let φ be a character of $SL_n(\mathbb{Z})$ for $n \geq 3$. Then, either*

(i) φ is the character of an irreducible finite dimensional representation of some congruence quotient $SL_n(\mathbb{Z}/N\mathbb{Z})$ for $N \geq 1$, or

(ii) φ is the trivial extension $\widetilde{\chi}$ of a character χ of the centre of $SL_n(\mathbb{Z})$..

Remark 4 No classification of the characters of the modular group $SL_2(\mathbb{Z})$ can be expected. Indeed, this group contains the free non-abelian group F_2 on two generators as normal subgroup. Every character of F_2 extends to a character on $SL_2(\mathbb{Z})$. Now, F_2 has a huge number of characters: if M is any finite factor with trace τ, every pair of unitaries in M defines a homomorphism $\pi : F_2 \to U(M)$ and a corresponding character $\tau \circ \pi$ on F_2.

The problem of the description of the characters of a discrete group Γ has been considered by several authors. E. Thoma [Thom64b] solved this problem for the infinite symmetric group S_∞ (see also [VerKe81]), H - L. Skudlarek [Skud76] for the group $\Gamma = GL(\infty, \mathbb{F})$, where $\mathbb{F}$ is a finite field, and D. Voiculescu [Voic76] for $\Gamma = U(\infty)$; see also [StVo75] and [Boye83]. A.A. Kirillov [Kiri65] described the characters of $\Gamma = GL_n(\mathbb{K})$ or $SL_n(\mathbb{K})$ for $n \geq 2$, where $\mathbb{K}$ is an infinite field (see also [Rose89] and [Ovci71]).

Our proof of Theorem 3 is based on an analysis of the restriction $\varphi|_V$ of a given character φ of $SL_n(\mathbb{Z})$ to various copies V of $\mathbb{Z}^{n-1}$. We will see that we have a dichotomy corresponding to the two different types of characters from Theorem 3: either the measure on the torus $\mathbb{T}^{n-1}$ associated to $\varphi|_V$ is atomic or this measure is the Lebesgue measure for every V. An important ingredient in our analysis is the solution of the congruence subgroup for $SL_n(\mathbb{Z})$ for $n \geq 3$.

The result of Theorem 3 can be interpreted as a classification of the traces on the full C^*-algebra $C^*(\Gamma)$ of $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$ (see Section 2).

E. Kirchberg asked in [Kirc93, Remark 8.2, page 487] whether the full C^*–algebra of $SL_4(\mathbb{Z})$ has a faithful trace. He was motivated by the fact that a positive answer to this question would imply a series of outstanding conjectures in the theory of von Neumann algebras (see Section 8). As a consequence of Theorem 3, we will see that the answer to Kirchberg's question is negative, namely:

Corollary 5 *The full C^*–algebra of $SL_n(\mathbb{Z})$ has no faithful tracial state for $n \geq 3$.*

In fact, we will prove the stronger result Corollary 19 below.

Recall that the reduced C^*-algebra $C^*_r(\Gamma)$ of a group Γ is the closure of the linear span of $\{\lambda_\Gamma(\gamma) \ : \ \gamma \in \Gamma\}$ in $\mathcal{L}(\ell^2(\Gamma))$ for the operator norm. Recall also that δ_e factorizes to a faithful tracial state on $C^*_r(\Gamma)$. The finite dimensional representations of $PSL_n(\mathbb{Z})$ do not factorize through $C^*_r(PSL_n(\mathbb{Z}))$, since $PSL_n(\mathbb{Z})$ is not amenable. As a consequence, Theorem 3 implies that δ_e is the unique tracial state on $C^*_r(PSL_n(\mathbb{Z}))$. This also follows from [BeCH95], where a different method is used.

Theorem 3 leaves open the problem of existence of *infinite*, semi-finite traces on $C^*(SL_n(\mathbb{Z}))$. We do not know whether such traces exist. Using [BeCH95], we can only show that no such trace exists on $C^*_r(PSL_n(\mathbb{Z}))$. In fact, this result is valid for a more general class of groups including $PSL_2(\mathbb{Z})$ (see Proposition 21 below).

This paper is organized as follows. Sections 2 and 3 are devoted to some general facts. The proof of Theorem 3 is spread over three sections: in

Section 4, we show that the proof splits into two cases which are then treated accordingly in Sections 5 and 6. In Section 7, we show that Theorem 1 is a consequence of Theorem 3. Corollary 5 is proved in Section 8 and Section 9 is devoted to a remark on the problem of the existence of infinite traces.

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2 Factor representations and characters

We review some general facts concerning the relationships between central positive definite functions on groups and factor representations. Details can be found in [Dix-C*, Chapters 6 and 17] or [Thom64a].

Let Γ be a discrete group. We are interested in representations of Γ in the unitary group of a finite von Neumann algebra.

Recall that a finite trace or a tracial state on a C^*–algebra A with unit 1 is a linear functional τ on A which has the property

$$\tau(xy) = \tau(yx) \qquad \text{for all} \quad x, y \in A,$$

which is positive (that is, $\tau(x^*x) \geq 0$ for all $x \in A$), and which is normalized by $\tau(1) = 1$. The trace τ is faithful if $\tau(x^*x) \neq 0$ for all $x \neq 0$.

Let M be a finite von Neumann algebra, with faithful trace normal τ. Let $\pi : \Gamma \to U(M)$ be a group homomorphism. The function $\varphi = \tau \circ \pi : \Gamma \to \mathbb{C}$ has the following properties:

(i) φ is central;

(ii) φ is positive definite;

(iii) $\varphi(e) = 1$.

Let $CP(\Gamma)$ denote the set of functions $\varphi : \Gamma \to \mathbb{C}$ with Properties (i), (ii) and (iii) above.

Ley $\varphi \in CP(\Gamma)$. Then there exist a finite von Neumann algebra M_φ, with faithful normal trace τ_φ, and a group homomorphism $\pi_\varphi : \Gamma \to U(M_\varphi)$ such that $\varphi = \tau_\varphi \circ \pi_\varphi$. Indeed, by GNS–construction, there exists a cyclic unitary representation π_φ of Γ on a Hilbert space $\mathcal{H}_\varphi$ with a cyclic unit vector ξ_φ such that

$$\varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi,\ \xi_\varphi\rangle \qquad \text{for all} \quad \gamma \in \Gamma.$$

Since φ is central, there exists another unitary representation ρ_φ of Γ on $\mathcal{H}_\varphi$ which commutes with π_φ (that is, $\pi_\varphi(\gamma)\rho_\varphi(\gamma') = \rho_\varphi(\gamma')\pi_\varphi(\gamma)$ for all $\gamma, \gamma' \in \Gamma$) and with the property that

$$\rho_\varphi(\gamma)\xi_\varphi = \pi_\varphi(\gamma^{-1})\xi_\varphi \qquad \text{for all} \quad \gamma \in \Gamma.$$

Let $M_\varphi = \pi_\varphi(\Gamma)''$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_\varphi)$ generated by $\pi_\varphi(\Gamma)$, where $\mathcal{S}' = \{T \in \mathcal{L}(\mathcal{H}_\varphi)\ :\ TS = ST \quad \text{for all} \quad S \in \mathcal{S}\}$ denotes the commutant of a subset $\mathcal{S}$ of $\mathcal{L}(\mathcal{H}_\varphi)$. The mapping

$$T \mapsto \langle T\xi_\varphi,\ \xi_\varphi\rangle \qquad \text{for all} \quad T \in M_\varphi$$

is a faithful normal trace τ_φ on M_φ and $\varphi = \tau_\varphi \circ \pi_\varphi$.

