COMBINATORIAL GEOMETRY OF GENERIC DEGENERATIONS OF QUADRATIC DIFFERENTIALS

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Abstract. We describe typical degenerations of quadratic differentials thus describing "generic cusps" of the moduli space of meromorphic quadratic differentials with at most simple poles. The part of the boundary of the moduli space which does not arise from "generic" degenerations is often negligible in problems involving information on compactification of the moduli space.

However, even for a typical degeneration one may have several short loops on the Riemann surface which shrink simultaneously. We explain this phenomenon, describe all rigid configurations of short loops, present a detailed description of analogs of desingularized stable curves arising here, and show how one can reconstruct a Riemann surface endowed with a quadratic differential which is close to a "cusp" by the corresponding point at the principal boundary.

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Introduction

0.1. **Saddle connections on flat surfaces.** We study flat metrics on a closed orientable surface of genus g, which have isolated conical singularities and linear holonomy restricted to $\{Id, -Id\}$. If the linear holonomy group is trivial, then the surface is referred to as a *translation surface*, such a flat surface corresponds to an Abelian differential ω on a Riemann surface. If the holonomy group is nontrivial, then such a flat surface arises from a meromorphic quadratic differential q with at most simple poles on a Riemann surface. In this paper, unless otherwise stated, a quadratic differential is *not* the square of an Abelian differential and a *flat surface*

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is the Riemann surface with the flat metric corresponding to an Abelian or to a quadratic differential.

It is natural to consider families of flat surfaces sharing the same combinatorial geometry: the genus, the number of singularities and the cone angles at singularities. Such families correspond to the *strata* $\mathcal{Q}(d_1,\dots,d_m)$ in the moduli space of quadratic differentials, where $d_i \in \{-1,0,1,2,3,\dots\}$ stands for the orders of singularities (simple poles, marked points, zeroes) of quadratic differentials. The collection $\alpha=\{d_1,\dots,d_m\}$ is called the *singularity data* of the stratum.

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior. For the flat metrics as described above, regular closed geodesics always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections. Thus, when some regular closed geodesic becomes short the corresponding saddle connection(s) become short as well. More generally, a degeneration of an Abelian or of a quadratic differential corresponds to collapse of some saddle connections.

Any saddle connection on a flat surface $S\in\mathcal{Q}(\alpha)$ persists under small deformations of S inside $\mathcal{Q}(\alpha)$. It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections. We say that a collection $\{\gamma_1,\dots,\gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1|:|\gamma_2|:\dots:|\gamma_n|$ of the lengths of all saddle connections in the collection.

0.2. **Degeneration of Abelian differentials.** In the case of Abelian differentials ω, rigid collections of saddle connections were studied in the paper [EMZ]. It was shown that all saddle connections in any rigid collection are *homologous*. In particular, they are all parallel and have equal length and either all of them join the same pair of distinct singular points, or they are all closed.

This implies that when the saddle connections in a rigid collection are contracted by a continuous deformation, the limiting flat surface generically decomposes into several components represented by nondegenerate flat surfaces $S'_1,\dots,S'_k$, where k might vary from one to the genus of the initial surface. Let $\mathcal{H}(\beta'_j)$ be the stratum ambient for S'_j. The stratum $\mathcal{H}(\beta')=\mathcal{H}(\beta'_1)\sqcup\dots\sqcup\mathcal{H}(\beta'_k)$ of disconnected flat surfaces $S'_1\sqcup\dots\sqcup S'_k$ is referred to as a *principal boundary stratum* of the stratum $\mathcal{H}(\beta)$. For any connected component of any stratum $\mathcal{H}(\beta)$ the paper [EMZ] describes all principal boundary strata; their union is called the *principal boundary* of the corresponding connected component of $\mathcal{H}(\beta)$.

The paper [EMZ] also presents the inverse construction. Consider any flat surface $S'_1\sqcup\dots\sqcup S'_k\in\mathcal{H}(\beta')$ in the *principal boundary* of $\mathcal{H}(\beta)$; consider a sufficiently small value of a complex parameter $\delta\in\mathbb{C}$. One can reconstruct the flat surface $S\in\mathcal{H}(\beta)$ endowed with a collection of homologous saddle connections $\gamma_1,\dots,\gamma_n$ such that $\int_{\gamma_i}\omega=\delta$, and such that the degeneration of S that consists of contracting the saddle connections γ_i in the collection gives the surface $S'_1\sqcup\dots\sqcup S'_k$. This inverse construction involves several *basic surgeries* of the flat structure. Given a disconnected flat surface $S'_1\sqcup\dots\sqcup S'_k$ one applies an appropriate surgery to each S'_j producing a surface S_j with boundary. The surgery depends on the parameter δ: the boundary of each S_j is composed of two geodesic segments of lengths $|\delta|$;

moreover, the boundary components of S_j and S_{j+1} are compatible, which allows one to glue the compound surface S from the collection of surfaces with boundary.

A collection $\gamma = \{\gamma_1, \dots, \gamma_n\}$ of homologous saddle connections determines the following data on combinatorial geometry of the decomposition $S \setminus \gamma$: the number of components, their boundary structure, the singularity data for each component, the cyclic order in which the components are glued to each other. These data are referred to as *configuration* of homologous saddle connections. A configuration $\mathcal{C}$ uniquely determines the corresponding boundary stratum $\mathcal{H}(\beta'_{\mathcal{C}})$.

The constructions above explain how configurations of homologous saddle connections on flat surfaces $S \in \mathcal{H}(\beta)$ determine the "cusps" of the stratum $\mathcal{H}(\beta)$. Consider a subset $\mathcal{H}_1^\varepsilon(\beta) \subset \mathcal{H}(\beta)$ of surfaces of area one having a saddle connection shorter than ε. Up to a subset $\mathcal{H}_1^{\varepsilon,thin}(\beta)$ of negligibly small measure the set $\mathcal{H}_1^\varepsilon(\beta)$ can be represented as a disjoint union over all admissible configurations $\mathcal{C}$ (i.e. as a union over different "cusps") of neighborhoods of the corresponding "cusps". When a configuration $\mathcal{C}$ is composed from homologous saddle connections joining distinct zeroes, the neighborhood of the corresponding cusp has the structure of a fiber bundle over the corresponding boundary stratum $\mathcal{H}(\beta'_{\mathcal{C}})$ with the fiber represented by an appropriate ramified cover over the Euclidean ε-disc. Moreover, the canonical measure in the corresponding connected component of $\mathcal{H}_1^{\varepsilon,thick}(\beta) = \mathcal{H}_1^\varepsilon(\beta) \setminus \mathcal{H}_1^{\varepsilon,thin}(\beta)$ decomposes as a product measure of the canonical measure in the boundary stratum and the Euclidean measure in the fiber, see [EMZ].

Remark. We warn the reader that the correspondence between the compactification of the moduli space of Abelian differentials and the Deligne—Mumford compactification of the underlying moduli space of curves is not straightforward. In particular, the desingularized stable curve corresponding to the limiting flat surface generically *is not* represented as the union of corresponding Riemann surfaces $S'_1, \dots, S'_k$: the stable curve might contain more components.

This paper concerns the study of similar phenomena in the case of quadratic differentials that are not squares of Abelian differentials. The extended version of this paper containing all proofs can be found in [MZ].

1. $\hat{\text{H}}$omologous saddle connections

A meromorphic quadratic differential q with at most simple poles on a Riemann surface S defines a canonical (ramified) double cover $p : \hat{S} \to S$ such that $p^*q = \omega^2$ is a square of an Abelian differential ω on $\hat{S}$. Let $P = \{P_1, \dots, P_m\} \subset S$ be the collection of singularities (zeroes and simple poles) of q; let $\hat{P} = p^{-1}(P)$ be the set of their preimages under the projection $p : \hat{S} \to S$.

Given an oriented saddle connection γ on S let γ', γ'' be its lifts to the double cover. If $[\gamma'] = -[\gamma'']$ as cycles in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$ we let $[\hat{\gamma}] := [\gamma']$, otherwise we define $[\hat{\gamma}]$ as $[\hat{\gamma}] := [\gamma'] - [\gamma'']$.

Definition 1. The saddle connections γ_1, γ_2 on a flat surface S defined by a quadratic differential q are $\hat{\text{h}}$omologous[1] if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$ under an appropriate choice of orientations of γ_1, γ_2.

[1]The notion "homologous in the relative homology with local coefficients defined by the canonical double cover induced by a quadratic differential" is unbearably bulky, so we introduced an abbreviation "$\hat{\text{h}}$omologous". We stress that the circumflex over the "h" is quite meaningful: as it is indicated in the definition, the corresponding cycles are homologous *on the double cover*.

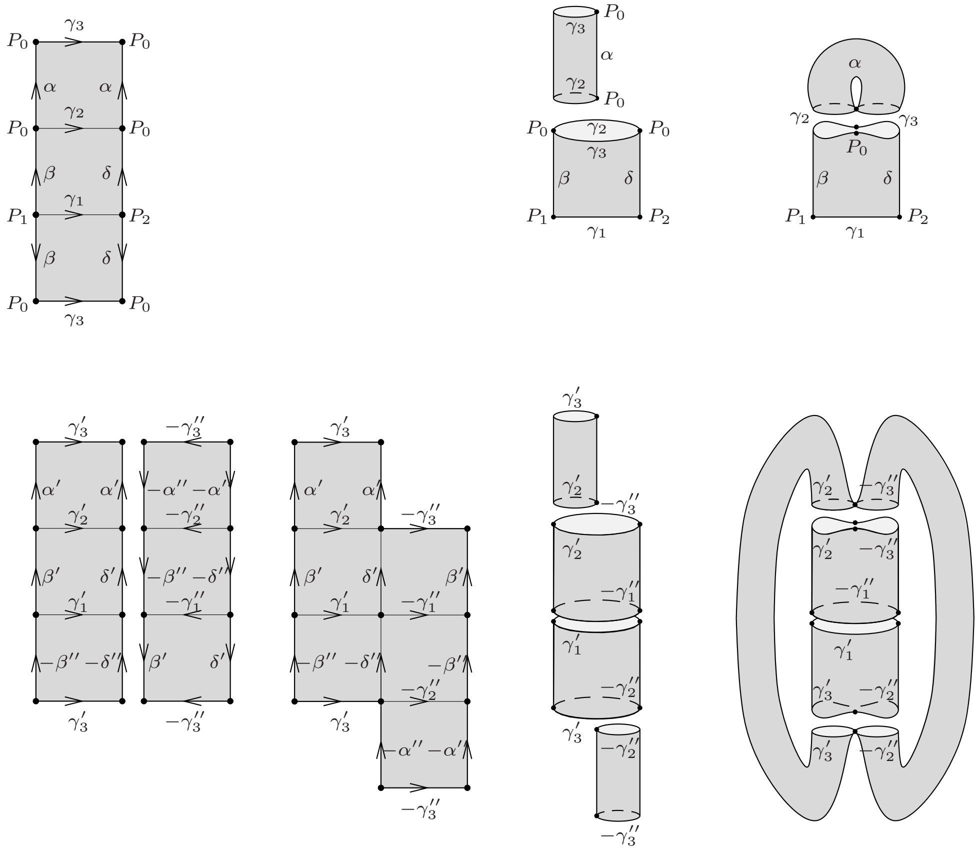


FIGURE 1. Saddle connections $\gamma_1, \gamma_2, \gamma_3$ on the torus (above picture) are homologous, though γ_1 is a segment joining distinct points and γ_2 and γ_3 are closed loops.

