Arakelov Geometry, Variational principles and Equidistribution of small points

Clay Mathematical Institute

Summer school on Arithmetic Geometry

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Antoine Chambert-Loir

antoine.chambert-loir@univ-rennes1.fr

IRMAR, Université de Rennes 1

1. Function fields *vs* Number fields

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	intersection theory on $\mathscr{X}$	Arakelov-Gillet-Soulé's arithmetic intersection theory

2. Metrized line bundles and heights of subvarieties

Green functions

Let X be a projective variety over a number field F.
D a very ample divisor, $D = H_0 \cap X$, $\iota\colon X \subset \mathbf{P}^n$.
Here $|\cdot|_v$ is any absolute value on F, $\mathbf{C}_v$ = completion of $\overline{F_v}$.

Definition. — The ***elementary Green function*** for D associated to the given embedding is the function on $X \setminus D(\mathbf{C}_v)$ given by

$$G_D(x) = -\log |x_0|_v / \max(|x_0|_v, \ldots, |x_n|_v), \quad \iota(x) = (x_0 : \cdots : x_n).$$

A (semipositive) Green function for D is a uniform limit of elementary Green functions.

Archimedean Green functions

It is better for analysis to replace $\max(|x_i|)$ by the hermitian norm $(\sum |x_i|^2)^{1/2}$.
Then, an elementary Green functions has a ***curvature form***, given on $X \setminus D(\mathbf{C})$ by

$$\omega = \mathrm{dd}^c G_D = \frac{\mathrm{i}}{\pi} \sum_{j,k} \frac{\partial^2 G_D}{\partial z_j \partial \overline{z_k}} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_k}.$$

It is the restriction to $X(\mathbf{C})$ of the Kähler form of $\mathbf{P}^n$.

Archimedean Green functions

When elementary Green functions converge to a semipositive Green function G_D, the associated forms converge to a closed positive current of type $(1,1)$ on $X(\mathbf{C})$ called the ***curvature current of*** G_D.

Conversely, one can prove that the semipositive Green functions are exactly the functions $g \leqslant 0$ such that (as distributions) $dd^c g + \delta_D$ is a positive current.

Archimedean Green functions

Similarly, the products ω_D^p converge to a positive current of type (p,p); for $p = \dim X$, one gets in particular a ***positive measure*** μ_D.

In the case of elementary Green functions, it is well known that $\int_X \omega_D^{\dim X}$ is the degree of X. Consequently, the mass of the measure μ_D is the degree of X (in the embedding corresponding to D).

Example: on $\mathbf{P}^1$, for $D = \infty = (1:0)$, $G_D((x:y)) = -\log\left|y\right| / \max(|x|, \left|y\right|)$ is an semipositive Green function with curvature

$$\omega_D(r\mathrm{e}^{\mathrm{i}\theta} : 1) = \delta_{r=1} \wedge \frac{1}{2\pi} \mathrm{d}\theta.$$

It is a probability measure.

Adelic Green functions

X, D very ample.
An adelic Green function for D is the data, for any place v of F, of a Green function $G_{D,v}$ at the place v with a compatibility assumption: there is an embedding of X inducing the (elementary) Green function $G_{D,v}$ for almost place v.

General case

Any divisor D is the difference $E-F$ of two very ample divisors. A (adelic) Green function for D is the difference of (adelic) Green functions for E and F.

These give rise to a representative of the height relative to $\mathscr{O}(D)$, as well as a decomposition of the height in local terms:

$$h_D(x) = \sum_v G_{D,v}(x), \qquad x \in X(F), \quad x \notin D.$$

Metrics on line bundles and Green functions

A metric on a line bundle is a family of norms on the stalks which varies continuously. For any local section s, one thus has a local continuous function $\|s\|$.
If $s' = fs$ is another local function, with $f \in \mathscr{O}_X$, one has $\left\|s'\right\|(x) = \left|f(x)\right| \|s\|(x)$.
Integrable metrics : such that $-\log\|s\|$ is a Green function for the divisor $\operatorname{div}(s)$.