Moreover, if $N_\varphi = \rho_\varphi(\Gamma)''$ is the von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_\varphi)$ generated by $\rho_\varphi(\Gamma)$, then

$$M'_\varphi = N_\varphi \qquad \text{and} \qquad N'_\varphi = M_\varphi.$$

In particular, the common centre of M_φ and N_φ is $M_\varphi \cap N_\varphi$.

As an important example, let $\varphi = \delta_e$ be the Dirac function at the group unit e. Then $\varphi \in CP(\Gamma)$. The unitary representations π_φ and ρ_φ associated to φ are the left and right regular representations λ_Γ and ρ_Γ on $\ell^2(\Gamma)$. Morever, M_φ is the von Neumann algebra $L(\Gamma)$ of Γ.

The set $CP(\Gamma)$ is a compact and convex subset of the vector space of all bounded functions on Γ, equipped with the weak *-topology. The set of extremal points $E(\Gamma)$ of $CP(\Gamma)$ is the set of all indecomposable central positive definite functions on Γ. By Choquet theory, every $\varphi \in CP(\Gamma)$ may be written as a integral

$$\varphi = \int_{E(\Gamma)} \psi d\mu(\psi)$$

for a probability measure μ on $E(\Gamma)$, at least when G is countable. For $\varphi \in CP(\Gamma)$, we have that M_φ is a factor if and only if $\varphi \in E(\Gamma)$. As an example, the Dirac function δ_e belongs to $E(\Gamma)$ if and only if Γ is an ICC group.

Let M be a finite von Neumann algebra, with faithful normal trace τ, and let $\pi : \Gamma \to U(M)$ be a homomophism such that $\pi(\Gamma)'' = M$. Observe that, if we set $\varphi = \tau \circ \pi \in CP(\Gamma)$, then, with the notation above, the mapping $\pi_\varphi(\gamma) \mapsto \pi(\gamma)$ extends to an isomorphism $M_\varphi \to M$ of von Neumann algebras.

A homomorphism $\pi : \Gamma \to U(M)$ for a finite factor M such that $\pi(\Gamma)'' = M$ will be called a finite factor representation of Γ. We say that two such representations $\pi_1 : \Gamma \to U(M_1)$ and $\pi_2 : \Gamma \to U(M_2)$ are quasi-equivalent if there exists an isomorphism $\Phi : M_1 \to M_2$ such that $\Phi(\pi_1(\gamma)) = \pi_2(\gamma)$ for all $\gamma \in \Gamma$. Summarizing the discussion above, we see that $E(\Gamma)$ classifies the finite factor representations of Γ, up to quasi-equivalence.

The set $E(\Gamma)$ parametrizes also the indecomposable traces on the full C^*-algebra of Γ. Recall that the full C^*-algebra $C^*(\Gamma)$ of Γ is the C^*-algebra with the universal property that every unitary representation of Γ on a Hilbert space $\mathcal{H}$ extends to a $*$–homomorphism $C^*(\Gamma) \to \mathcal{L}(\mathcal{H})$. The algebra $C^*(\Gamma)$ can be realized as completion of the group algebra $\mathbb{C}[\Gamma]$ under the norm

$$\left\| \sum_{\gamma\in\Gamma} c_\gamma \gamma \right\| = \sup \left\{ \left\| \sum_{\gamma\in\Gamma} c_\gamma \pi(\gamma) \right\| \ : \ \pi \in \mathrm{Rep}(\Gamma) \right\},$$

where $\mathrm{Rep}(\Gamma)$ denotes the set of (equivalence classes of) cyclic unitary representations of Γ.

We will view Γ as a subgroup of the group of unitaries in $C^*(\Gamma)$ by means of the canonical embedding $\Gamma \to C^*(\Gamma)$. Every trace on $C^*(\Gamma)$ defines by restriction to Γ an element of $CP(\Gamma)$. Conversely, every $\varphi \in CP(\Gamma)$ extends to a trace on $C^*(\Gamma)$, since, as seen above, $\varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi, \ \xi_\varphi \rangle$ and π_φ is a unitary representation of Γ.

3 Some subgroups of $SL_n(\mathbb{Z})$

Let n is a fixed integer with $n \geq 2$. For a pair of integers (i, j) with $1 \leq i \neq j \leq n$, denote by e_{ij} the corresponding elementary matrix, that is, the $(n \times n)$-matrix with 1's on the diagonal, 1 at the (i, j)-entry, and 0 elsewhere. It is well-known that $SL_n(\mathbb{Z})$ is generated by

$$\{e_{ij} \ : \ 1 \leq i \neq j \leq n\}.$$

Moreover, for $n \geq 3$, any two elementary matrices are conjugate inside $SL_n(\mathbb{Z})$. Indeed, observe that the matrix

$$s_{ij} = e_{ij} e_{ji}^{-1} e_{ij} \in SL_n(\mathbb{Z})$$

permutes the i-th and the j-th standard unit vectors of $\mathbb{Z}^n$, up to a sign. Hence, if e_{kl} and e_{pq} are two elementary matrices, conjugation by a suitable product of matrices of the form s_{ij} will carry e_{kl} into e_{pq} or e_{pq}^{-1}. Now, e_{pq} and e_{pq}^{-1} are conjugate via a suitable diagonal matrix in $SL_n(\mathbb{Z})$, when $n \geq 3$.

The proof of the following two lemmas is by straightforward computation. We will always view an element $a \in \mathbb{Z}^n$ as column vector. Its transpose a^t is then a row vector. We denote by $e_1, \ldots, e_n$ the standard unit vectors in $\mathbb{Z}^n$.

Lemma 6 *Let k be a non-zero integer and let $i, j \in \{1, \ldots, n\}$ with $1 \leq i \neq j \leq n$. The centralizer of e_{ij}^k in $SL_n(\mathbb{Z})$ consists of all matrices with εe_i as i-th column and εe_j^t as j-th row for $\varepsilon \in \{\pm 1\}$.* ■

For instance, the centralizer of e_{12}^k is the subgroup of all matrices of the form

$$\begin{pmatrix} \varepsilon & * & * & \cdots & * \\ 0 & \varepsilon & 0 & \ldots & 0 \\ 0 & * & * & \cdots & * \\ & \vdots & \vdots & \ddots & \\ 0 & * & * & * & * \end{pmatrix},$$

for $\varepsilon \in \{\pm 1\}$.

For $j \in \{1, \ldots, n\}$, let $V_j \cong \mathbb{Z}^{n-1}$ be the subgroup generated by

$$\{e_{ij} \ : \ 1 \leq i \leq n, i \neq j\};$$

for instance, V_1 is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ * & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ * & 0 & \ldots & 1 \end{pmatrix}.$$

Lemma 7 *The normalizer of V_j in $SL_n(\mathbb{Z})$ is the subgroup G_j of all matrices in $SL_n(\mathbb{Z})$ with εe_j as j-th row for $\varepsilon \in \{\pm 1\}$.*■

Thus, for instance, the normalizer G_1 of V_1 is the group of all matrices

$$\begin{pmatrix} \varepsilon & 0 & \cdots & 0 \\ * & & & \\ \vdots & & A & \\ * & & & \end{pmatrix},$$

where $A \in GL_{n-1}(\mathbb{Z})$ and $\varepsilon = \det A$.