We begin with the following example which illustrates many of the main ideas.

Example 1. Consider three unit squares, or rather a rectangle 1×3 and glue a torus from it as indicated at the top left corner of Figure 1. Identifying the three corresponding sides β, γ_1 and δ of the two bottom squares we obtain a "pocket" with two "corners" P_1 and P_2 at the bottom and with two "corners" P_0 at the boundary on top. Identifying the points P_0 we obtain a "pocket" with a "figure-eight" boundary (the bottom fragment of the top right picture at Figure 1). Identifying the sides α of the remaining square we obtain a cylinder which we glue to the previous fragment. Topologically the surface thus obtained is a torus. Metrically this torus has three conical singularities. Two of them ("the corners P_1, P_2 of the pocket") have cone angle π; the third conical singularity P_0 has cone angle 4π. Such a flat torus gives us a point in the stratum $\mathcal{Q}(2, -1, -1)$.

The bottom picture illustrates the canonical double covering over the above torus. The cycle γ_2' is homologous to γ_3' on the double cover and the cycle γ_2'' is homologous to γ_3''. This implies that the cycles $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$ on the double cover are homologous to the waist curve of the thick cylinder fragment of the right bottom picture. Thus, the saddle connections γ_1, γ_2 and γ_3 are $\hat{\text{h}}$omologous, though γ_1 is a segment joining distinct points P_1 and P_2, and γ_2, γ_3 are the closed loops with the base point P_0.

It essentially follows from the definition that $\hat{\text{h}}$omologous saddle connections are parallel on S and that their lengths either coincide or differ by a factor of two. The following simple statement proved in appendix A characterizes rigid collections of saddle connections on a flat surface with nontrivial linear holonomy.

Proposition 1. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are $\hat{\text{h}}$omologous.*

There is an obvious geometric test for deciding when saddle connections γ_1, γ_2 on a translation surface S are homologous: it is sufficient to check whether $S \setminus (\gamma_1 \cup \gamma_2)$ is connected or not (provided $S \setminus \gamma_1$ and $S \setminus \gamma_2$ are connected). It is slightly less obvious to check whether saddle connections γ_1, γ_2 on a flat surface S with nontrivial linear holonomy are $\hat{\text{h}}$omologous or not. In particular, a pair of closed saddle connections might be homologous in the usual sense, but not $\hat{\text{h}}$omologous; a pair of closed saddle connections might be $\hat{\text{h}}$omologous even if one of them represents a loop homologous to zero, and the other does not; finally, a saddle connection joining a pair of *distinct* singularities might be $\hat{\text{h}}$omologous to a saddle connection joining a singularity to itself.

The following criterion tells when two saddle connections are $\hat{\text{h}}$omologous and what is the structure of the complement $S \setminus (\gamma_1 \cup \gamma_2)$.

Theorem 1. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. Two saddle connections γ_1, γ_2 on S are $\hat{\text{h}}$omologous if and only if they have no interior intersections and one of the connected components of the complement $S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Moreover, if such a component exists, it is unique.*

2. Graph of connected components

A collection γ of $\hat{\text{h}}$omologous saddle connections $\gamma = \{\gamma_1, \dots, \gamma_n\}$ divides S into simpler surfaces S_j with boundary. We associate to any such decomposition a graph $\Gamma(S, \gamma)$. The vertices of the graph correspond to the connected components S_j of $S \setminus (\gamma_1 \cup \dots \cup \gamma_n)$. We denote the vertices corresponding to cylinders (if any) by small circles "$\circ$". The remaining vertices are labelled with a "$+$" sign if the corresponding surface S_j has trivial linear holonomy and with a "$-$" sign if it does not. We do not label the vertices of "$\circ$"-type: it is easy to see that the cylinders always have trivial linear holonomy.

The edges of the graph are in the one-to-one correspondence with the saddle connections γ_i. Each saddle connection γ_i is on the boundary of either one or two surfaces. If γ_i is on the boundary of pair of surfaces, it corresponds to an edge joining the corresponding vertices. If γ_i is on the boundary of only one surface, then it corresponds to an edge of the graph which joins the vertex to itself; such an edge contributes 2 to the valence of the vertex.

Remark 1. The union $\gamma = \gamma_1 \cup \dots \cup \gamma_n$ of saddle connections can be considered as a graph γ embedded into the surface S. By definition $\Gamma(S, \gamma)$ is a graph *dual* to γ. Namely, $\Gamma(S, \gamma)$ can be realized as graph embedded into the surface S in the following way. A vertex of $\Gamma(S,\gamma)$ corresponding to a connected component S_j of $S\setminus\gamma$ is mapped to a point v_j located in the interior of the corresponding surface with boundary S_j. The line representing the image of an edge of $\Gamma(S,\gamma)$ corresponding to a saddle connection γ_i has a single transversal intersection with γ_i in some interior point; this line does not intersect itself nor any other such line nor some other saddle connection $\gamma_{i'}$, where $i'\neq i$, in an interior point.

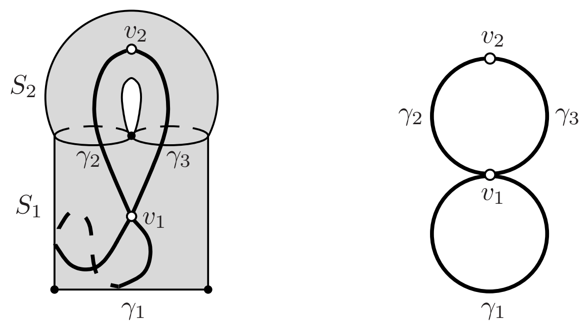


FIGURE 2. Graph $\Gamma(S,\gamma)$ of connected components

Example 2. Consider the surface S and the collection γ of $\hat{\text{h}}$omologous saddle connections $\{\gamma_1,\gamma_2,\gamma_3\}$ as in example 1 above (see figure 1). The complement $S\setminus\gamma$ has two connected components; both represented by flat cylinders. The graph $\Gamma(S,\gamma)$ contains two vertices, both of the "○"-type, and three edges. The graph $\Gamma(S,\gamma)\subset S$ is dual to the graph $\gamma\subset S$, see figure 2.

It follows from the definition of $\hat{\text{h}}$omologous saddle connections that their lengths are either the same or differ by a factor of two. Having a collection γ of $\hat{\text{h}}$omologous saddle connections $\gamma_1,\dots,\gamma_n$ we can normalize the length of the shortest one to 1. Then the other saddle connections have lengths either 1 or 2, which endows the edges of the graph Γ with the weights 1 or 2.

The theorem below classifies all possible graphs corresponding to nonempty collections of $\hat{\text{h}}$omologous saddle connections.

Theorem 2. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles; let γ be a collection of $\hat{\text{h}}$omologous saddle connections $\{\gamma_1,\dots,\gamma_n\}$, and let $\Gamma(S,\gamma)$ be the graph of connected components encoding the decomposition $S\setminus(\gamma_1\cup\dots\cup\gamma_n)$.*

The graph $\Gamma(S,\gamma)$ either has one of the basic types listed below or can be obtained from one of these graphs by placing additional "○"-vertices of valence two at any subcollection of edges subject to the following restrictions. At most one "○"-vertex may be placed at the same edge; a "○"-vertex cannot be placed at an edge adjacent to a "○"-vertex of valence 3 *if this is the edge separating the graph.*

The graphs of basic types, presented in Figure 3, are given by the following list:

a) *An arbitrary (possibly empty) chain of "+"-vertices of valence two bounded by a pair of "−"-vertices of valence one;*
b) *A single loop of vertices of valence two having exactly one "−"-vertex and arbitrary number of "+"-vertices (possibly no "+"-vertices at all);*