Example: Fubiny-Study metric on $\mathscr{O}(1)$ on $\mathbf{P}^n$. A global section s_P of $\mathscr{O}(d)$ corresponds to a homogenous polynomial P of degree d. Then

$$\|s_P\|(x_0 : \cdots : x_n) = \frac{|P(x_0, \ldots, x_n)|}{\left(|x_0|^2 + \cdots + |x_n|^2\right)^{d/2}}.$$

Metrics on line bundles and Green functions

Notations: $\overline{\mathscr{L}} = (\mathscr{L}, (\|\cdot\|_v))$.

The $(1,1)$ form/current defined by the Green function $-\log\|s\|$ is denoted $c_1(\overline{\mathscr{L}})$.

The isomorphism classes of line bundles on X endowed with an integrable metric form an abelian group $\widehat{\mathrm{Pic}}(X)$ (for tensor product and t.p. of metrics), the forget-the-metric map is a surjective homomorphism to $\mathrm{Pic}(X)$.

For any morphism $f\colon X \to Y$, there is a homomorphism $f^*\colon \widehat{\mathrm{Pic}}(Y) \to \widehat{\mathrm{Pic}}(X)$ given by the pull-back of line bundles and pull-back of metrics.

Degrees and heights: Degrees

Assume L is a very ample line bundle on a projective variety X; let $Z \subset X$ be a subvariety of dimension p. The degree of Z with respect to L is given by the formula from intersection theory:

$$\deg_L(Z) = (c_1(L)^p | Z) \qquad (c_1(L) = \text{first Chern class}).$$

Inductive definition:

$$\deg_L(Z) = \deg_L(\operatorname{div}(s|_Z)), \qquad \text{where } s \in \Gamma(L),\ s|_Z \neq 0.$$

Degrees and heights: Heights

Assume that $\overline{\mathscr{L}}$ is a very ample line bundle with metric given by an elementary Green function G_D. If Z is a subvariety of X not contained in D, then the height of Z can be defined by a formula in arithmetic intersection theory:

$$h_{\overline{\mathscr{L}}}(Z) = (\widehat{c}_1(\overline{\mathscr{L}})^{p+1}|Z) = (\widehat{c}_1(\overline{\mathscr{L}})^p|Z \cap D)) + \int_{\mathbf{P}^n(\mathbf{C})} G_D \omega_D^p \delta_Z$$

where ω_D is the curvature form of the Green function G_D. The first term is given by “classical” intersection theory in $\mathbf{P}^n_{\mathbf{Z}}$; the second is the analytic input which takes care of the archimedean places.

Heights

The general machinery of heights is defined from this "elementary" case, using multilinearity and limit processes (which make crucial use of the positivity properties of elementary Green functions).

For points, if s is a section of $\mathscr{L}$ such that $s(P) \neq 0$,

$$h(P) = \frac{1}{[F(P):F]} h([P]) = \frac{1}{[F(P):F]} \sum_w \log \| s(P) \|_w^{-1},$$

where w varies over (normalized) absolute values of $F(P)$. One thus recovers classical height functions.

Projection formula: for $f\colon Y \to X$ and $Z \subset Y$, one has

$$h_{f^*\overline{\mathscr{L}}}(Z) = h_{\overline{\mathscr{L}}}(f_* Z).$$

(Recall: $f_* Z = \deg(f|_Z) f(Z)$ if $f\colon Z \to f(Z)$ is finite.)

Dynamical systems: Canonical metrics

Let $(X, f, \mathscr{L})$ be a polarized dynamical system over a number field F. Fix an isomorphism $\varepsilon\colon f^*\mathscr{L} \simeq \mathscr{L}^d$, with $d \geqslant 2$.
By mimicking the limit process defining the canonical height $\hat{h}_f$, one can show that there is a ***unique*** semipositive metric on $\mathscr{L}$ such that ε becomes an ***isometry***.
From the point of view of Green functions, fix a non-zeo global section s of $\mathscr{L}$, let $D = \operatorname{div}(s)$. Then ε defines a rational function α such that $f^*D = dD + \operatorname{div}(\alpha)$. There is a unique Green function $\hat{G}_D$ for D such that

$$\hat{G}_D(f(x)) = d\hat{G}_D(x) - \log|\alpha(x)|$$

for any point x such that $x \not\in D$ and $f(x) \not\in D$.