Up to a subgroup of index two, G_j is isomorphic to the semi-direct product $SL_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1}$ for the natural action of $SL_{n-1}(\mathbb{Z})$ on $\mathbb{Z}^{n-1}$.

We will have also to consider the transpose subgroups V_i^t generated by

$$\{e_{ij} \ : \ 1 \leq j \leq n, j \neq i\}.$$

Observe that $V_j \cap V_i^t$ is the copy of $\mathbb{Z}$ generated by e_{ij} for $i \neq j$. The normalizer of V_i^t in $SL_n(\mathbb{Z})$ is of course the group G_i^t. Observe also that

$$V_j \subset G_i^t \qquad \text{and} \qquad V_i^t \subset G_j$$

for all $i \neq j$.

We will refer to subgroups of the form V_j and V_i^t as to the copies of $\mathbb{Z}^{n-1}$ inside $SL_n(\mathbb{Z})$.

4 Proof of Theorem 3: A preliminary reduction

The starting point of the proof of Theorem 3 is the following classification from [Burg91, Proposition 9] of the measures on the n-dimensional torus $\mathbb{T}^n$ which are invariant under the natural action of $SL_n(\mathbb{Z})$; for a more elementary proof in the case $n = 2$, see [DaKe79].

Lemma 8 ***([Bur])*** *Let $n \geq 2$ be an integer. Let μ be a $SL_n(\mathbb{Z})$–invariant ergodic probability measure on the Borel subsets of $\mathbb{T}^n$. Then either μ is concentated on a finite $SL_n(\mathbb{Z})$–orbit or μ is the normalized Lebesgue measure on $\mathbb{T}^n$.*

Recall that a point $x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ has a finite $SL_n(\mathbb{Z})$–orbit if and only if $x \in \mathbb{Q}^n/\mathbb{Z}^n$.

Let $n \geq 3$ and let

$$\varphi : SL_n(\mathbb{Z}) \to \mathbb{C}$$

be an indecomposable central positive definite function on $SL_n(\mathbb{Z})$, fixed throughout the proof.

As in Section 2, let π and ρ be the corresponding commuting factor representations of Γ on the Hibert space $\mathcal{H}$ with cyclic vector ξ such that

$$\varphi(\gamma) = \langle \pi(\gamma)\xi, \ \xi \rangle = \langle \rho(\gamma^{-1})\xi, \ \xi \rangle, \qquad \text{for all} \quad \gamma \in \Gamma.$$

Fix any copy $V = V_j$ or $V = V_j^t$ of $\mathbb{Z}^{n-1}$ inside $SL_n(\mathbb{Z})$ and consider the restriction $\varphi\big|_V$ to V.

As φ is central, $\varphi\big|_V$ is a G–invariant positive definite function on V, where

$$G = G_j \qquad \text{or} \qquad G = G_j^t$$

is the normalizer of V in $SL_n(\mathbb{Z})$. Since G contains a copy of the semi-direct product $SL_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1}$ (for the usual action in case $V = V_j$ and for the inverse transpose of the usual action in case $V = V_j^t$), we have

$$\varphi(Ax) = \varphi(x) \qquad \text{for all} \quad x \in \mathbb{Z}^{n-1}, \quad A \in SL_{n-1}(\mathbb{Z}).$$

Thus, by Bochner's theorem, $\varphi\big|_V$ is the Fourier transform of a $SL_{n-1}(\mathbb{Z})$–invariant probability measure on the torus

$$\mathbb{T}^{n-1} \cong \widehat{V}.$$

Let $(\mathcal{O}_i)_{i\geq 1}$ denote the sequence of finite $SL_{n-1}(\mathbb{Z})$–orbits in $\mathbb{T}^{n-1}$. For each $i \geq 1$, denote by $\mu_{\mathcal{O}_i}$ the uniform distribution on $\mathcal{O}_i$, that is, the probability measure

$$\mu_{\mathcal{O}_i} = \frac{1}{|\mathcal{O}_i|} \sum_{\chi \in \mathcal{O}_i} \delta_\chi$$

on $\mathbb{T}^{n-1}$. Lemma 8 shows that μ has a decomposition as a convex combination

$$\mu = t_\infty^{(V)} \mu_\infty + \sum_{i \geq 1} t_i^{(V)} \mu_{\mathcal{O}_i} \qquad \text{with} \qquad t_\infty^{(V)} + \sum_{i \geq 1} t_i^{(V)} = 1,\ t_\infty^{(V)} \geq 0,\ t_i^{(V)} \geq 0,$$

where μ_∞ is the normalized Lebesgue measure on $\mathbb{T}^{n-1}$. Thus, we obtain a corresponding decomposition of $\varphi\big|_V$

$$\varphi\big|_V = t_\infty^{(V)} \delta_e + \sum_{i \geq 1} t_i^{(V)} \psi_{\mathcal{O}_i} \qquad \text{with} \qquad t_\infty^{(V)} + \sum_{i \geq 1} t_i^{(V)} = 1,\ t_\infty^{(V)} \geq 0,\ t_i^{(V)} \geq 0,$$

where $\psi_{\mathcal{O}_i}$ is the Fourier transform of the measure $\mu_{\mathcal{O}_i}$.

By general theory, we have a corresponding decomposition of $\mathcal{H}$ into a direct sum of $\pi(V)$–invariant subspaces

$$\mathcal{H} = \mathcal{H}_\infty^V \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^V$$

where $\mathcal{H}_\chi^V$ is the subspace on which V acts according to the character χ, that is,

$$\mathcal{H}_\chi^V = \{\eta \in \mathcal{H} \;\; \pi(v)\eta = \chi(v)\eta \qquad \text{for all} \quad v \in V\}$$

and where $\mathcal{H}_\infty^V$ is a subspace on which $\pi(V)$ is a multiple of the regular representation λ_V of V. Observe that some of these subspaces may be $\{0\}$. Observe also that, since the representation ρ commutes with π, each of the subspaces $\mathcal{H}_\chi^V$ and $\mathcal{H}_\infty^V$ is invariant under the whole of $\rho(SL_n(\mathbb{Z}))$.

We claim that we have the following dichotomy.

Lemma 9 *We have*

- *either* $\mathcal{H} = \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^V$ *for every copy* V *of* $\mathbb{Z}^{n-1}$ *in* $SL_n(\mathbb{Z})$, *or*
- $\mathcal{H} = \mathcal{H}_\infty^V$ *for every copy* V *of* $\mathbb{Z}^{n-1}$ *in* $SL_n(\mathbb{Z})$.

Proof Let V be a copy of $\mathbb{Z}^{n-1}$ with $\mathcal{H}_\infty^V \neq \{0\}$. We will show that

$$\mathcal{H} = \mathcal{H}_\infty^W \qquad \text{for every copy } W \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).$$

Clearly, this will prove the lemma.

• *First step:* Let W be a copy of $\mathbb{Z}^{n-1}$ for which we assume that $V \cap W \neq \{0\}$. We claim that $\mathcal{H}_\infty^V = \mathcal{H}_\infty^W$.