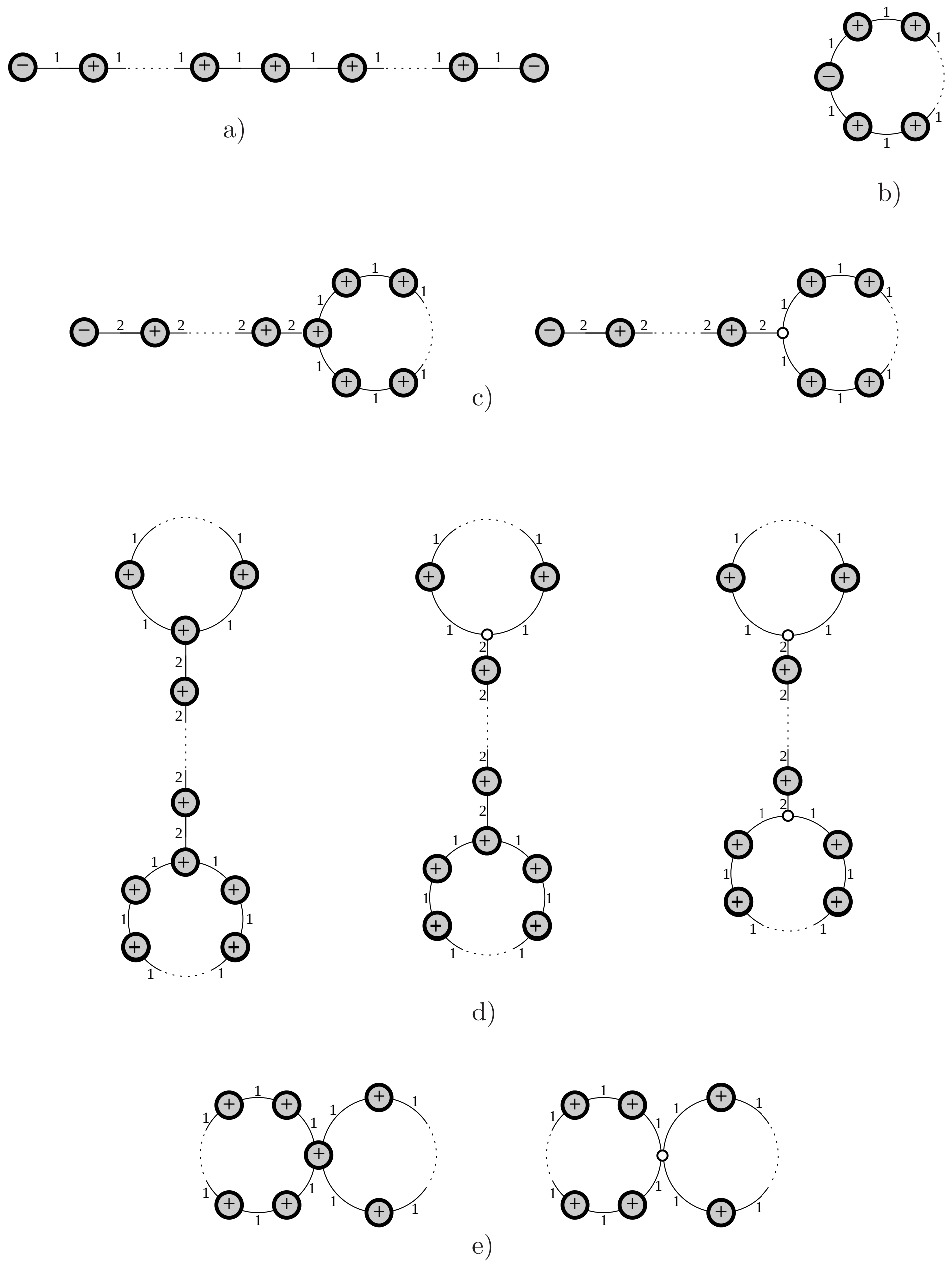


FIGURE 3. Classification of admissible graphs.

c) *A single chain and a single loop joined at a vertex of valence three. The graph has exactly one "$-$"-vertex of valence one; it is located at the end of the chain. The vertex of valence three is either a "$+$"-vertex, or a "$\circ$"-vertex (vertex of the cylinder type). Both the chain, and the cycle may have* *in addition an arbitrary number of "+"-vertices of valence two (possibly no "+"-vertices at all);*

d) *Two nonintersecting cycles joined by a chain. The graph has no "−"-vertices. Each of the two cycles has a single vertex of valence three (the one where the chain is attached to the cycle); this vertex is either a "+"-vertex or a "○"-vertex. If both vertices of valence three are "○"-vertices, the chain joining two cycles is nonempty: it has at least one "+"-vertex. Otherwise, each of the cycles and the chain may have arbitrary number of "+"-vertices of valence two (possibly no "+"-vertices of valence two at all);*

e) *"Figure-eight" graph: two cycles joined at a vertex of valence four, which is either a "+"-vertex or a "○"-vertex. All the other vertices (if any) are the "+"-vertices of valence two. Each of the two cycles may have arbitrary number of such "+"-vertices of valence two (possibly no "+"-vertices of valence two at all).*

Every graph listed above corresponds to some flat surface S and to some collection of saddle connections γ.

3. Boundary singularities

It is convenient to consider a closed surface with boundary S_j^{comp} canonically associated to S_j by taking the natural *compactification* of S_j. Note, that S_j^{comp} need not be the same as the *closure* of S_j in S. For example, if we cut a surface S along a single saddle connection γ joining a pair of distinct singularities we obtain a surface S_1 whose compactification is a surface with boundary composed of a pair of parallel distinct geodesics of the same length, while the closure of $S_1 = S \setminus \gamma_1$ in S coincides with S. The closure of S_j in S is obtained from the compactification S_j^{comp} of S_j by identification of some boundary points (if necessary), or by identification of some boundary saddle connections (if necessary).

3.1. **Ribbon graph.** Given a vertex v of a finite graph Γ consider a tree Γ_v obtained as a small neighborhood of v in Γ in the natural topology of a one-dimensional cell complex. The tree Γ_v together with the canonical mapping of the graphs $\Gamma_v \to \Gamma$ will be referred to as the *boundary* of v. The number of edges of Γ_v is exactly the valence of v (and hence is at most 4 for the graphs from figure 3).

Suppose that the boundary of S_j^{comp} has $r = r(j)$ connected components (called for brevity *boundary components*). Every boundary component is composed of a closed chain of saddle connections $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$, where $1 \le i \le r$. The case $p(i) = 1$ is not excluded: a boundary component might be composed of a single saddle connection. The canonical orientation of S_j^{comp} determines the orientation of every boundary component $\mathcal{B}_i$ of ∂S_j^{comp} and hence determines the cyclic order

$$\to \gamma_{j_{i,1}} \to \dots \to \gamma_{j_{i,p(i)}} \to \tag{1}$$

on every such chain; by convention we let $j_{i,p(i)+1} := j_{i,1}$. Thus, we get a natural decomposition of the set of edges of Γ_{v_j} into a disjoint union of subsets, each endowed with a cyclic order,

$$\{\to \gamma_{j_{1,1}} \to \gamma_{j_{1,2}} \to \dots \to \gamma_{j_{1,p(1)}} \to\} \sqcup \dots \sqcup \{\to \gamma_{j_{r,1}} \to \dots \to \gamma_{j_{r,p(r)}} \to\} \tag{2}$$

It is convenient to encode such combinatorial structure by a *local ribbon graph* $\mathbb{G}_{v_j}$ which is defined in the following way.

Consider a realization of $\Gamma(S,\gamma)$ by an embedded graph dual to the graph γ in S (see remark 1 above). For every vertex v_j of $\Gamma(S,\gamma)$ we get an induced embedding $\Gamma_{v_j} \hookrightarrow S_j^{comp}$. Let a connected component $\mathcal{B}_i$ of ∂S_j^{comp} be represented by a chain (1) of saddle connections. A tubular neighborhood in S_j^{comp} of the union of the corresponding edges $\{\gamma_{j_{i,1}} \cup \dots \cup \gamma_{j_{i,p(i)}}\}$ of $\Gamma_{v_j} \subset S_j^{comp}$ (as in the left picture of figure 4) inherits the canonical orientation of S. This orientation induces a natural cyclic order on the edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ of Γ_{v_j}. We choose the embedding $\Gamma_{v_j} \hookrightarrow S_j^{comp}$ in such way that turning counterclockwise around v_j (considered as a point of S_j^{comp}) we see the edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ appear in the cyclic order (1).

When the boundary ∂S_j^{comp} contains several connected components, the ribbon graphs corresponding to different components overlap at v_j (as in the left picture of figure 4). However, it is easy to make them disjoint by a small deformation, subject to an appropriate choice of the initial embedding $\Gamma_{v_j} \hookrightarrow S_j^{comp}$. From now on we shall always assume that the embedding is chosen appropriately.

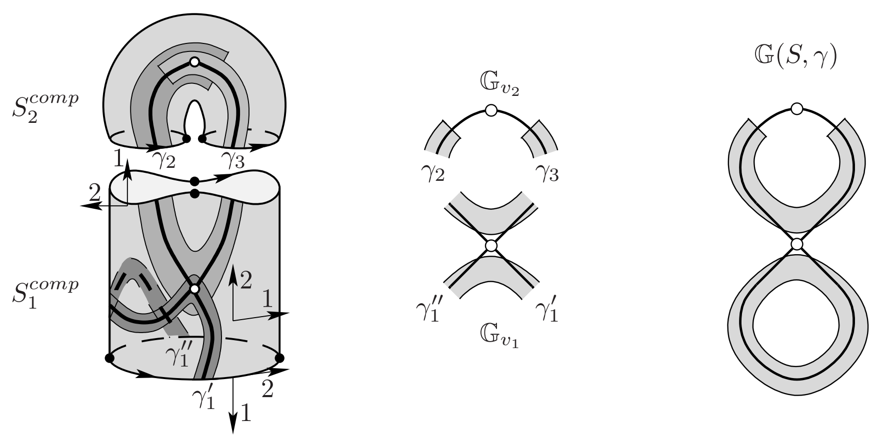


FIGURE 4. Compactifications S_1^{comp}, S_2^{comp} of connected components of $S \setminus \gamma$, the associated local ribbon graphs $\mathbb{G}_{v_1}, \mathbb{G}_{v_2}$ and the global ribbon graph $\mathbb{G}(S,\gamma)$

Example 3. Consider once again the surface S and the collection γ of $\hat{\text{h}}$omologous saddle connections $\{\gamma_1,\gamma_2,\gamma_3\}$ as in example 1, see figure 1. In example 2 we have constructed the associated graph $\Gamma(S,\gamma)$, see figure 2.

The complement $S \setminus \gamma$ has two connected components; their compactifications S_1^{comp}, S_2^{comp} are represented by a pair of flat cylinders. Each of the two connected components of the boundary of S_2^{comp} (the top cylinder in figure 4) is formed by a single saddle connection, so we get $\partial S_2^{comp} = \{\gamma_2\} \sqcup \{\gamma_3\}$. Each of the two connected components of the boundary of S_1^{comp} (the bottom cylinder in figure 4) is formed by a pair of saddle connections, so we get $\partial S_2^{comp} = \{\gamma_2 \to \gamma_3\} \sqcup \{\gamma_1' \to \gamma_1''\}$. The orientation of the boundary components induced by the canonical orientation of S is indicated in the left picture.

The picture in the center of figure 4 shows the corresponding local ribbon graphs and the picture on the right shows the global ribbon graph $\mathbb{G}(S,\gamma)$ for this example.