Then the measure $c_1(\overline{\mathscr{L}})^{\dim X}$ (defined by the limit process) is a multiple of the canonical measure μ_f.

Dynamical systems: height of preperiodic subvarieties

We now have a ***canonical height*** for any subvariety defined by $\hat{h}_{\mathscr{L}} = h_{\overline{\mathscr{L}}}$, if $\overline{\mathscr{L}}$ is a canonical metric on $\mathscr{L}$.

If $V \subset X$ is a preperiodic subvariety, then $\hat{h}_{\mathscr{L}}(V) = 0$.

Indeed, if $V = f(V)$,

$$h_{\overline{\mathscr{L}}}(f_* V) = h_{f^*\overline{\mathscr{L}}}(V) = h_{f^*\overline{\mathscr{L}}^d}(V) = d^{\dim V+1} h_{f^*\overline{\mathscr{L}}^d}(V)$$

Moreover,

$$\deg_{\mathscr{L}}(f_* V) = \deg_{f^*\mathscr{L}}(V) = \deg_{\mathscr{L}^d}(V) = d^{\dim V} \deg_{\mathscr{L}}(V),$$

so $f_* V = d^{\dim V} V$. Consequently, $h_{\overline{\mathscr{L}}}(V) = d h_{\overline{\mathscr{L}}}(V)$, whence the result since $d \neq 1$.

Dynamical systems: height of preperiodic subvarieties

If $V \subset X$ is a preperiodic subvariety, then $\hat{h}_{\mathscr{L}}(V) = 0$.

In view of what is known for points, it is natural to expect that the converse holds.

Theorem (Zhang). — The height $\hat{h}_{\mathscr{L}}$ is nonnegative. Moreover, these statements are equivalent:

- Bogomolov's conjecture holds for (X, f)
- Any subvariety $V \subset X$ such that $\hat{h}_{\mathscr{L}}(V) = 0$ is preperiodic.

This theorem follows from the fact, due to Zhang, that the heights of points and of subvarieties are strongly related.

3. Arithmetic positivity

Global sections: geometric analogue

Consider a fibration $\mathscr{X} \to B$ over a projective curve, a line bundle $\mathscr{L}$ on $\mathscr{X}$.
Positivity of degrees of sections $\sigma_P \colon B \to \mathscr{X}$ is related to the existence of global sections of $\mathscr{L}$. Indeed: if $\mathscr{L}$ has a global section $s \neq 0$, then $\deg_{\mathscr{L}}(\sigma_P) = \sigma_P^* c_1(\mathscr{L}) = (\operatorname{div} s . \sigma_P) \geqslant 0$ for any point P such that $P \not\in \operatorname{div}(s)$.
If the restriction of $\mathscr{L}$ to the generic fiber $X/\mathbf{C}(B)$ is ample, it is likely to have sections s. Their divisors, extended to $\mathscr{X}$, will have a horizontal part H and vertical components V in fibres of $\mathscr{X} \to B$.
If $\operatorname{div}(s) = H + d[\mathscr{X}_t] + \dots$, then $\operatorname{ord}_t s(P) \geqslant d$ for any point P such that $s(P) \neq 0$, hence $\|s\|_t \leqslant e^{-d}$.
Conclusion. The inequality $\|s\|_t \leqslant 1$ (for all $t \in B$) means precisely that s extends to a global section of $\mathscr{L}$, at least if $\mathscr{X}$ is normal.

Global sections: arithmetic case

X variety over a number field F,
$\overline{\mathscr{L}}$ a line bundle on X with an adelic metric.

