Global $\epsilon$-factors for Symmetric Products of the Kloosterman Sheaf

Lei Fu
Institute of Mathematics, Nankai University, Tianjin, P. R. China
leifu@nankai.edu.cn

This is a joint work with Daqing Wan. Let $p \neq 2$ be a prime number, $\mathbb{F}_p$ the finite field with $p$ elements, $\ell$ a prime number different from $p$, and $\psi : \mathbb{F}_p \to \overline{\mathbb{F}_q}^*$ a nontrivial additive character. The Kloosterman sheaf $KL_2$ is a lisse $\overline{\mathbb{Q}}_l$-sheaf of rank 2 on $G_m(\mathbb{F}_q) = \mathbb{F}_q^*$, $\text{Tr}(F_x, KL_{2,x})$ is equal to the Kloosterman sum

$$- \sum_{x \in \mathbb{F}_q^*} \psi \left( \text{Tr}_{\mathbb{F}_x/\mathbb{F}_p} \left( \lambda + \frac{x}{\lambda} \right) \right),$$

where $F_x$ is the geometric Frobenius element at the point $x$. Let $j : G_m \to \mathbb{P}^1$ be the open immersion. For each positive integer $k$, the $L$-function

$$L_k(T) = L(\mathbb{P}^1, j_*(\text{Sym}^k(KL_2)), T)$$

of the $k$-th symmetric product of $KL_2$ satisfies a functional equation of the form

$$L_k(T) = cT^\delta L \left( \frac{1}{p^{k+1}}, T^\delta \right),$$

where

$$c^2 = p^{(k+1)\delta},$$

and $\delta = -\chi(\mathbb{P}^1, j_*(\text{Sym}^k(KL_2)))$. The value of $\delta$ has been determined by us before. Based on numerical calculation, R. Evans made conjectures on the sign of $c$ for $k = 7, 11, 13$. We calculate $c$ for all $k$. A corollary of our result is the following:

**Proposition.** If $k$ is even and $p > 2$, the sign of $c$ is always 1. If $k$ is odd and $p > k$, the sign of $c$ is

$$(-1)^{\frac{k-1}{2}} \prod_{j \in \{0,1,\ldots,\lfloor \frac{k}{2} \rfloor \}} p, \frac{p}{j+1} \left( \frac{(-1)^j(2j+1)}{p} \right).$$

In particular, this implies the conjecture of Evans.

The key step is to determine the (arithmetic) local monodromy at $\infty$ of the Kloosterman sheaf using local Fourier transformations and Laumon’s stationary phase principle. We then calculate local $\epsilon$-factors for symmetric products of the Kloosterman sheaf using explicit reciprocity laws. Finally we get the formula for $c$ using Laumon’s product formula.