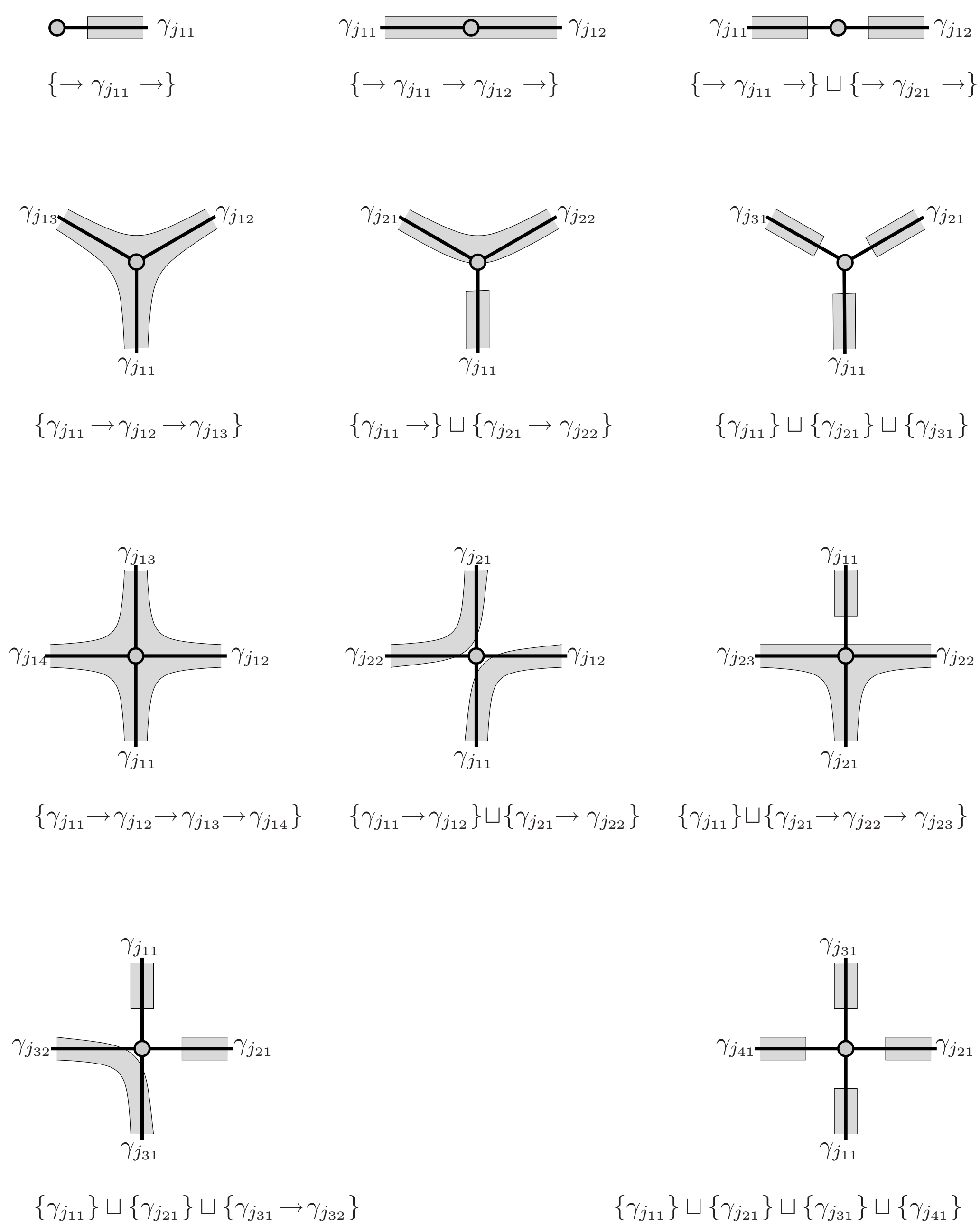


FIGURE 5. All local ribbon graphs $\mathbb{G}_v$ of valences from one to four

For vertices v of valence $1, 2, 3, 4$ figure 5 gives a complete list of all possible partitions of the edges of Γ_v into a disjoint union of subsets endowed with a cyclic order and of the corresponding local ribbon graphs $\mathbb{G}_v$. Note that the canonical orientation of S induces the counterclockwise ordering of the edges of Γ_v.

3.2. **Boundary singularities.** Let S_j be a connected component of the decomposition $S \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$; let S_j^{comp} be its compactification, and let a connected component $\mathcal{B}_i$ of the boundary ∂S_j^{comp} be represented by a chain (1) of saddle connections. The common endpoint of γ_{j_i} and $\gamma_{j_{i+1}}$ is called the *boundary singularity* of S_j^{comp}. Since all saddle connections $\gamma_1, \dots, \gamma_n$ are parallel, the corresponding angle between γ_{j_i} and $\gamma_{j_{i+1}}$ is an integer multiple of π. There might be also several conical singularities in the interior of S_j^{comp}; they are called *interior singularities*.

Definition 2. If the total angle at a boundary singularity is $(k+1)\pi$ the *order of the boundary singularity* is defined to be k, and the *parity of the boundary singularity* is defined to be the parity of k. If the total angle at an interior singularity is $(d+2)\pi$ the *order of the interior singularity* is defined to be d.

The order of the interior singularity coincides with the order of the zero (simple pole) of the corresponding germ of a quadratic differential. By convention, boundary singularities, and their orders will always refer to the compactification S_j^{comp}.

When S_j is represented by a "+"-vertex of the graph $\Gamma(S,\gamma)$, we include the parities of the boundary singularities in our combinatorial structure represented by the embedded local ribbon graph $\mathbb{G}_{v_j}$. Let $\mathcal{B}_i$ be a connected component of the boundary ∂S_j^{comp} constituted by a chain (1) of saddle connections. The edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ of the embedded graph $\Gamma_{v_j} \hookrightarrow S_j^{comp}$ subdivide a neighborhood of v_j in S_j into $p(i)$ sectors. To each sector bounded by a pair of consecutive edges $\gamma_{j_{i,l}}$ and $\gamma_{j_{i,l+1}}$ we associate the parity of the order $k_{j_{i,l}}$ of the corresponding boundary singularity of S_j^{comp}: of the common endpoint of the consecutive saddle connections $\gamma_{j_{i,l}} \to \gamma_{j_{i,l+1}}$ in $\mathcal{B}_i$.

Any connected component S_j of the decomposition $S \setminus \{\gamma_1, \dots, \gamma_n\}$ determines the following combinatorial data which we refer to as the *boundary type* of S_j: the structure (2) of the local ribbon graph $\mathbb{G}_{v_j}$ as in figure 5; an embedding $\Gamma_{v_j} \hookrightarrow \Gamma(S,\gamma)$ and a collection of parities of boundary singularities of S_j.

Theorem 3. *Consider a decomposition of a flat surface S as in theorem 2. Every connected component S_j of the decomposition has one of the boundary types presented in figure 6 and all indicated boundary types are realizable.*

The dotted lines in figure 6 indicate pairs of edges of a vertex $v \in \Gamma(S,\gamma)$ of valence 3 or 4, which are joined by a loop in the graph $\Gamma(S,\gamma)$ (see figure 3) and encode in this way the embedding $\Gamma_{v_j} \hookrightarrow \Gamma(S,\gamma)$.

Remark 2. We use the following convention on indexation of local ribbon graphs in figure 6: the first symbol represents the type ("+", "−", or "∘") of the vertex v_j in the graph $\Gamma(S,\gamma)$; the second symbol is the valence of v_j; the number after a dot is the number of boundary components of S_j. An extra letter "a, b, c" is employed when it is necessary to distinguish different embedded local ribbon graphs sharing the same vertex type, valence and number of boundary components.

The first part of the statement of theorem 3 which claims that every connected component of the decomposition has one of the boundary types in figure 6 is quite elementary. The statement about the realizability of all boundary types presented in figure 6 is much less trivial; it follows from theorem 4.

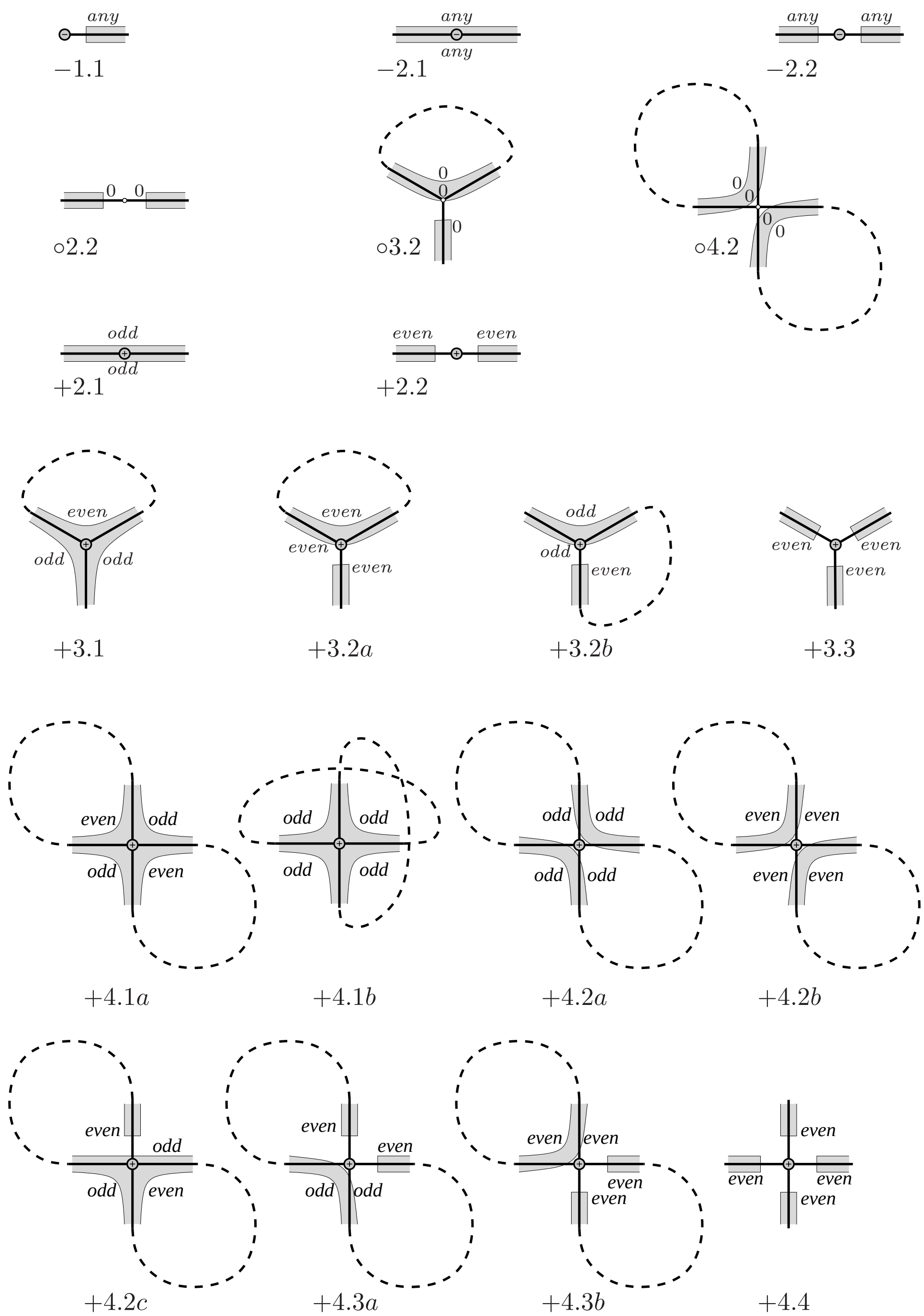


Figure 6. Classification of embedded local ribbon graphs

4. Configurations of $\hat{\text{h}}$omologous saddle connections

We formalize the data on combinatorial geometry of $S \setminus \gamma$ in definition below.