Definitions. —
$H^0(X, \overline{\mathscr{L}}) = \{s \in \Gamma(X, \mathscr{L}),\ \|s\|_v \leqslant 1 \text{ for all places } v\}$.
$h^0(X, \overline{\mathscr{L}}) = \log \# H^0(X, \overline{\mathscr{L}})$.
$\chi(X, \overline{\mathscr{L}}) = \log(\operatorname{vol}(B)/\operatorname{covol}(\Lambda))$, where $B \subset \Gamma(X, \mathscr{L})_{\mathbf{R}}$ is the unit ball (for the supremum norm at archimedean places) and Λ is the lattice of sections having sup norm $\leqslant 1$ at finite places.

The quantity $\chi(X, \overline{\mathscr{L}})$ is an analogue of $\deg \pi_* \mathscr{L}$ and is related to $h^0(X, \overline{\mathscr{L}})$ by ***Minkowski's inequality:***

$$h^0(X, \overline{\mathscr{L}}) \geqslant \chi(X, \overline{\mathscr{L}}) - \operatorname{rank} \Gamma(X, \mathscr{L}) \log 2,$$

itself an analogue of Riemann inequality.

Hilbert-Samuel formulas

In algebraic geometry, the Hilbert-Samuel formula relates $\Gamma(X,\mathscr{L}^k)$ to intersection theory:

If $\mathscr{L}$ is relatively ample wrt $\pi\colon \mathscr{X} \to B$, $n = \dim \mathscr{X}$, then

$$\deg_B \pi_*\mathscr{L}^k \sim \frac{1}{n!}(c_1(\mathscr{L})^n|\mathscr{X})k^n.$$

The arithmetic analogue is due to Gillet-Soulé, with refinements by Zhang:

If $\mathscr{L}$ is ample on X with a semipositive adelic metric, and $n = \dim X$, then

$$\chi(X,\overline{\mathscr{L}}^k) \sim \frac{1}{(n+1)!}(\widehat{c}_1(\overline{\mathscr{L}})^{n+1}|X)k^{n+1}.$$

Positivity of heights

The combination of the arithmetic analogue of Hilbert-Samuel and Minkowski has the following consequence: ***if*** $\overline{\mathscr{L}}$ ***is semipositive and*** $h_{\overline{\mathscr{L}}}(X) > 0$ ***then some power of*** $\overline{\mathscr{L}}$ ***has a global section*** s ***of sup norm*** $\leqslant 1$ ***at all places.*** Then, the height of any point P outside of the divisor of this section, given by the formula

$$h_{\overline{\mathscr{L}}}(P) = -\sum_v \log \| s(P) \|_v \,,$$

is nonnegative.

Remark. Zhang proved a converse (harder) inequality: if the height of points is positive, so is the height of the variety.

Zhang's fundamental inequality

Rescaling the metrics, one obtains the following fundamental inequality:

Assume that $\mathscr{L}$ is ample, with a semipositive metric. Let (P_k) be a ***generic*** sequence of points in $X(\overline{F})$. Then,

$$\liminf h_{\overline{\mathscr{L}}}(P_k) \geqslant \frac{(\widehat{c}_1(\overline{\mathscr{L}})^{n+1}|X)}{(n+1)(c_1(\mathscr{L})^n|X)} = \frac{h_{\overline{\mathscr{L}}}(X)}{(\dim X+1)\deg_{\mathscr{L}}(X)}.$$

"Generic" means that for any subvariety $V \subsetneq X$, $P_k \notin V$ for k big enough.

We will see that there is equidistribution for generic sequences achieving the equality.

4. The variational principle and equidistribution

Equality implies equidistribution !

Equidistribution was first observed in the case of Abelian varieties by Szpiro, Ullmo and Zhang.
The idea for the proof is to apply Zhang's inequality to a perturbation of $\overline{\mathscr{L}}$. Fix an embedding of F in $\mathbf{C}$, some function smooth function φ on $X(\mathbf{C})$ and denote by $\overline{\mathscr{L}}(\varepsilon f)$ the metrized line bundle obtained from $\overline{\mathscr{L}}$ by multiplying the metric by $\exp(-\varepsilon\varphi)$ at the given place.