Indeed, $V \cap W$ is the copy of $\mathbb{Z}$ generated by the appropriate elementary matrix. We have two decompositions of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_\infty^V \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^V \qquad \text{and} \qquad \mathcal{H} = \mathcal{H}_\infty^W \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^W.$$

Consider the restriction of π to $V \cap W$. Each one of the subspaces

$$\bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^V \qquad \text{and} \qquad \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^W$$

has a decomposition into a direct sum of subspaces under which $\pi(V \cap W)$ acts according to a character of $V \cap W$.

On the other hand, the representation $\pi|_{V\cap W}$ restricted to $\mathcal{H}_\infty^V$ or to $\mathcal{H}_\infty^W$ is a multiple of the regular representation $\lambda_{V\cap W}$, since $\lambda_V|_{V\cap W}$ and $\lambda_w|_{V\cap W}$ are mutiples of $\lambda_{V\cap W}$. It follows that we necessarily have $\mathcal{H}_\infty^V = \mathcal{H}_\infty^W$.

• *Second step:* Let W be now an arbitrary copy of $\mathbb{Z}^{n-1}$. We claim that we still have $\mathcal{H}_\infty^V = \mathcal{H}_\infty^W$.

Indeed, as is readily verified, we can find two copies W^1 and W^2 of $\mathbb{Z}^{n-1}$ with

$$V \cap W^1 \neq \{0\}, \quad W^1 \cap W^2 \neq \{0\}, \quad \text{and} \quad W^2 \cap W \neq \{0\}.$$

Therefore, by the first step, we have

$$\mathcal{H}_\infty^V = \mathcal{H}_\infty^{W^1}, \quad \mathcal{H}_\infty^{W^1} = \mathcal{H}_\infty^{W^2} \qquad \mathcal{H}_\infty^{W^2} = \mathcal{H}_\infty^W,$$

so that $\mathcal{H}_\infty^V = \mathcal{H}_\infty^W$.

• *Third step:* We claim that $\mathcal{H}_\infty^V = \mathcal{H}$.

Indeed, by the second step, we have

$$\mathcal{H}_\infty^V = \mathcal{H}_\infty^W \qquad \text{for every copy } W \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z})..$$

Since $\mathcal{H}_\infty^W$ is invariant under $\pi(W)$, it follows that $\mathcal{H}_\infty^V$ is invariant under $\pi(SL_n(\mathbb{Z}))$.

On the other hand, $\mathcal{H}_\infty^V$ is also invariant under $\rho(SL_n(\mathbb{Z}))$. Since π is a factor representation and since $\mathcal{H}_\infty^V \neq \{0\}$, the claim follows. ■

We have to consider separately the two possible decompositions of $\mathcal{H}$ given by the previous lemma. We will see that the first one corresponds to a character of a congruence quotient, and that the second one to a character induced from the centre.

5 Proof of Theorem 3: First case

With the notation from the last section, we assume in this section that

$$\mathcal{H} = \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}_\chi^V \qquad \text{for every copy } V \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).$$

We claim that there exists some integer $N \geq 1$ such that π is trivial on the congruence normal subgroup

$$\Gamma(N) = \{\gamma \in SL_n(\mathbb{Z}) \ : \ \gamma \equiv I \mod N\}.$$

Let $\gamma_0, \gamma_1, \ldots, \gamma_d$ denote the elementary matrices in $SL_n(\mathbb{Z})$, where $d = n(n-1) - 1$.

For every $k \in \{0, \dots, d\}$, we have a decomposition

$$\mathcal{H} = \bigoplus_{\alpha \in \mathbb{Q}/\mathbb{Z}} \mathcal{H}_{\alpha}^{\gamma_k}$$

of $\mathcal{H}$ under the action of the unitary operator $\pi(\gamma_k)$, where $\mathcal{H}_{\alpha}^{\gamma_k}$ is the eigenspace (possibly equal to $\{0\}$) of $\pi(\gamma_k)$ corresponding to α.

Lemma 10 *There exists an integer $N \geq 1$ such that $\pi(\gamma_0^N), \pi(\gamma_1^N), \dots, \pi(\gamma_d^N)$ have a non-zero common invariant vector in $\mathcal{H}$.*

Proof Let M be the factor generated by $\pi(\Gamma)$ and denote by τ the trace on M defined by φ.

Write the elements in $\mathbb{Q}/\mathbb{Z}$ as a sequence $\{\alpha_i)_{i \geq 1}$. For every $i \geq 1$, let

$$p_i : \mathcal{H} \to \mathcal{H}_{\alpha_i}^{\gamma_0}$$

denote the orthogonal projection onto $\mathcal{H}_{\alpha_i}^{\gamma_0}$. Observe that $p_i \in M$ (in fact, p_i belongs to the abelian von Neumann algebra generated by $\pi(\gamma_0)$). We have $\tau(p_i) \in [0, 1]$ and $\sum_{i \geq 1} \tau(p_i) = 1$, since $\sum_{i \geq 1} p_i = I$,

Let ε be a real number with

$$0 < \varepsilon < 1/2^d.$$

There exists $i_0 \geq 1$ such that

$$\sum_{i=1}^{i_0} \tau(p_i) \geq 1 - \varepsilon.$$

Since elements in $\mathbb{Q}/\mathbb{Z}$ have finite order, we can find an integer $N \geq 1$ such that

$$\alpha_i^N = 1 \qquad \text{for all} \quad i \in \{1, \dots, i_0\}.$$

Then $\pi(\gamma_0^N)$ acts as the identity on

$$\bigoplus_{i=1}^{i_0} \mathcal{H}_{\alpha_i}^{\gamma_0}.$$

For $l \in \{0, 1, \dots, d\}$, let $\mathcal{H}^{\gamma_l^N}$ be the subspace of $\pi(\gamma_l^N)$–invariant vectors in $\mathcal{H}$. We claim that

$$\mathcal{H}^{\gamma_0^N} \cap \mathcal{H}^{\gamma_1^N} \cap \dots \cap \mathcal{H}^{\gamma_d^N} \neq \{0\}.$$

For every $k \in \{0, 1, \dots, d\}$, let q_k denote the orthogonal projection onto

$$\mathcal{H}^{\gamma_0^N} \cap \mathcal{H}^{\gamma_1^N} \cap \cdots \cap \mathcal{H}^{\gamma_k^N}.$$

It is clear that $q_k \in M$.

We claim that

$$\tau(q_k) \geq 1 - 2^k \varepsilon \qquad \text{for all} \quad k = 0, 1, \dots, d. \tag{1}$$

Once proved, this will imply that

$$\tau(q_d) \geq 1 - 2^d \varepsilon > 0,$$

and hence $q_d \neq 0$ since τ is faithful on M; this will finish the proof of the lemma.

To prove (1), we proceed by induction on k. Since

$$\bigoplus_{i=1}^{i_0} \mathcal{H}_{\alpha_i}^{\gamma_0} \subset \mathcal{H}^{\gamma_0^N},$$

we have $q_0 \geq \sum_{i=1}^{i_0} p_i$. Hence,

$$\tau(q_0) \geq \sum_{i=1}^{i_0} \tau(p_i) \geq 1 - \varepsilon,$$

and this proves (1) in the case $k = 0$.

Let $k \geq 1$ and assume that

$$\tau(q_{k-1}) \geq 1 - 2^{k-1} \varepsilon. \tag{2}$$

Set

$$\mathcal{K} = \mathcal{H}^{\gamma_0^N} \cap \mathcal{H}^{\gamma_1^N} \cap \cdots \cap \mathcal{H}^{\gamma_{k-1}^N}$$

and set $q = q_{k-1}$, the orthogonal projection on $\mathcal{K}$.