Definition 3. The following combinatorial structure is called a *configuration of $\hat{h}$omologous saddle connections*.

(1) A finite graph Γ endowed with a labelling of each vertex by one of the symbols "+", "−, or "∘", of one of the types described in theorem 2 (see figure 3).

(2) For any vertex v of the graph Γ an embedded ribbon graph $\mathbb{G}_v$ (encoding the decomposition of Γ_v into a disjoint union of subsets, called *boundary components*, each endowed with a cyclic order; see equation (2)) of one of the types described in theorem 3 (see figure 6).

(3) For every "+"-vertex v of Γ and for every pair of consecutive elements $\gamma_{i,l} \to \gamma_{i,l+1}$ of $\mathbb{G}_v$ (called *boundary singularities*) an associated parity (even or odd) as in figure 6.

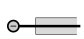
(4) For every vertex v of Γ and for every boundary singularity of $\mathbb{G}_v$ a nonnegative integer $k_{i,l}$ (referred to as the *order of the boundary singularity*) satisfying the following conditions. The order of the boundary singularity respects the *parity* associated to the corresponding boundary singularity as in (3) when v is of the "+"-type; the order of any boundary singularity of any vertex of the "∘"-type is equal to zero. The sum $D_i + 2 = k_{i,1} + \cdots + k_{i,p(i)}$ of orders of boundary singularities along any boundary component $\mathcal{B}_i$ of v satisfies $D_i \geq 0$ for a vertex of "+"-type and $D_i \geq -1$ for a vertex of "−"-type.

(5) For every vertex v of Γ of "−"-type an unordered (possibly empty) collection of integers $\{d_1, \dots, d_{s(v)}\}$, where $d_l \in \{-1, 1, 2, 3, \dots\}$; for every vertex v of Γ of "+"-type an unordered (possibly empty) collection of positive even integers $\{d_1, \dots, d_{s(v)}\}$, where $d_l \in \{2, 4, \dots\}$. In both cases these collections of integers (called *orders of interior singularities*) satisfy the following compatibility conditions with the collection of boundary singularities of $\mathbb{G}_v$:

$$-4 \;\leq\; \left(\sum d_l + \sum D_i\right) \;\equiv\; 0 \mod 4\,,$$

where the first sum is taken over all interior singularities and the second sum is taken over all boundary components $\mathcal{B}_i$ of $\mathbb{G}_v$.

(6) When the vertex v is of the "−"-type the couple [unordered collection of interior singularities, unordered collection of boundary singularities] is in addition not allowed to belong to the following exceptional list:

	$[\varnothing,\{2\}]$; $[\{-1\},\{3\}]$; $[\{1\},\{1\}]$; $[\{-1,1\},\{2\}]$ $[\{1\},\{5\}]$; $[\{3\},\{3\}]$; $[\{1,3\},\{2\}]$; $[\varnothing,\{6\}]$; $[\{4\},\{2\}]$
	$[\varnothing,\{2,0\}]$; $[\varnothing,\{1,1\}]$; $[\{-1\},\{0,3\}]$; $[\{-1\},\{1,2\}]$ $[\{1\},\{0,1\}]$; $[\{1,-1\},\{0,2\}]$; $[\{1,-1\},\{1,1\}]$ $[\{3,1\},\{2,0\}]$; $[\{3,1\},\{1,1\}]$; $[\{3\},\{3,0\}]$; $[\{3\},\{2,1\}]$ $[\{1\},\{5,0\}]$; $[\{1\},\{4,1\}]$; $[\{1\},\{3,2\}]$; $[\{4\},\{2,0\}]$; $[\{4\},\{1,1\}]$ $[\varnothing,\{6,0\}]$; $[\varnothing,\{5,1\}]$; $[\varnothing,\{4,2\}]$; $[\varnothing,\{3,3\}]$
	$[\varnothing,\{2,2\}]$; $[\varnothing,\{1,3\}]$; $[\{-1\},\{2,3\}]$; $[\{1\},\{1,2\}]$; $[\{-1,1\},\{2,2\}]$ $[\varnothing,\{3,5\}]$; $[\{1\},\{2,5\}]$; $[\{3\},\{2,3\}]$; $[\{1,3\},\{2,2\}]$ $[\varnothing,\{2,6\}]$; $[\{4\},\{2,2\}]$

The above definition might be viewed as an instruction for a “lego game”. Having an infinite stock of elementary “lego bricks” of twenty different kinds (we mean the embedded local ribbon graphs of twenty types presented at figure 6) one constructs the entire building following the plan given by the global graph from figure 3. After that one “decorates the building” with an arbitrary collection of integers matching the parities of the “lego bricks” and satisfying some elementary conditions.

Parts (1)–(2) of the definition describe the combinatorial geometry of the building; conditions (4)–(6) impose elementary restrictions on the collection of orders of singularities. Note that the *parities* of the boundary singularities are encoded in the “lego bricks”. Thus, condition (3) makes a bridge between the geometry (1)–(2) of the ribbon graph and the arithmetic (4)–(6) of the collection of integers representing the orders of singularities. See the Main Theorem for a formal statement and Appendix B for an explicit illustration of this approach.

4.1. **Singularity data corresponding to a configuration.** Any two flat surfaces realizing the same configuration $\mathcal{C}$ of $\hat{\ }$homologous saddle connections belong to the same stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. The singularity data α are defined by the configuration $\mathcal{C}$ as follows.

First note that any configuration $\mathcal{C}$ determines a natural *global ribbon graph* $\mathbb{G}$ in the following way. We have defined a structure of a local ribbon graph for a small neighborhood Γ_v of every vertex $v \in \Gamma$. For every vertex v of Γ we have a ribbon going along a germ of every edge of $\Gamma_v \subset \Gamma$ in direction from v to the center of the edge. Note that all local ribbon graphs carry the canonical orientation induced from the canonical orientation of the embodying plane. For every edge of Γ we can extend the ribbons from the endpoints towards the center of the edge and glue them together respecting the canonical orientation. Applying this procedure to all edges of Γ we get a global ribbon graph endowed with the canonical orientation.

Consider the global ribbon graph $\mathbb{G}$ as a surface with boundary. The boundary components of this surface are in a one-to-one correspondence with the subset of those conical points of S which serve as the endpoints of the saddle connections γ_i in the collection $\gamma_1, \dots, \gamma_n$. The orders of the corresponding singularities are calculated as follows. For any connected component $(\partial\mathbb{G})_m$ of its boundary define an integer b_m as

$$b_m = \sum_{\substack{\text{boundary singularities}\\ \text{which belong to } (\partial\mathbb{G})_m}} (k_{i,l}+1) \quad - 2 \tag{3}$$

The set with multiplicities α can be defined now as

$$\alpha = \left(\bigcup_{\substack{\pm\text{-vertices}\\ v_j \in \Gamma(\mathcal{C})}} \text{interior singularities of } v_j \right) \bigcup \left(\bigcup_{\substack{\text{components } (\partial\mathbb{G})_m\\ \text{of the boundary}\\ \text{of } \mathbb{G}(\mathcal{C})}} b_m \right) \tag{4}$$

Example 4. The configurations $\mathcal{C}$ presented in the left picture of figure 7 has 8 saddle connections $\gamma = \{\gamma_1 \cup \dots \cup \gamma_8\}$; the surface $S \setminus \gamma$ decomposes into 7 connected components $S_1 \sqcup \dots \sqcup S_7$. Two components are represented by cylinders and thus have no interior singularities. Among the remaining five components three have no interior singularities and are denoted with $\varnothing$, one has one interior singularity of order 2, and one has two interior singularities of orders 4. Thus, we get

$$\bigcup_{\substack{\pm\text{-vertices}\\ v_j\in\Gamma(\mathcal{C})}} \text{interior singularities of } v_j = \{2,4,4\}$$

The boundary of the global ribbon graph $\mathbb{G}$ has two components $(\partial\mathbb{G})_1$ and $(\partial\mathbb{G})_2$ which correspond to two conical singularities P_1 and P_2 of S. The saddle connections $\gamma_5, \gamma_6, \gamma_7$ join P_2 to itself; the other saddle connections join P_1 to itself. Turning counterclockwise around the point P_l, $l = 1, 2$, we see geodesic rays parallel to γ_i appear in the same order as they appear when we follow the corresponding component $\mathcal{B}_l$ in the positive direction. Denoting by "x" the geodesic rays which do not belong to the configuration we get the following (cyclically ordered) list for the zero P_2:

$$\gamma_5 x \gamma_6 x \gamma_7 \gamma_7 x \gamma_6 x \gamma_5$$

We have 10 geodesic rays; this corresponds to the cone angle 10π matching our formula for the order b_2 of the zero P_2:

$$(0+1)+(1+1)+(1+1)+(1+1)+(1+1)+(0+1)-2 = 8\,,$$

The analogous list for P_1 is as follows

$$xx\gamma_1 x\gamma_2 x\gamma_3 xxxxx\gamma_4\gamma_4 xxxxxxxxx\gamma_3\gamma_8\gamma_8 x\gamma_2 xxx\gamma_1$$

The number of consecutive "x" coincides with the order of the corresponding boundary singularity (see definition 2). Thus, at P_1 we find 32 geodesic rays parallel to γ_i, which corresponds to the cone angle 32π, and the order b_1 of P_1 equals to

$$(2{+}1){+}(1{+}1){+}(1{+}1){+}(5{+}1){+}(0{+}1){+}(9{+}1){+}(0{+}1){+}(0{+}1){+}(1{+}1){+}(3{+}1){-}2{=}30$$

Finally, we get the following set with multiplicities:

$$\alpha = (2, 4, 4, 8, 30).$$

The surface S has genus $g = 13$; the configuration $\mathcal{C}$ represents the stratum $\mathcal{Q}(2, 4, 4, 8, 30)$. Note, that the picture on the right represents the same configuration as the picture of the left.