Formulas. For $P \in X(\overline{F})$,

$$h_{\overline{\mathscr{L}}(\varepsilon\varphi)}(P) = h_{\overline{\mathscr{L}}}(P) + \varepsilon \int_{X(\mathbf{C})} \varphi\delta(P)$$

$$h_{\overline{\mathscr{L}}(\varepsilon\varphi)}(X) = h_{\overline{\mathscr{L}}}(X) + (n+1)\varepsilon \int_{X(\mathbf{C})} \varphi c_1(\overline{\mathscr{L}})^{\dim X} + \mathrm{O}(\varepsilon^2).$$

Variational principle

Let (P_k) be a generic sequence of points. Assume equality in Zhang's inequality and let ν a limit value of the sequence $\delta(P_k)$.

Assume also that $\overline{\mathscr{L}}(\varepsilon\varphi)$ ***is semipositive for small*** $\boldsymbol{\varepsilon}$**.**

Then

$$\begin{aligned}\liminf h_{\overline{\mathscr{L}}(\varepsilon\varphi)}(P_k) &= \liminf h_{\overline{\mathscr{L}}}(P_k) + \varepsilon \int_{X(\mathbf{C})} \varphi\nu \\ &= \frac{h_{\overline{\mathscr{L}}}(X)}{(n+1)\deg_{\mathscr{L}}(X)} + \varepsilon \int_{X(\mathbf{C})} \varphi\nu \geqslant \frac{h_{\overline{\mathscr{L}}(\varepsilon\varphi)}(X)}{(n+1)\deg_{\mathscr{L}}(X)} \\ &\geqslant \frac{h_{\overline{\mathscr{L}}}(X)}{(n+1)\deg_{\mathscr{L}}(X)} + \varepsilon \frac{1}{\deg_{\mathscr{L}}(X)} \int_{X(\mathbf{C})} \varphi c_1(\overline{\mathscr{L}})^n + \mathrm{O}(\varepsilon^2).\end{aligned}$$

Variational principle

Hence, denoting $\mu_{\overline{\mathscr{L}}} = c_1(\overline{\mathscr{L}})^n / \deg_{\mathscr{L}}(X)$, one has

$$\varepsilon \int_{X(\mathbf{C})} \varphi\nu \geqslant \varepsilon \int_{X(\mathbf{C})} \varphi\mu_{\overline{\mathscr{L}}} + \mathrm{O}(\varepsilon^2).$$

When $\varepsilon \to 0$ by positive or negative values, we obtain the desired equality:

$$\int_{X(\mathbf{C})} \varphi\nu = \int_{X(\mathbf{C})} \varphi \frac{c_1(\overline{\mathscr{L}})^n}{\deg_{\mathscr{L}}(X)} = \int_{X(\mathbf{C})} \varphi\mu_{\overline{\mathscr{L}}}.$$

This shows that the sequence $\delta(P_k)$ has only one possible limit value, $\mu_{\overline{\mathscr{L}}}$. By compactness of the space of probability measures on $X(\mathbf{C})$, $\delta(P_k)$ converges to $\mu_{\overline{\mathscr{L}}}$.

Usability of the variational principle

We assumed that for any smooth function φ, $\overline{\mathscr{L}}(\varepsilon\varphi)$ ***is semipositive for small*** ε**.**

However, this assumption is quite strong and never satisfied in practice, except for Abelian varieties where this method was discovere, and lead to a proof of Bogomolov's conjecture!

For dynamical systems, $\mu_{\overline{\mathscr{L}}}$ is the canonical measure μ_f and one can show (Berteloot, Dupont, Loeb) that it is characteristic of the examples of Lattès, *i.e.* those dominated by an Abelian variety.

Curves

Autissier proved that the assumption is not necessary in the case of curves. In that case, this implies an equidistribution theorem for sequences achieving equality in Zhang's inequality.

Theorem (Autissier). — Let $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ be an endomorphism of degree $d \geqslant 2$. For any sequence (P_k) of distinct points in $\mathbf{P}^1(\overline{\mathbf{Q}})$ such that $\hat{h}_f(P_k) \to 0$, the measures $\delta(P_k)$ converge to the canonical measure μ_f on $\mathbf{P}^1(\mathbf{C})$.