Since any two elementary matrices are conjugate, we have $\gamma_k = s\gamma_0 s^{-1}$ for some element $s \in SL_n(\mathbb{Z})$. Observe that

$$\mathcal{H}^{\gamma_k^N} = \pi(s)\mathcal{H}^{\gamma_0^N}.$$

Consider the operator

$$T = (1 - q)\pi(s^{-1})q$$

on $\mathcal{H}$. Observe that $T \in M$. For $\eta \in \mathcal{H}$, we have $T(\eta) = 0$ if and only if $\pi(s^{-1})q(\eta) \in \mathcal{K}$, that is, if and only $q(\eta) \in \pi(s)\mathcal{K}$. Hence

$$\operatorname{Ker} T = (\mathcal{K} \cap \pi(s)\mathcal{K}) \oplus \mathcal{K}^{\perp}. \tag{3}$$

Let

$$p_{\operatorname{Ker} T} : \mathcal{H} \to \operatorname{Ker} T$$

be the orthogonal projection on $\operatorname{Ker} T$. Then $p_{\operatorname{Ker} T} \in M$, since $T \in M$. Moreover, since the range of T is contained in $\mathcal{K}^{\perp}$, we have

$$\tau(1-q) \geq \tau(I) - \tau(p_{\operatorname{Ker} T}) = 1 - \tau(p_{\operatorname{Ker} T}).$$

Hence, by (2),

$$\tau(p_{\operatorname{Ker} T}) \geq 1 - 2^{k-1}\varepsilon. \tag{4}$$

We have, by (3)

$$\tau(p_{\operatorname{Ker} T}) = \tau(p_{\mathcal{K}\cap\pi(s)\mathcal{K}}) + \tau(1-q),$$

where $p_{\mathcal{K}\cap\pi(s)\mathcal{K}} \in M$ is the orthogonal projection on $\mathcal{K} \cap \pi(s)\mathcal{K}$. Now,

$$\mathcal{K} \cap \pi(s)\mathcal{K} \subset \mathcal{K} \cap \pi(s)\mathcal{H}^{\gamma_0^N} = \mathcal{K} \cap \mathcal{H}^{\gamma_k^N}.$$

Since q_k is the orthogonal projection on $\mathcal{K} \cap \mathcal{H}^{\gamma_k^N}$, it follows in view of (2) and (4) that

$$\begin{aligned} \tau(q_k) &\geq \tau(p_{\mathcal{K}\cap\pi(s)\mathcal{K}}) \\ &= \tau(p_{\operatorname{Ker} T}) - (1 - \tau(q)) \\ &\geq (1 - 2^{k-1}\varepsilon) - 2^{k-1}\varepsilon = 1 - 2^k\varepsilon. \end{aligned}$$

This proves the claim (1) and finishes the proof of the lemma. ∎

Corollary 11 *Under the assumption made at the beginning of this section, there exists an irreducible representation π_0 of the congruence quotient*

$$SL_n(\mathbb{Z})/\Gamma(N^2) \cong SL_n(\mathbb{Z}/N^2\mathbb{Z})$$

such that φ is the (normalized) character of π_0 lifted to $SL_n(\mathbb{Z})$, where N is as in Lemma 10.

Proof By the previous lemma, the subspace $\mathcal{K}$ of the common invariant vectors under $\pi(\gamma_0^N), \pi(\gamma_1^N), \ldots, \pi(\gamma_d^N)$ is non-zero. Let Γ be the subgroup of $SL_n(\mathbb{Z})$ generated by

$$\{\gamma_0^N, \gamma_1^N, \ldots, \gamma_d^N\}.$$

By [Tits76, Proposition 2], Γ contains the congruence normal subgroup $\Gamma(N^2)$.

Consider the subspace

$$\mathcal{H}^{\Gamma(N^2)} = \{\eta \in \mathcal{H} \ : \ \pi(\gamma)\eta = \eta \qquad \text{for all} \quad \gamma \in \Gamma(N^2)\},$$

of $\pi(\Gamma(N^2))$-invariant vectors. Then $\mathcal{H}^{\Gamma(N^2)} \neq \{0\}$ since $\mathcal{K} \subset \mathcal{H}^{\Gamma(N^2)}$. Moreover, $\mathcal{H}^{\Gamma(N^2)}$ is invariant under $\pi(SL_n(\mathbb{Z}))$, as $\Gamma(N^2)$ is normal in $SL_n(\mathbb{Z})$.

On the other hand, $\mathcal{H}^{\Gamma(N^2)}$ is clearly invariant under $\rho(SL_n(\mathbb{Z}))$. It follows that

$$\mathcal{H}^{\Gamma(N^2)} = \mathcal{H}.$$

Hence, π factorizes through the finite group $SL_n(\mathbb{Z})/\Gamma(N^2)$. It follows that $\mathcal{H}$ is finite-dimensional, that π is a equivalent to a multiple $m\pi_0$ of an irreducible representation π_0 of $SL_n(\mathbb{Z})/\Gamma(N^2)$ with $m = \dim(\pi_0)$, and that φ is the normalized character of π_0. ■

6 Proof of Theorem 3: Second case

With the notation as in Section 4, we assume now that

$$\mathcal{H} = \mathcal{H}_\infty^V \qquad \text{for every copy } V \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).$$

This is equivalent to:

$$\varphi|_V = \delta_e \qquad \text{for every copy } V \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).$$

Let χ_φ be the unitary character of the centre $C = \{\pm I\}$ of $SL_n(\mathbb{Z})$ such that

$$\varphi(z\gamma) = \chi_\varphi(z)\varphi(\gamma)\text{for all} \quad z \in C, \gamma \in SL_n(\mathbb{Z}).$$

We claim that

$$\varphi(\gamma) = \begin{cases} 0 & \text{if } \gamma \in SL_n(\mathbb{Z}) \setminus C \\ \chi_\varphi(\gamma) & \text{if } \gamma \in C. \end{cases}$$

The following proposition, which is of independent interest, will play a crucial rôle.

Proposition 12 *Every matrix $\gamma \in SL_n(\mathbb{Z})$ is conjugate to the product $g_1 g_2 g_3$ of three matrices of the form*

$$g_1 = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & * & \\ 0 & * & \cdots & * \end{pmatrix} \in G_1^t\ , \qquad g_2 = \begin{pmatrix} * & * & \cdots & * \\ \vdots & & * & \\ * & * & \cdots & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_n$$

and

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & 0 & \cdots & 1 \end{pmatrix} \in V_1$$

Proof

• *First step:* We first claim that γ is conjugate to a matrix γ_1 with first column of the form $(*, 0, *, 0, \ldots, 0)^t$. This is Lemma 1 in [Bren60]. The result is proved by conjugating γ by permutation matrices (with sign ajusted) and by elementary matrices of the type e_{ij} with $1 < i \neq j \leq n$.

So, we can assume that the first column of γ is of the form $(k, 0, l, 0, \ldots, 0)^t$ for $k, l \in \mathbb{Z}$.