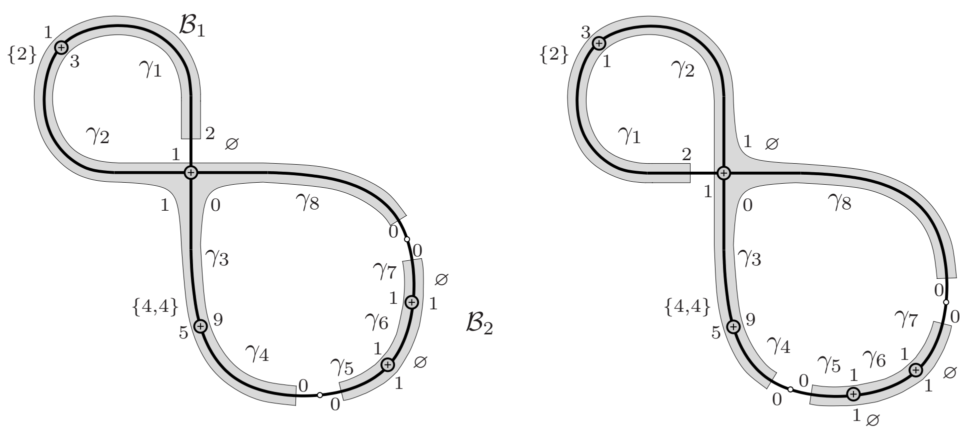


FIGURE 7. An example of a configuration

Remark 3. The example above gives an idea of how can one construct all configurations (in the sense of definition 3) for a given stratum $\mathcal{Q}(\alpha)$ of meromorphic quadratic differentials with at most simple poles. This algorithm is discussed in more details in appendix B, where as an illustration we present a complete list of all configurations of $\hat{\ }$homologous saddle connections for holomorphic quadratic differentials in genus two.

5. Principal boundary

Analogously to the case of Abelian differentials a configuration $\mathcal{C}$ of $\hat{\ }$homologous saddle connections determines the corresponding principal boundary stratum $\mathcal{Q}(\alpha'_{\mathcal{C}})$ or $\mathcal{H}(\beta'_{\mathcal{C}})$. Namely, to each boundary component $\mathcal{B}_i$

$$\{\rightarrow \gamma_{j_{i,1}} \rightarrow \cdots \rightarrow \gamma_{j_{i,p(i)}} \rightarrow\}$$

of every "+" or "−"-vertex v_j of the graph $\Gamma(\mathcal{C})$ (i.e. to each connected component of the corresponding local ribbon graph $\mathbb{G}_j$) we assign a number

$$D_{j_i} = k_{j_{i,1}} + \cdots + k_{j_{i,p(i)}} - 2, \tag{5}$$

where $k_{j_{i,1}}, \dots, k_{j_{i,p(i)}}$ are the orders of the boundary singularities corresponding to this boundary component. It is not difficult to prove that the number D_{j_i} is always a nonnegative even integer whenever v_j is a "+"-vertex. To every "+"-vertex v_j of the graph $\Gamma(\mathcal{C})$ we assign the stratum

$$\mathcal{H}(\beta'_j) = \mathcal{H}\left(\frac{d_1}{2}, \dots, \frac{d_{s(j)}}{2}, \frac{D_1}{2}, \dots, \frac{D_{r(j)}}{2}\right) \tag{6}$$

of holomorphic Abelian differentials, where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities of v_j. Note that conditions (4) and (5) in definition 3 of a configuration of $\hat{\ }$homologous saddle connections imply that the entries of β'_j are integers and that their sum is even, so the stratum $\mathcal{H}(\beta'_j)$ is nonempty.

We assign to a "−"-vertex v_j the stratum

$$\mathcal{Q}(\alpha'_j) = \mathcal{Q}(d_1, \dots, d_{s(j)}, D_1, \dots, D_{r(j)}) \tag{7}$$

of meromorphic quadratic differentials with at most simple poles, where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities of v_j. Note that condition (5) in definition 3 of a configuration of $\hat{\ }$homologous saddle connections guarantees that the sum of entries of α'_j defined above equals 0 modulo 4, while condition (6) guarantees that $\alpha' \notin \big\{(\varnothing, \{-1,1\}, \{3,1\}, \{4\}\big\}$, which implies that the stratum $\mathcal{Q}(\alpha'_j)$ is nonempty.

Given a configuration $\mathcal{C}$ we assign to every "±"-vertex of the graph Γ the corresponding stratum. When Γ does not contain "−" vertices we get a stratum $\mathcal{H}(\beta'_{\mathcal{C}})$ of disconnected translation surfaces $S'_1 \sqcup \cdots \sqcup S'_k$, where $S'_j \in \mathcal{H}(\beta'_j)$, $j = 1, \dots, k$. Otherwise we get a stratum $\mathcal{Q}(\alpha'_{\mathcal{C}})$ of disconnected flat surfaces $S'_1 \sqcup \cdots \sqcup S'_k$, where $S'_j \in \mathcal{H}(\beta'_j)$ when S'_j is represented by a "+"-vertex and $S'_j \in \mathcal{Q}(\alpha'_j)$ when S'_j is represented by a "−"-vertex. The resulting stratum is called the *principal boundary stratum* corresponding to the admissible configuration $\mathcal{C}$.

Example 5. Let us compute the principal boundary stratum corresponding to the configuration from example 4, see figure 7. The components represented by cylinders, encoded by ∘-vertices do not contribute to the principal boundary: they shrink and disappear. The vertex v_1 of valence four has type $+4.2c$, see figure 6; the corresponding local ribbon graph $\mathbb{G}_{v_1}$ has two connected components, $r(1) = 2$, which correspond to two connected components $\mathcal{B}_1, \mathcal{B}_2$ of the boundary ∂S_1^{comp}. The corresponding zeroes of the induced Abelian differential on S_1' are calculated in terms of $D_1 = 2-2 = 0$ and $D_2 = 1+0+1-2 = 0$, see (5). Since S_1^{comp} does not have interior singularities, the corresponding closed flat surface S_1' is a torus with two marked points, $S_1' \in \mathcal{H}(0,0)$, see (6).

The remaining four vertices of $\Gamma(S,\gamma)$ have type $+2.1$; the boundary of each of the corresponding components $S_2, \dots, S_4$ is connected. Applying formulae (5) and (6) we get the following list of surfaces S_j':

$$S_2' \in \mathcal{H}\left(\frac{2}{2},\ \frac{3+1-2}{2},\right) \qquad S_3' \in \mathcal{H}\left(\frac{4}{2},\ \frac{4}{2},\frac{5+9-2}{2},\right) \qquad S_4', S_5' \in \mathcal{H}\left(\frac{1+1-2}{2}\right)$$

The corresponding principal boundary stratum is

$$\mathcal{H}(0,0) \sqcup \mathcal{H}(1,1) \sqcup \mathcal{H}(6,2,2) \sqcup \mathcal{H}(0) \sqcup \mathcal{H}(0)$$

6. Main Theorems

As a key technical tool we use in [MZ] some basic surgeries which depend continuously on a small complex parameter $\delta \in \mathbb{C}$ (responsible for the length and direction of the saddle connections which form the boundary) and on an additional discrete parameter having finitely many values. The theorem below makes a bridge between the formal combinatorial constructions discussed above and the geometry of the moduli spaces of quadratic differentials.

We denote by $\mathcal{Q}_1^{\varepsilon}(\alpha) \subset \mathcal{Q}_1(\alpha)$ the subset of those flat surfaces of area one, which have at least one saddle connection of length at most ε.

Theorem 4. *For each configuration $\mathcal{C}$ of $\hat{\text{h}}$omologous saddle connections as in definition 3, let Γ be the graph of connected components corresponding to this configuration. Let $\mathcal{Q}(\alpha'_{\mathcal{C}})$ (or $\mathcal{H}(\beta'_{\mathcal{C}})$) be the boundary stratum corresponding to the configuration $\mathcal{C}$.*

For any flat surface $S' \in \mathcal{Q}(\alpha'_{\mathcal{C}})$ (correspondingly in $\mathcal{H}(\beta'_{\mathcal{C}})$), and any sufficiently small value of the complex parameter δ, if one applies the basic surgeries to the connected components of S' and assembles a closed surface S from the resulting surfaces with boundary according to the structure of the graph $\Gamma(\mathcal{C})$, then the result is a surface in $\mathcal{Q}^{\varepsilon}(\alpha)$.

Similar to the case of Abelian differentials, we denote by $\mathcal{Q}_1^{\varepsilon,thick}(\alpha) \subset \mathcal{Q}_1(\alpha)$ the subset of those flat surfaces of area one, which have a collection of $\hat{\text{h}}$omologous saddle connections of length at most ε and no other short saddle connection. Here "short" means, of length less than $\lambda\varepsilon^r$ for some parameters $\lambda \geq 1$ and $0 < r \leq 1$, where the values of the parameters depend on the stratum. Then one can show that any surface in $\mathcal{Q}_1^{\varepsilon,thick}(\alpha)$ can be obtained by this construction.

We put theorem 2, theorem 3 and theorem 4 together in one statement which may be considered as our main theorem.

We say that a collection γ of $\hat{\text{h}}$omologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ on a flat surface $S \in \mathcal{Q}(\alpha)$ is *in general position* if there are no other saddle connections on S parallel to saddle connections in the collection γ. It follows from proposition 3 stated in appendix A that for almost all flat surfaces in any stratum any collection of $\hat{\text{h}}$omologous saddle connections is in general position. This implies, that we can always put a collection of $\hat{\text{h}}$omologous saddle connections in general position by an arbitrary small deformation of the flat surface inside the stratum.

Main Theorem. *Any collection γ of $\hat{h}$omologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ in general position on a flat surface $S \in \mathcal{Q}(\alpha)$ naturally defines a corresponding configuration $\mathcal{C}(S,\gamma)$.*

Any "formal" configuration of $\hat{h}$omologous saddle connections as in definition 3 corresponds to some actual collection of $\hat{h}$omologous saddle connections on an appropriate flat surface.

Proof. By theorem 2 any collection γ of $\hat{h}$omologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ on a flat surface $S \in \mathcal{Q}(\alpha)$ naturally defines a graph of connected components $\Gamma(S,\gamma)$ (structure 1 of a configuration). According to theorem 3, for every vertex v of $\Gamma(S,\gamma)$ the collection γ also defines a local ribbon graph (structure 2 of a configuration) as well as the orders d_l and $k_{i,l}$ of all interior and boundary singularities. By theorem 3, for vertices of "+"-type, the orders $k_{i,l}$ of the boundary singularities are compatible with the corresponding parities (structures 3 and 4 of a configuration). It is not difficult to verify the lower bounds for the sums D_i of orders of boundary singularities and the necessary condition of the compatibility of the orders of interior singularities with the orders of boundary singularities formalized as structure 5. The list of nonrealizable singularity data for the vertices of the "−"-types presented in structure 6 of a configuration can be obtained from the list of empty strata, see [MS2]. This completes the proof of the first part of the statement.