Arithmetic big-ness

In Algebraic geometry, a theorem of Siu asserts that powers of $L \otimes M^{-1}$ have many global sections when L and M are ample on X and $c_1(L)^n > n c_1(M) c_1(L)^{n-1}$ (where $n = \dim X$).

Theorem (Yuan). — For $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ "arithmetically ample",

$$\begin{aligned} &\chi(X, (\overline{\mathscr{L}} \otimes \overline{\mathscr{M}}^{-1})^k) \\ &\quad \geqslant \frac{1}{(n+1)!}\Big((\widehat{c}_1(\overline{\mathscr{L}})^{n+1}|X) - (n+1)(\widehat{c}_1(\overline{\mathscr{L}})^n \widehat{c}_1(\overline{\mathscr{M}})|X)\Big) k^{n+1} \\ &\qquad + \mathrm{o}(k^{n+1}). \end{aligned}$$

This allows to apply the variational method in all cases!

Equidistribution for dynamical systems

We finally have described the proof of the following theorem:

Let $(X, \mathscr{L}, f)$ be a polarized dynamical system over a number field $F \subset \mathbf{C}$. Let $\hat{h}_f$ and μ_f denote the ***canonical height function*** on $X(\overline{F})$ and the ***canonical probability measure*** on $X(\mathbf{C})$.
Let (P_k) be a sequence of points of $X(\overline{F})$ satisfying the following assumptions:

1. $\hat{h}_f(P_k) \to 0$;
2. for any subvariety $V \subsetneq X$, $P_k \not\in V$ for k big enough.

Then the discrete measures $\delta(P_k)$ converge to the measure μ_f.

5. p-adic equidistribution

Question

p prime number
$\mathbf{C}_p$ completion of the algebraic closure of $\mathbf{Q}_p$, the field of p-adic numbers

If X is a variety over a number field F and $x \in X(\overline{\mathbf{Q}})$, one defines a measure μ_x on $X(\mathbf{C}_p)$ as before: the average of the Dirac masses at conjugates of x.
Pick a height function on X, e.g. associated to a polarized dynamical system $(X, \overline{\mathscr{L}}, f)$.

Question. — What is the p-adic limit distribution of points of "small" height?

An adequate answer requires to use p-adic analytic spaces ***in the sense of Berkovich.***

Berkovich spaces

Berkovich introduced new analytic spaces on ultrametric fields.
They contain the "classical" points, but many other, corresponding to ***multiplicative semi-norms*** on affinoid algebras.
They are complicated topological spaces, but well behaved.
For example, the analytic space associated to a smooth connected algebraic variety is locally compact, path-connected, locally contractible,…

The unit disk over $\mathbf{C}_p$

Analytic functions: $\mathscr{A} = \mathbf{C}_p\{z\}$, formal series whose coefficients in $\mathbf{C}_p$ tend to 0.
Gauss norm: $\|f\| = \max|a_n|$ if $f = \sum a_n z^n$.

The unit disk: $\mathscr{M}(\mathscr{A})$ = set of all multiplicative "bounded" semi-norms (which extend the valuation of $\mathbf{C}_p$).

A "classical" point z of the disk corresponds to the semi-norm $f \mapsto \left|f(z)\right|$.
The ***Gauss point*** corresponds to the Gauss norm $\|\cdot\|$.
It is indeed multiplicative! (Gauss's Lemma)

Bounded means $v(f) \leqslant C\|f\|$ for some constant $C > 0$.
(Here, $C = 1$ necessarily.)

The unit disk: topology

The topology of the disk is the coarsest topology such that all maps $f \mapsto v(f)$, for $v \in \mathscr{M}(\mathscr{A})$, are continuous.
It makes of the disk a topological space which is ***compact*** and metrizable. This is a compactification of the unit ball in $\mathbf{C}_p$.

It is moreover ***connected*** and ***contractible.*** (In fact, it looks like a “real tree”: any two points are linked by a unique injective path.)
Example: $t \mapsto v_t$, where

$$v_t(f) = \max_n |a_n| t^n = \sup_{|z| \leqslant t} \left| f(t) \right|, \qquad 0 \leqslant t \leqslant 1$$

is a path which links $v_0 = 0$ to the Gauss point v_1.