• *Second step:* There exists a matrix $\gamma_1 \in G_n$ such that the first column of $\gamma_1\gamma$ is $(k, 1, l, 0, \ldots, 0)^t$. Indeed, since $\gcd(k, l) = 1$, there exist $p, q \in \mathbb{Z}$ such that $pk + ql = 1$. We can take

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p & 1 & q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in G_n.$$

• *Third step:* There exists a matrix $\gamma_2 \in G_1^t \cap G_n$ such that the first column of $\gamma_2\gamma_1\gamma$ is $(1, 1, l, 0, \ldots, 0)^t$. Indeed, we can take

$$\gamma_2 = \begin{pmatrix} 1 & 1-k & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in G_n.$$

• *Fourth step:* There exists a matrix $\gamma_3 \in V_1$ such that the first column of $\gamma_3\gamma_2\gamma_1\gamma$ is $(1,0,0,0,\ldots,0)^t$. Indeed, we can take

$$\gamma_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -l & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in V_1.$$

By the last step, $\gamma_4 = \gamma_3\gamma_2\gamma_1\gamma \in G_1^t$. We have

$$\gamma_4\gamma\gamma_4^{-1} = \gamma_4(\gamma_1^{-1}\gamma_2^{-1})\gamma_3^{-1}.$$

The claim follows, since $\gamma_1^{-1}\gamma_2^{-1} \in G_n$ and $\gamma_3^{-1} \in V_1$. ■

Remark 13 In the case $n \geq 4$, the previous proposition can be improved: every $\gamma \in SL_n(\mathbb{Z})$ is conjugate to a product $g_1g_2 \in G_1^tG_n$. Indeed, in this case, the matrix γ_3 in the fourth step of the proof belongs to G_n and hence

$$\gamma_4\gamma\gamma_4^{-1} = \gamma_4(\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}) \in G_1^tG_n.$$

Returning to the proof of Theorem 3, the previous proposition implies that it suffices to show that

$$\varphi(\gamma) = 0 \qquad \text{for all} \quad \gamma \in G_1^tG_nV_1 \quad \text{with} \quad \gamma \notin C.$$

For this, several preliminary steps will be needed.

We will use several times the following elementary lemma.

Lemma 14 *Let Γ be a group and $(\pi, \mathcal{H})$ a unitary representation of Γ. Let $\psi = \langle \pi(\cdot)\xi, \xi\rangle$ be an associated positive definite function such that $\psi = \delta_e$. Then, for every sequence $(g_k)_{k\in\mathbb{N}}$ of pairwise distinct elements $g_k \in \Gamma$, the sequence $(\pi(g_k)\xi)_{k\in\mathbb{N}}$ converges weakly to 0 in $\mathcal{H}$.*

Proof For $k, l \in \mathbb{N}$ with $k \neq l$, we have

$$\begin{aligned} \langle \pi(g_k)\xi, \pi(g_l)\xi\rangle &= \langle \pi(g_l^{-1}g_k)\xi, \xi\rangle \\ &= \psi(g_l^{-1}g_k) = 0. \end{aligned}$$

Therefore, $(\pi(g_k)\xi)_{k\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$ and the claim follows.$\blacksquare$

The first step in this part of the proof of Theorem 3 is to show that

$$\varphi(\gamma) = 0 \qquad \text{for all} \quad \gamma \in G_1^t \cup G_n \quad \text{with} \quad \gamma \notin C.$$

For elements x, y in a group, let $[x, y]$ denote the commutator $x^{-1}y^{-1}xy$.

Lemma 15 *Let V be a copy of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ and let G be the normalizer of V. Then $\varphi(\gamma) = 0$ for every $\gamma \in G \setminus C$.*

Proof Write

$$V = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n,$$

where $x_1, \ldots, x_n$ are the elementary matrices contained in V.

Let $\gamma \in G \setminus C$. We claim that there exists $i \in \{1, \ldots, n\}$ such that

$$x_i^{-k}\gamma x_i^k \neq x_i^{-l}\gamma x_i^l \qquad \text{for all} \quad k, l \in \mathbb{Z},\ k \neq l.$$

Indeed, otherwise there would exist non-zero integers k_i such that γ is in the centralizer of $x_i^{k_i}$ for all $i \in \{1, \ldots, n\}$. This would imply that $\gamma \in C$ (see Lemma 6).

The commutators $[\gamma, x_i^k]$ belong to V and are pairwise distinct. Hence, by Lemma 14, the sequence $(\pi([\gamma, x_i^k])\xi)_{k\in\mathbb{N}}$ is weakly convergent to 0 in $\mathcal{H}$. For $k \in \mathbb{N}$, we have

$$\begin{aligned} \varphi(\gamma) &= \varphi(x_i^{-k}\gamma x_i^k) \\ &= \varphi(\gamma[\gamma, x_k]) \\ &= \langle \pi([\gamma, x_i^k])\xi, \pi(\gamma^{-1})\xi \rangle. \end{aligned}$$

Hence,

$$\varphi(\gamma) = \lim_k \langle \pi([\gamma, x_i^k])\xi, \pi(\gamma^{-1})\xi \rangle = 0,$$

as claimed. $\blacksquare$

The next step is to show that

$$\varphi(\gamma) = 0 \qquad \text{for all} \quad \gamma \in G_1^t G_n \quad \text{with} \quad \gamma \notin C.$$

Lemma 16 *Let V, W be two copies of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ with $V \cap W \neq \{0\}$. Let G, H be the normalizers of V and W, respectively. Let $\gamma = gh$ with $g \in G$, $h \in H$, and $\gamma \notin C$. Then $\varphi(\gamma) = 0$.*

Proof If $g \in C$ or $h \in C$, then $\gamma \in G$ or $\gamma \in H$ and then $\varphi(\gamma) = 0$, by Lemma 15. Hence, we can assume that $g \notin C$ and $h \notin C$.

Let x denote the elementary matrix such that

$$V \cap W = \langle x \rangle.$$

It is readily verified that, for $k \in \mathbb{Z} \setminus \{0\}$, the centralizer of x^k is contained in $G \cap H$. Hence, we can assume that γ does not belong to this centralizer, that is, that the elements $x^{-k}\gamma x^k$ are pairwise distinct.

We have

$$x^{-k}\gamma x^k = x^{-k} g x^k x^{-k} h x^k = g[g, x^k] x^{-k} h x^k,$$

Set $y_k = [g, x^k]x^{-k}hx^k$. Observe that $V \subset G \cap H$. Since $[g, x^k] \in V$, we have $y_k \in H$. Moreover, the elements y_k are pairwise distinct, since

$$y_k = g^{-1}x^{-k}\gamma x^k.$$

Hence, again by Lemma 14, the sequence $(\pi(y_k)\xi)_{k\in\mathbb{N}}$ is weakly convergent to 0 in $\mathcal{H}$. As in the previous lemma, it follows that

$$\varphi(\gamma) = \lim_k \varphi(x^{-k}\gamma x^k) = \lim_k \langle \pi(y_k)\xi, \pi(g^{-1})\xi\rangle = 0. \blacksquare$$

We will also need the following consequence of Lemma 16.

Lemma 17 *Let V, W two copies of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ with $V \cap W \neq \{0\}$. Let G, H be the normalizers of V and W, respectively. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of pairwise distinct elements in GH. Then $(\pi(\gamma_k)\xi)_{k\in\mathbb{N}}$ converges weakly to* 0 *in $\mathcal{H}$.*

Proof Observe that $(\pi(\gamma_k)\xi)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{H}$. Therefore, it suffices to show that every subsequence $(\pi(\gamma_{k_i})\xi)_{i\in\mathbb{N}}$ of $(\pi(\gamma_k)\xi)_{k\in\mathbb{N}}$ has a subsequence which weakly converges to 0.