The realizability of all formal configurations immediately follows from theorem 4. □

7. Final comments, open problems, applications

The thick part $\mathcal{Q}_1^{\varepsilon, thick}(\alpha)$ decomposes into a disjoint union

$$\mathcal{Q}_1^{\varepsilon, thick}(\alpha) = \bigsqcup_{\text{configurations } \mathcal{C}} \mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$$

of (not necessarily connected) components corresponding to admissible configurations; the surfaces in any such component of $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ share the same configuration $\mathcal{C}$ of $\hat{h}$omologous saddle connections. Following the lines of the paper [EMZ] one could extend theorem 4 and prove that up to a defect of a very small measure, for every configuration $\mathcal{C}$ there is an integer $M(\mathcal{C})$ such that $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ is a (ramified) covering of order $M(\mathcal{C})$ over the following space. The space is a fiber bundle over the boundary stratum $\mathcal{Q}_1(\alpha'_{\mathcal{C}})$ (correspondingly $\mathcal{H}_1(\beta'_{\mathcal{C}})$). It has a Euclidean ε-disc as a fiber when $\mathcal{C}$ does not contain cylinders, and the space $\mathcal{H}_1^{\varepsilon}(0, \dots, 0)$ when $\mathcal{C}$ contains cylinders (number of marked points on the torus equals the number of cylinders). In both cases it is easy to express the measure on $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ in terms of the product measure on the fiber bundle, and compute the volume of $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ in terms of volumes of the strata, and using the Siegel—Veech formula compute the constants $c_{\mathcal{C}}$.

However, the evaluation of the constants M (which depend on the configuration $\mathcal{C}$) requires some additional work. In particular, if the corresponding surgeries (see theorem 4) are nonlocal (i.e. those, which use a path on a surface) one needs to study the dependence of the resulting surface on the homotopy type of the path. These and related issues will be discussed in the forthcoming paper [B2].

Another subject which we do not discuss in this paper is the individual study of the connected components of the strata of quadratic differentials: different connected components of the same stratum $\mathcal{Q}(\alpha)$ have their individual lists of admissible configurations, graphs, boundary strata, etc. In particular, one can use the lists of admissible configurations to determine the connected component to which a given flat surface belongs. For example, a saddle connection joining the zero and the simple pole on any flat surface from the component $\mathcal{Q}^{ir}(9,-1)$ has a ĥomologous saddle connection joining the zero to itself, while analogous saddle connections on surfaces from the complementary connected component $\mathcal{Q}^{reg}(9,-1)$ may have multiplicity one. The existing invariant called the *Rauzy class* used to distinguish these components is rather complicated, see [L]. Configurations of ĥomologous saddle connections for nonconnected strata will be studied in the papers [B1] and [B2].

Given a billiard in a rational polygon Π, one can build a translation surface $\hat{S}$ from an appropriate number $2N$ of copies of Π such that geodesics on S will project to the billiard trajectories in Π. Taking N copies instead of $2N$ one obtains a flat surface with $\mathbb{Z}/2\mathbb{Z}$-holonomy with the same properties of geodesics. In some cases this latter construction is more advantageous. In the paper [AEZ] there is the study of billiards in polygons whose angles are multiples of $\pi/2$. Identifying two copies of such polygons by their boundaries one obtains a flat surface corresponding to a meromorphic quadratic differential on $\mathbb{C}P^1$ with at most simple poles. The results of this paper are used to classify closed billiard trajectories and generalized diagonals in the paper [AEZ], see also [B1].

Acknowledgements

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Appendix A. Long saddle connections

In this appendix we study collections of ĥomologous saddle connections when they are not necessarily short.

The next proposition follows immediately from definition 1 and the notion of configuration.

Proposition. *Let $\gamma(S_0) = \{\gamma_1, \dots, \gamma_n\}$ be a collection of ĥomologous saddle connections on a flat surface S_0 in $\mathcal{Q}(\alpha)$. Let a flat surface S be obtained by a sufficiently small continuous deformation of S_0 in $\mathcal{Q}(\alpha)$ and $\gamma(S)$ the corresponding collection of saddle connections. Then all saddle connections in the collection $\gamma(S)$ are ĥomologous. The configuration $\mathcal{C}(S, \gamma(S))$ defined by the collection $\gamma(S)$ of ĥomologous saddle connections on S coincides with the initial configuration $\mathcal{C}(S_0, \gamma(S_0))$.*

We recall the definition of the natural $GL(2;\mathbb{R})$-invariant measure in the stratum $\mathcal{Q}(\alpha)$. Let $\hat{P} = p^{-1}(P)$ be the collection of preimages of the singularities of a flat surface $S \in \mathcal{Q}(\alpha)$. Let $H_1^-(\hat{S}, \hat{P}; \mathbb{Z})$ be the subgroup in the relative homology group of $\hat{S}$, odd with respect to the involution τ. Similarly, let $H^1_-(\hat{S},\hat{P};\mathbb{C})$ be the subspace in the relative cohomology odd with respect to the involution τ (i.e. the invariant subspace corresponding to the eigenvalue -1 of the induced linear involution $\tau^*: H^1(\hat{S},\hat{P};\mathbb{C}) \to H^1(\hat{S},\hat{P};\mathbb{C})$). We can choose a basis in $H_1^-(\hat{S},\hat{P};\mathbb{Z})$ obtained as lifts $\hat{\gamma}_i$, $i=1,\dots,\dim_{\mathbb{C}}\mathcal{Q}(\alpha)$, of a collection of saddle connections on S. For any surface near S the affine holonomy vectors $\int_{\hat{\gamma}}\omega$ serve as local coordinates for $\mathcal{Q}(\alpha)$. We define a measure $d\nu(S)$ on $\mathcal{Q}(\alpha)$ as Lebesgue measure defined by these coordinates, normalized so that the volume of a fundamental domain of the integer lattice in

$$H^1_-(\hat{S},\hat{P};\mathbb{Z}\oplus i\mathbb{Z}) \subset H^1_-(\hat{S},\hat{P};\mathbb{C})$$

is equal to one.

Remark. Note that the Abelian differential ω on $\hat{S}$ has a regular point at the preimage $P_i' \in p^{-1}(P_i)$ of a simple pole P_i of the quadratic differential q on S. Consider the set $\tilde{P} \subseteq \hat{P}$ obtained by removing these regular points. It is easy to see that the canonical homomorphism $H^1_-(\hat{S},\hat{P};\mathbb{C}) \to H^1_-(\hat{S},\tilde{P};\mathbb{C})$ induced by the inclusion $\tilde{P} \subseteq \hat{P}$ is actually an isomorphism. Thus, it does not matter which of two sets $\hat{P},\tilde{P}$ is used to define the coordinate charts.

By definition, a configuration $\mathcal{C}$ of $\hat{\text{h}}$omologous saddle connections is admissible for a given connected component $\mathcal{Q}^c(\alpha)$ of the stratum $\mathcal{Q}(\alpha)$ if there is *at least one* flat surface $S_0 \in \mathcal{Q}^c(\alpha)$ and at least one collection γ of $\hat{\text{h}}$omologous saddle connections $\gamma = \{\gamma_1,\dots,\gamma_n\}$ on S_0 realizing $\mathcal{C}$. Consider any surface S in the same connected component $\mathcal{Q}^c(\alpha)$. By $N_{\mathcal{C}}(S,L)$ denote the number of collections $\gamma = \{\gamma_1,\dots,\gamma_n\}$ of $\hat{\text{h}}$omologous saddle connections on S defining the same configuration $\mathcal{C}(S,\gamma) = \mathcal{C}$ and such that $\max_{1\le i\le n}|\gamma_i| \le L$. The results in [EM] imply the following statement.

Proposition 2. *For almost every flat surface S in the connected component $\mathcal{Q}^c(\alpha)$ containing S_0 the following limit exists*

$$\lim_{L\to+\infty}\frac{N_{\mathcal{C}}(S,L)}{L^2} = c_{\mathcal{C}}(S)$$

and is strictly positive. Moreover, for almost all surfaces S in $\mathcal{Q}^c(\alpha)$ this limit is the same, $c_{\mathcal{C}}(S) = const_{\mathcal{C}}$. (This limit is called Siegel–Veech constant.)

In particular, any admissible configuration is presented on almost every flat surface in the corresponding connected component of the stratum by numerous collections of $\hat{\text{h}}$omologous saddle connections.

Proof. Let $\mathcal{C}$ be an admissible configuration of $\hat{\text{h}}$omologous saddle connections. Let $\gamma = \{\gamma_1,\dots,\gamma_n\}$ be a collection of $\hat{\text{h}}$omologous saddle connections on the flat surface S_0 representing configuration $\mathcal{C}$. Choose some saddle connection γ_i corresponding to an edge of weight 1 of the graph $\Gamma(S,\gamma)$; such edge always exists, see figure 3. We associate to the collection γ a pair of vectors $\pm\vec{v}(\gamma) \in \mathbb{R}^2$ setting $v = \int_{\gamma_i}\omega \in \mathbb{C} \cong \mathbb{R}^2$. For every surface S in the same connected component we consider the discrete subset $V_{\mathcal{C}}(S)$ by taking the union $V_{\mathcal{C}}(S) = \cup \pm v(\gamma)$ over all collections of $\hat{\text{h}}$omologous saddle connections γ realizing $\mathcal{C}$.

It is easy to see that the set $V_{\mathcal{C}}(S)$ satisfies axioms (A), (B), (C_μ) in [EM]. Proposition 2 now follows from the general results in [EM] and from theorem 4 which implies that the Siegel–Veech constant $const_{\mathcal{C}}$ is nonzero. $\square$

Proposition 3. *For almost every flat surface in any stratum, two saddle connections are parallel if and only if they are ĥomologous.*

Proof. If saddle connections γ_1 and γ_2 are parallel, then $\int_{\hat\gamma_1}\omega = r\int_{\hat\gamma_2}\omega$ for r real. If γ_1 and γ_2 are not ĥomologous then the homology classes of the lifts $\hat\gamma_1$ and $\hat\gamma_2$ are independent in $H_1^-(\hat S, \hat P; \mathbb{Z})$. Then the above equation holds only for a set of measure zero in $H^1_-(\hat S, \hat P; \mathbb{C})$. Taking a countable union of sets of measure zero corresponding to possible pairs of cycles and different coordinate charts, we see that two nonĥomologous saddle connections on S are parallel only for a set of S of measure zero. □

Proof of proposition 1. Suppose that there are two saddle connections γ_1, γ_2 in the collection which are not ĥomologous. Then the corresponding periods $\int_{\hat\gamma_1}\omega$ and $\int_{\hat\gamma_1}\omega$ correspond to two independent coordinates in a small neighborhood of the initial flat surface, and hence they can be deformed independently. Since the length $|\gamma|$ equals $|\int_{\hat\gamma}\omega|$ or $1/2|\int_{\hat\gamma}\omega|$ (depending on whether γ is homologous to zero or not), we conclude that a collection containing two nonĥomologous saddle connections cannot be rigid.