A p-adic analogue of Bilu's theorem

Bilu's theorem is concerned with the complex distribution of points of small Weil height. Then the limit measure is the integration measure on the unit circle.
In the p-adic case:

Theorem. — Let (x_n) be a sequence of distinct points in $\mathbf{P}^1(\overline{\mathbf{Q}})$ such that $h_{\mathrm{Weil}}(x_n) \to 0$. The sequence (μ_{x_n}) of measures on $(\mathbf{P}^1)^{\mathrm{an}}$ converges to the Dirac measure at the (non-classical) point which corresponds to the Gauss norm on polynomials.

("As usual," the projective line is obtained by glueing two disks.)

Measures (elementary metrics)

Let X be a variety, dimension n, over a number field F, $\mathscr{L}$ a very ample line bundle on X together with an elementary metric, associated to an embedding $\iota\colon X \hookrightarrow \mathbf{P}^N_F$. Let $\mathscr{X}$ denote the Zariski closure of X in the projective space $\mathbf{P}^N_{\mathfrak{o}_F}$.
Assume for simplicity that $\mathscr{X}$ is normal.
X = special fibre of $\mathscr{X}$ at a finite place v.
X_j = irreducible components of X; m_j = multiplicity.

Fact. — X_j is the reduction of a canonical point ξ_j of the Berkovich space X^{an} over F_v.

Compute degrees: $D_j = \deg(\mathsf{X}_j)$

Measures (elementary metrics)

Definition of a measure on X_v^{an}. —

$$c_1(\overline{\mathscr{L}})_v^d = \sum_j m_j D_j \delta_{\xi_j}$$

where $D_j = \deg(\mathsf{X}_j)$, and ξ_j is the Berkovich point which reduces to the generic point of X_j.

⇒ Inspired by a paper by Kani who was considering finitely additive measures on the Boolean algebra generated by the affinoids of a curve.

Example: **Weil height**. $\mathscr{X} = \mathbf{P}^1$, $\mathscr{L} = \mathscr{O}(1)$.

$c_1(\overline{\mathscr{L}})_v$ = Dirac measure at the Gauss point.

Measures (integrable metrics)

Let X be a variety over a number field F, dimension n
$\overline{\mathscr{L}}$ a line bundle on X together with an integrable metric.
In the limit process giving rise to integrable metrics from elementary ones, the defined measures converge to a measure $c_1(\overline{\mathscr{L}})^n$ on X_v^{an}.

For example, given a polarized dynamical systems $(X, \mathscr{L}, f)$, one obtains a measure $\mu_{f,v}$ analogous to the canonical measure in the archimedean case, satisfying the relations $f_*\mu_f = \mu_f$ and $f^*\mu_f = d^{\dim X}\mu_f$.

Measures (integrable metrics)

Properties:

- the total mass of $c_1(\overline{\mathscr{L}})^n$ is equal to $\deg_{\mathscr{L}}(X) = (c_1(\mathscr{L})^n|X)$;
- if $\overline{\mathscr{L}}$ is “semipositive”, then $c_1(\overline{\mathscr{L}})^n$ is a positive measure.
- symmetric multilinear extension;
- compatibility with morphisms and products.

Notice: We defined $c_1(\overline{\mathscr{L}})^n$; I have no idea of what $c_1(\overline{\mathscr{L}})$ could be in general.

Equidistribution

With these definitions of measures, the formulas in Arakelov geomety that we used can be rewritten so as to treat all places on equal foot.
This implies that the variational principle can be used again (essentially verbatim) and we obtain a proof of the following theorem:

Let $(X, \mathscr{L}, f)$ be a polarized dynamical system over a number field $F \subset \mathbf{C}_p$. Let (P_k) be a sequence of points of $X(\overline{F})$ satisfying the following assumptions:

1. $\hat{h}_f(P_k) \to 0$;
2. for any subvariety $V \subsetneq X$, $P_k \notin V$ for k big enough.

Then the discrete measures $\delta(P_k)$ converge to the measure μ_f on X_p^{an}.