For $i \in \mathbb{N}$, write $\gamma_{k_i} = g_{k_i}h_{k_i}$ for $g_{k_i} \in G$ and $h_{k_i} \in H$.

Since C is finite and since the elements γ_{k_i} are pairwise distinct, we can find a subsequence of $(\gamma_{k_i})_i$, still denoted by $(\gamma_{k_i})_i$, such that $\gamma_{k_j}^{-1}\gamma_{k_i} \notin C$ for all $i \neq j$. It follows that

$$g_{k_j}^{-1} g_{k_i} h_{k_i} h_{k_j}^{-1} \notin C \qquad \text{for all} \quad i \neq j.$$

From Lemma 16, we deduce that, for all $i \neq j$,

$$\begin{aligned}\varphi(\gamma_{k_j}^{-1}\gamma_{k_i}) &= \varphi(h_{k_j}^{-1}g_{k_j}^{-1}g_{k_i}h_{k_i}) \\ &= \varphi(g_{k_j}^{-1}g_{k_i}h_{k_i}h_{k_j}^{-1}) \\ &= 0,\end{aligned}$$

since $g_{k_j}^{-1}g_{k_i} \in G$ and $h_{k_i}h_{k_j}^{-1} \in H$. As in the proof of Lemma 14, this shows that $(\pi(\gamma_{k_i})\xi)_i$ weakly converges to 0. ■

We can now conclude the proof of Theorem 3. Let $\gamma \in SL_n(\mathbb{Z}) \setminus C$. We want to show that $\varphi(\gamma) = 0$.

By Proposition 12, we can assume that $\gamma = g_1g_2g_3$ for matrices of the form

$$g_1 = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & * & \\ 0 & * & \cdots & * \end{pmatrix} \in G_1^t\ , \qquad g_2 = \begin{pmatrix} * & * & \cdots & * \\ \vdots & & * & \\ * & * & \cdots & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_n$$

and

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & 0 & \cdots & 1 \end{pmatrix} \in V_1.$$

If $g_3 \in G_n$, then γ is a non-central element in $G_1^tG_n$, and it follows from Lemma 16 that $\varphi(\gamma) = 0$. We can therefore assume that $g_3 \notin G_n$, that is, $a_n \neq 0$.

Let x be the elementary matrix $e_{2,n}$, thus

$$x = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $x \in G_1^t \cap G_n$ and the centralizer of every power x^k for $k \neq 0$ is contained in G_n. Hence, if γ is contained in the centralizer of some power x^k for $k \neq 0$, the claim follows from Lemma 15. We can therefore assume that

$$x^{-k}\gamma x^k \neq x^{-l}\gamma x^l \qquad \text{for all} \quad k \neq l.$$

We compute that

$$x^{-k}g_3x^k = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 + ka_n & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Hence $x^{-k}g_3x^k = \alpha_k\beta$, where

$$\alpha_k = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ ka_n & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Observe that $\alpha_k \in G_n$ for every k. We have

$$\begin{aligned} x^{-k}\gamma x^k &= x^{-k}g_1g_2g_3x^k \\ &= (x^{-k}g_1x^k)(x^{-k}g_2x^k)(x^{-k}g_3x^k) \\ &= (x^{-k}g_1x^k)(x^{-k}g_2x^k)\alpha_k\beta. \end{aligned}$$

Now, since $x \in G_1^t \cap G_n$, we have $x^{-k}g_1x^k \in G_1^t$ and $x^{-k}g_2x^k\alpha_k \in G_n$. It follows that

$$x^{-k}\gamma x^k\beta^{-1} \in G_1^tG_n \qquad \text{for every} \quad k.$$

Set

$$\gamma_k = x^{-k}\gamma x^k\beta^{-1}.$$

Since γ is not in the centralizer of x^k, we have $\gamma_k \neq \gamma_l$ for all $k \neq l$. Hence, by Lemma 17, the sequence $(\pi(\gamma_k))_{k\in\mathbb{N}}$ converges weakly to 0. It follows that

$$\begin{aligned} \varphi(\gamma) &= \lim_k \varphi(\beta x^{-k}\gamma x^k\beta^{-1}) \\ &= \lim_k \varphi(\beta\gamma_k) \\ &= \lim_k \langle\pi(\beta\gamma_k)\xi, \xi\rangle \\ &= \lim_k \langle\pi(\gamma_k)\xi, \pi(\beta^{-1})\xi\rangle \\ &= 0. \end{aligned}$$

This concludes the proof of Theorem 3.■

7 Deducing Theorem 1 from Theorem 3

Let $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. Let M be a finite factor, with trace τ, and let $\pi : \Gamma \to U(M)$ be a group homomorphism such that $\pi(\Gamma)'' = M$. Then $\varphi = \tau \circ \pi$ is a character of Γ.

Assume that M is finite dimensional. Let $\pi_\varphi : \Gamma \to U(M_\varphi)$ be the finite factor representation associated to φ (see Section 2). The mapping $\pi_\varphi(\gamma) \mapsto \pi(\gamma)$ extends to an isomorphism $M_\varphi \to M$ of von Neumann algebras. Hence M_φ is finite dimensional and, by Theorem 3, φ is the character of an irreducible finite dimensional representation of some congruence quotient $SL_n(\mathbb{Z}/N\mathbb{Z})$ for $N \geq 1$. It follows that π factorizes through $SL_n(\mathbb{Z}/N\mathbb{Z})$.

Assume now that M is infinite dimensional. By Theorem 3, we have $\varphi = \widetilde{\chi}$ for a character χ of the centre C. If n is odd, let $\Lambda = \Gamma$ and, if n is even, let $\Lambda = \Gamma(N)$ be a congruence subgroup for $N \geq 3$. Then Λ has finite index in Γ and $\Lambda \cap C = \{e\}$. We therefore have $\varphi|_\Lambda = \delta_e$. The GNS-representation of Λ corresponding to δ_e is the regular representation λ_Λ which generates the von Neumann algebra $L(\Lambda)$. The mapping $\lambda_\Lambda(\gamma) \mapsto \pi(\gamma)$ extends to a normal homomorphism $L(\Lambda) \to M$.

Remark 18 Observe that the conclusion in (ii) of Theorem 1 is that $\pi|_\Lambda$ extends to $L(\Lambda)$ and not just to $U(L(\Lambda))$. P. de la Harpe pointed out to me that this is a stronger statement: a homomorphism $U(M_1) \to U(M_2)$ between the unitary groups of two finite factors M_1, M_2 does not necessarily extend to an algebra homomorphism $M_1 \to M_2$. As a simple example, take $M_1 = M_2(\mathbb{C})$ and $M_2 = M_4(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. The group homomorphism $\pi : U(2) \to U(4), g \mapsto g \otimes g$ does not extend to an algebra homomorphism $M_2(\mathbb{C}) \to M_4(\mathbb{C})$.

8 A question of Kirchberg

A conjecture of Kirchberg [Kirc93, Section 8, (B4)] is:

The full C^*-algebra $C^*(SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}))$ of the direct product $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ has a faithful tracial state.

As shown in [Kirc93], this problem is in fact equivalent to a series of outstanding conjectures, among them the following one which was suggested by Connes in [Conn76, page 105]:

Every factor of type II_1 with separable predual is a subfactor of the ultrapower R_ω of the hyperfinite factor R of type II_1.