The necessity of the condition in proposition 1 is proved. Sufficiency immediately follows from the fact that the lengths of ĥomologous saddle connections are either the same or differ by a factor of two. □

Appendix B. List of configurations in genus 2

Using definition 3, theorem 4 and following constructions suggested in examples 4 and 5 one can construct a complete list of configurations for any given stratum $\mathcal{Q}(\alpha)$. In this section we present an outline of the algorithm and list all configurations for holomorphic quadratic differentials in genus 2.

There are two natural parameters measuring "complexity" of singularity data $\alpha = \{d_1, \dots, d_m\}$: the genus g of a flat surface S in $\mathcal{Q}(\alpha)$ and the number N of simple poles on S (i.e. the number of conical points with the cone angle π). Having a configuration $\mathcal{C}$ denote by N' the number of interior singularities of order -1 corresponding to this configuration and by $g'_1, \dots, g'_k$ the genera of surfaces $S'_1, \dots, S'_k$ corresponding to the principal boundary $\mathcal{Q}(\alpha'_{\mathcal{C}})$ (correspondingly $\mathcal{H}(\beta'_{\mathcal{C}})$ when $\mathcal{C}$ does not have "$-$"-vertices). It is easy to see that the number of simple poles on S (i.e. the number of entries "-1" of α) might vary from N' to $N'+4$, and that the genus g might vary from $\sum_{j=1}^k g'_j$ to $\sum_{j=1}^k g'_j + 2$ (see [B1] for an explicit expression of $g(S)$ in terms of genera g'_j of components and of a structure of the global ribbon graph). Thus, having fixed the upper bounds for g and N, we confine the list of corresponding configurations to a finite one.

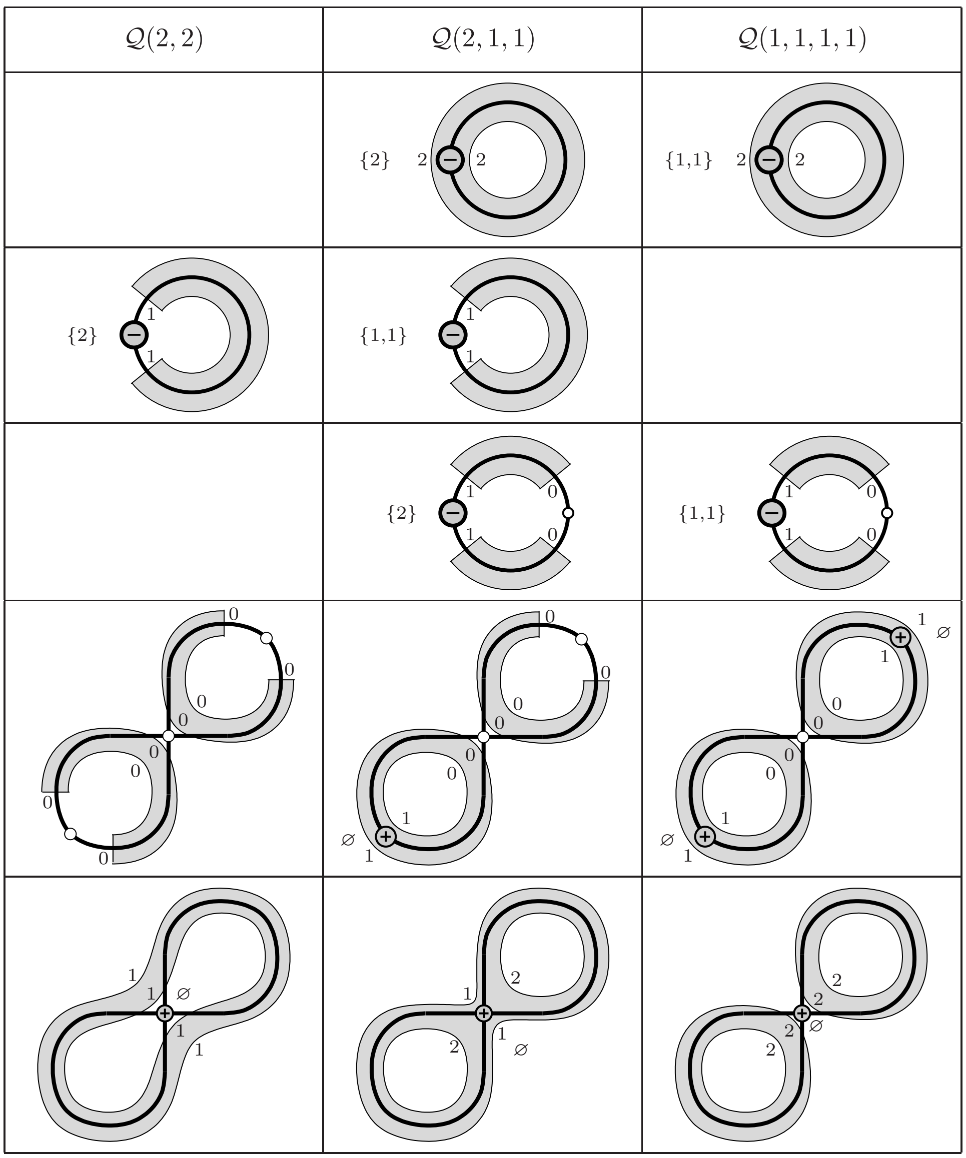
A naive algorithm of enumeration of all configurations for a given stratum $\mathcal{Q}(\alpha)$ can be represented as follows. Let $g = g(\alpha)$ be the genus corresponding to the singularity data α,

$$d_1 + \dots + d_m = 4g(\alpha) - 4$$

Consider complete lists of (possibly disconnected) strata $\mathcal{H}(\beta')$ of genera $g-2$, $g-1$, g. These lists are finite and can be easily constructed. Consider complete lists of (possibly disconnected) strata $\mathcal{Q}(\alpha')$ of genera $g-2, g-1, g$ such that α' contains from $N-4$ to N entries "-1" and at most two connected components α'_i, α'_j representing strata of quadratic differentials $\mathcal{Q}(\alpha'_i)$, $\mathcal{Q}(\alpha'_j)$ (the remaining connected components are represented by strata of holomorphic differentials $\mathcal{H}(\alpha'_l)$). These lists are also finite and can be easily constructed. Add the empty set to these lists when $0 \le g \le 2$.

For every entry $\alpha' = \alpha'_1 \sqcup \cdots \sqcup \alpha'_k$ (correspondingly β') as above consider all possible ways to organize the set $\{\alpha'_1, \ldots, \alpha'_k\}$ into one of the graphs as in figure 3, in such way that vertices corresponding to the strata $\mathcal{H}(\alpha'_j)$, $\mathcal{H}(\beta'_j)$ have "+"-type, and vertices corresponding to the strata $\mathcal{Q}(\alpha'_j)$ have "−"-type. Using these basic graphs, construct all possible "extended" graphs adding vertices of the "∘"-type as described in theorem 2.

For every vertex of every graph as above consider all possible structures of an embedded local ribbon graph as in figure 6.



At the current stage we have already chosen $\alpha' = \{\alpha'_1, \dots, \alpha'_k\}$ (correspondingly β'), the graph Γ, the bijection of $\{\alpha'_1, \dots, \alpha'_k\}$ (correspondingly $\{\beta'_1, \dots, \beta'_k\}$) with the set of vertices of Γ compatible with the structure of "+" and "−"-vertices, and the structure of a local ribbon graph for every vertex of Γ. Now for every local ribbon graph $\mathbb{G}_j$ representing a "+" or "−"-vertex S_j consider all possible ways to arrange orders of interior singularities and of boundary singularities of S_j in a way compatible with conditions (3)–(6) of definition 3 and with equation (6) for the corresponding singularity data β'_j (correspondingly equation (7) for the singularity data α'_j). By "compatibility" with equations (6)–(7) we mean that singularity data computed by these equations should produce β'_j (correspondingly α'_j) possibly completed with several (from 1 to r_j) entries "0" (where r_j is the number of connected components of the local ribbon graph $\mathbb{G}_j$).

From the resulting lists of configurations extract those which correspond to the required singularity data α.

As an example we present a complete list of configurations of $\hat{\text{h}}$omologous saddle connections for holomorphic quadratic differentials in genus 2. We are grateful to Alex Eskin, who helped us to test completeness of this list.

References

[AEZ] J. Athreya, A. Eskin, A. Zorich, *Rectangular billiards and volumes of spaces of quadratic differentials*, in progress.

[B1] C. Boissy, *Degenerations of meromorphic quadratic differentials on $\mathbb{C}P^1$ and on hyperelliptic Riemann surfaces*, to appear.

[B2] C. Boissy, *Configurations of $\hat{\text{h}}$omologous saddle connections for quadratic differentials in exceptional strata*, in progress.

[EM] A. Eskin, H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory and Dynamical Systems, **21** (2) (2001), 443–478.

[EMZ] A. Eskin, H. Masur, A. Zorich, *Moduli spaces of Abelian differentials: the principal boundary, counting problems and the Siegel–Veech constants*, Publications Mathématiques de l'IHÉS, **97** (1) (2003), 61–179.

[HM] J. Hubbard, H. Masur, *Quadratic differentials and foliations*, Acta Math., **142** (1979), 221–274.

[K] M.Kontsevich, Lyapunov exponents and Hodge theory. "The mathematical beauty of physics" (Saclay, 1996), (in Honor of C. Itzykson) 318–332, Adv. Ser. Math. Phys., 24, World Sci. Publishing, River Edge, NJ, 1997.

[L] E. Lanneau, *Connected components of the moduli spaces of quadratic differentials*, Ph.D Thesis.

[M] H. Masur, *Interval exchange transformations and measured foliations*. Ann. of Math., **115** (1982), 169-200.

[MS1] H. Masur, J. Smillie, *Hausdorff dimension of sets of nonergodic foliations*, Ann. of Math., **134** (1991), 455-543.

[MS2] H. Masur, J. Smillie, *Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms*, Comment. Math. Helvetici, **68** (1993), 289–307.

[MZ] H. Masur, A. Zorich, *Multiple Saddle Connections on Flat Surfaces and Principal Boundary of the Moduli Spaces of Quadratic Differentials*, `arXiv:math.GT/0402197`, 52 pp.

[V] W. A. Veech, *Moduli spaces of quadratic differentials*, Journal d'Analyse Math. **55** (1990), 117–171.

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