A positive answer to the following question of Kirchberg [Kirc93, Remark 8.2] would imply the conjecture above:

Does $C^*(SL_4(\mathbb{Z}))$ have a faithful tracial state?

Indeed, $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ embedds as a subgroup of $SL_4(\mathbb{Z})$, for instance, via the mapping

$$SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \ni (\gamma_1, \gamma_2) \to \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in SL_4(\mathbb{Z}).$$

A faithful tracial state on $C^*(SL_4(\mathbb{Z}))$ would give, by restriction, a faithful tracial state on $C^*(SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}))$.

We proceed to show that the answer to this question is negative. In fact, the following stronger result will be proved. We will consider the copy

$$\Lambda = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} \ : \ \gamma \in SL_2(\mathbb{Z}) \right\} \cong SL_2(\mathbb{Z})$$

of $SL_2(\mathbb{Z})$ inside $SL_n(\mathbb{Z})$.

Corollary 19 *Let $n \geq 3$ and set $\Gamma = SL_n(\mathbb{Z})$. Let φ be a tracial state on $C^*(\Gamma)$. Then $\varphi|_{C^*(\Lambda)}$ is not faithful.*

Proof Let π be the cyclic unitary representation of Γ corresponding to φ. By Theorem 3, π decomposes as a direct sum

$$\pi_\infty \oplus \bigoplus_i \sigma_i \,,$$

where π_∞ is a multiple of the regular representation λ_Γ, and where every representation σ_i factorizes through some congruence quotient $\Gamma/\Gamma(N_i)$.

Let $\mathrm{Rep}_{\mathrm{cong}}(\Gamma)$ denote the set of all unitary representations of Γ which factorize through some congruence quotient. In fact, as a consequence of the positive answer to the congruence subgroup problem, $\mathrm{Rep}_{\mathrm{cong}}(\Gamma)$ coincides with the set of all finite dimensional unitary representations of Γ (see [Bekk99, Proposition 2]). This implies (see, for instance, [Bekk99, Proposition 1]) that

$$\bigcap_{\sigma \in \mathrm{Rep}_{\mathrm{cong}}(\Gamma)} C^* - \mathrm{Ker}\, \sigma \subset C^* - \mathrm{Ker}\, \lambda_\Gamma,$$

where $C^* - \mathrm{Ker}\,\sigma$ denotes the kernel in $C^*(\Gamma)$ of the extension of a unitary representation σ of Γ.

We consider now the restriction $\pi|_\Lambda$ of π to Λ. Observe that

$$\mathrm{Rep}_{\mathrm{cong}}(\Lambda) = \{\sigma|_\Lambda \ : \ \sigma \in \mathrm{Rep}_{\mathrm{cong}}(\Gamma)\}\,.$$

Since $C^* - \mathrm{Ker}\,\lambda_\Lambda = C^* - \mathrm{Ker}(\lambda_\Gamma|_\Lambda)$, we have

$$\bigcap_{\sigma\in\mathrm{Rep}_{\mathrm{cong}}(\Lambda)} C^* - \mathrm{Ker}\,\sigma \subset C^* - \mathrm{Ker}\,\lambda_\Lambda,$$

It follows from Selberg's inequality $\lambda_1 \geq 3/16$ (see [Bekk99, Lemma 3]) and from the fact that $SL_2(\mathbb{Z})$ does not have Kazhdan's Property (T) that $\mathrm{Rep}_{\mathrm{cong}}(\Lambda)$ does not separate the points of $C^*(\Lambda)$, that is,

$$\bigcap_{\sigma\in\mathrm{Rep}_{\mathrm{cong}}(\Lambda)} C^* - \mathrm{Ker}\,\sigma \neq \{0\}.$$

Hence, we have

$$\begin{aligned}
C^* - \mathrm{Ker}(\pi|_\Lambda) &= C^* - \mathrm{Ker}(\pi_\infty|_\Lambda) \cap \bigcap_i C^* - \mathrm{Ker}(\sigma_i|_\Lambda) \\
&= C^* - \mathrm{Ker}\,\lambda_\Lambda \cap \bigcap_i C^* - \mathrm{Ker}(\sigma_i|_\Lambda) \\
&\supset C^* - \mathrm{Ker}\,\lambda_\Lambda \cap \bigcap_{\sigma\in\mathrm{Rep}_{\mathrm{cong}}(\Lambda)} C^* - \mathrm{Ker}\,\sigma \\
&= \bigcap_{\sigma\in\mathrm{Rep}_{\mathrm{cong}}(\Lambda)} C^* - \mathrm{Ker}\,\sigma
\end{aligned}$$

and $C^* - \mathrm{Ker}(\pi|_\Lambda) \neq \{0\}$. This clearly implies that $\varphi|_\Lambda$ is not faithful. ■

Remark 20 The previous result does not hold for $n = 2$. Indeed, as was shown in [Choi80, Corollary 9], $C^*(SL_2(\mathbb{Z})$ has a faithful trace. In fact a stronger result is proved in [Choi80, Theorem 7]: $C^*(SL_2(\mathbb{Z})$ is residually finite dimensional, that is, the finite dimensional representations of $SL_2(\mathbb{Z})$ separate the points of $C^*(SL_2(\mathbb{Z}))$.

It is shown in [LuSh04] that other interesting groups have a residually finite dimensional full C^*-algebra; this is, for instance, the case for fundamental groups of surfaces.

9 A remark on semi-finite traces

As mentioned in the introduction, it is conceivable that semi-finite, infinite traces exist on $C^*(PSL_n(\mathbb{Z}))$ for $n \geq 3$. The following result implies that no such trace factorizes through the reduced C^*-algebra $C^*_r(PSL_n(\mathbb{Z}))$ for any integer $n \geq 2$.

Proposition 21 *Let G be a connected real semisimple Lie group without compact factors and with trivial centre. Let Γ be a Zariski-dense subgroup of G. Then the tracial state δ_e is, up to a scalar multiple, the unique semi-finite trace on $C^*_r(\Gamma)$. In particular, $C^*_r(\Gamma)$ has no normal factor representation of type II_∞.*

Proof Let $\varphi : C^*_r(\Gamma)^+ \to [0, \infty]$ be a semi-finite trace on the set of positive elements of $C^*_r(\Gamma)$.

We use an observation from [Rose89, page 583]. It is well-known that there exist a non-zero two-sided ideal $\mathfrak{m}$, called the ideal of definition of φ, and a linear functional on $\mathfrak{m}$ which coincides with φ on $\mathfrak{m}^+$ (see [Dix-C*, Proposition 6.1.2]). Now, by [BeCH95], $C^*_r(\Gamma)$ is simple, that is, $C^*_r(\Gamma)$ has no non-trivial two-sided (closed or non-closed) ideals. Hence, $\mathfrak{m} = C^*_r(\Gamma)$ and φ is a finite trace. By [BeCH95], δ_e is the unique tracial state on $C^*_r(\Gamma)$ and the claim follows.■

Examples of Zariski dense subgroups Γ of a group G as in the previous proposition include all lattices in G. So Proposition 21 applies, for instance, when $\Gamma = PSL_n(\mathbb{Z})$ for $n \geq 2$ or when Γ is the fundamental group of an oriented compact surface of genus ≥ 2.

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